Coherent structures in an energy-enstrophy theory for axisymmetric flows

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This paper is dedicated to Professor Alexander Chorin on the occasion of his 65th birthday.

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Abstract

The equilibrium statistical mechanics of a version of an energy-enstrophy theory for the axisymmetric Euler equations is solved exactly in the sense that a configurational integral is calculated in closed form. Under the assumption that the energy and the enstrophy (mean squared azimuthal vorticity) are conserved, a long range version of Kac's Spherical Model with logarithmic interaction is derived and solved exactly in the zero total circulation case in the standard thermodynamic limit. The spherical model formulation is based on the fundamental observation that the conservation of enstrophy expressed microcanonically is mathematically equivalent to Kac's spherical constraint. Two-point vorticity correlations are calculated exactly in two qualitatively different phases separated by $\tilde{\beta}_* = 0$. Physical interpretations of the results in this paper are obtained and applied to the relaxed end-states of turbulent high Reynolds number round jets. The negative temperature phase is a very high energy (flow rate) and relatively low enstrophy turbulent flow state which has the form of a axially uniform axial round jet plus a long wavelength axial perturbation. The positive temperature phase corresponds to a low energy and high enstrophy (shear) flow state which consists of a disordered vorticity distribution.

Keywords: axisymmetric vorticity equation, statistical equilibrium energy enstrophy theory, spherical model, 2-D turbulence, inverse cascade

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1. Introduction

The principal aim of this paper is to formulate and solve exactly an energy-enstrophy statistical mechanics theory for incompressible axisymmetric flows. Axisymmetric flows are intermediate in complexity between the 2-D Euler equations and the 3-D Euler equations, due to the presence of a geometric stretching term which is absent in 2-D flows. Three-dimensional Euler flows have the additional complexity of the vortex stretching term. The study of self-organized structures in axisymmetric flows is relevant to many technological applications concerning combustion and sound generation [1]. Large scale coherent structures play an important role in these engineering applications because they are key objects in the turbulent flow field that control the mixing between the unburned fuel/air mixture and the reacted products. A good review for mixing layers and jets is Ho and Huerre [6], and Crow and Champagne [7] respectively.

Another physically interesting axisymmetric inviscid problem that can be addressed using the approach proposed in this paper, is the vortex pinch-off process as a relaxed end-state of axisymmetric vortex ring formation [8], [9]. In fact, Mohseni [10] used a Miller-Robert statistical equilibrium approach to solve this problem. Our method which is based on the exactly-solvable Spherical Model formulation gives exact expressions for the all important partition function Z and two point vorticity correlations in the vortex pinch-off problem.

Turbulent jets are typically high Reynolds numbers phenomena and often not axisymmetric, adding to the difficulty of their study. Moreover, the jet half width $r_{1/2}(x)$ and the center line velocity $u(0,x)$ of a self-similar region in a high Reynolds number turbulent jet are given by

$$\begin{aligned} r_{1/2}(x) &= S(x-x_0) \\ \frac{u(0,x)}{u_e} &= 2B\frac{R}{x-x_0} \end{aligned} \tag{1.1}$$

where B, S, u_e, x_0 and R are constants [4], [5]. These expressions imply that the jet width grows linearly with x independent of the Reynolds number, which is another source of technical difficulty in its theoretical (analytical and numerical) analysis. Despite many technical difficulties, it is still possible to attempt an exact solution of a suitably reduced and idealized model which will be discussed next.

Since viscous turbulent axisymmetric flows are very complicated problems that are impossible to solve analytically using current mathematical methods, we will work on a simpler model. We will make two simplifying assumptions: (1) the viscosity is zero, and (2) the flow is axisymmetric. In most engineering applications of turbulent axisymmetric flows, transients are not important and only the steady-state matters. More precisely, it is the statistically stationary steady-state that one is interested in studying. When taking the viscosity in the model to zero, statistically stationary steady states must necessarily tend to statistical equilibrium states in the corresponding zero viscosity statistical mechanics model. The assumption of axisymmetry allows us to reduce the problem to essentially a 2D inviscid problem for which there is a well-known Hamiltonian formulation in terms of a stream function, and for which, a exactly-solvable energy-enstrophy theory has been recently developed.

Many equilibrium statistical mechanics theories for 2D turbulence have been proposed in the past decades. These include the Miller-Robert theory [12], [13] where an infinite number of Cassimirs are explicitly included in the Gibbs ensemble, and earlier theories due to Montgomery and Joyce [14] and Lundgren [15]. Of the many equilibrium statistical mechanics models for quasi-2D turbulence (cf. Chorin [17]), the energy-enstrophy theory [18], [20], [21] is special because it has been shown recently to be not only exactly-solvable in the sense of having closed form expressions for its partition function but also statistically exact with respect to higher order enstrophy constraints (see Majda and Holen [3]). That it is exactly-solvable is easily seen when we prove below (see section 3) that Kraichnan's model is equivalent to the Gaussian model. We will discuss in section 3 some of its properties in relation to a well-known defect of the Gaussian Model called the low temperature catastrophe. This low temperature defect is one of the motivatons for re-formulating the Energy-Enstrophy theories in terms of the Spherical Model [24] which corresponds to a probability measure that is canonical in the energy and micro-canonical in the enstrophy constraint. In fact the spherical constraint or equivalently, the micro-canonical enstrophy constraint modifies the original energy-enstrophy theory so as to prevent the so-called low and negative temperature catastrophes that Gaussian models are prone to (cf. Lim [25]).

Another even more important motivation for using a microcanonical enstrophy constraint instead of the traditional canonical constraint in Kraichnan's model is to avoid making what amounts to the assumption that the equilibrium statistical behaviour of the 2D and axisymmetric Euler model in unbounded flow domains is governed by Gaussian i.i.d. statistics. We will prove below that the canonical enstrophy constraint is equivalent to this assumption of Gaussian i.i.d. statistics. Although Gaussian i.i.d. statistics is correct for some equilibrium statistical mechanics models for bounded flow domains in a nonextensive continuum limit, there is no evidence that it is correct for flows in unbounded domains in the standard or extensive thermodynamic limit. Because we explicitly avoided making the assumption of Gaussian i.i.d. statistics, this Spherical model formulation of the energy-enstrophy theory is not a mean field theory unlike the Miller-Robert [12], [13] and preceding theories [14], [15].

The results reported here address the issues of coherent structures and inverse energy cascade for the idealized axisymmetric jet and also other inviscid axisymmetric flows, such as periodic vortex rings [26]. A subsequent paper will be devoted to the Spherical Model approach to the periodic vortex ring problem. Using a Gibbs ensemble which is canonical in the energy and micro-canonical or sharp in the enstrophy (also known as the Spherical model formulation) we extend the energy-enstrophy theory to axisymmetric inviscid flows. We calculate in closed-form, two-point azimuthal vorticity correlations in several distinct regimes of flow kinetic energy and azimuthal enstrophy space. $T_* = \infty$ is the boundary between coherent structures at negative temperatures and a random state at positive temperatures. In fact the negative temperature states are very high energy and relatively low enstrophy turbulent flows. On the other hand, the positive temperature states at $0 < T < \infty$ have low energy and relatively high enstrophy.

We find that for negative temperatures, the spin-spin correlations $\langle x_i x_j \rangle$ are dominated by the largest wavelength eigenfunctions, which indicated the presence of a large coherent structure, which is related to the inverse cascade of energy to large scales [28]. On the other hand, the correlations for high positive temperatures indicated that the equilibrium macrostates consist of a random mix of vorticity of opposite signs.The equilibrium statistical mechanics temperature T in these results is none other than a Lagrange multipier associated with energy.

These formulations and exact solutions are based on the fundamental observation that in a extensive or standard thermodynamic limit, the microcanonical enstrophy (mean squared vorticity) constraint is mathematically equivalent to Kac's spherical constraint and the equilibrium statistics of the new version of the energy-enstrophy theory can be obtained by analyzing the exact solutions for a long-range lattice vortex gas known as the Spherical model [24]. Moreover, it will be clear from the discussion below that this lattice Hamiltonian formulation for the equilibrium statistical mechanics of ideal axisymmetric flows, is based on a Eulerian-type (or fixed grid) numerical approximation of the azimuthal vorticity distribution in the steady-state Euler equations. Indeed, as the number M of lattice sites in these models tend to ∞ in an extensive thermodynamic limit, this family of lattice Hamiltonians approaches the steady-state Euler equations for axisymmetric flows. It will be explained below that in order to study phase transitions, one needs to take a thermodynamic or continuum limit of a suitable family of lattice models. In the case of fluid turbulence in unbounded flow domains, the appropriate limit is an extensive continuum limit.

In the following sections we will set up the problem and comment on the physical relevance of equilibrium statistical mechanics for the zero swirl axisymmetric problem. Then, we will formulate the correct energy-enstrophy theory for this problem and write down the Gibbs ensemble that is the principal object of investigation. We will then demonstrate that the equilibrium statistical mechanics of the energy-enstrophy theory for this problem is mathematically equivalent to a version of Kac's Spherical model. The exact solution for the partition function of the Spherical model gives exact expressions for the free energy and two-point correlation functions for the axisymmetric flow problem. We will devote a section to the physical interpretations of the mathematically exact results in this paper. One of the significant physical conclusions is that for very high energies and relatively low enstrophy the two-point correlations strongly indicate a relaxed flow state in the form of a nearly uniform axial round jet whose axial velocity profile has a shear curve which is given by the graph of the Bessel's function up to its first node. This flow state is stable according to a version of Rayleigh's stability criterion for plane Poiseuille flow.

2. Axisymmetric flows

We will focus only on the zero swirl case and leave the problem of axisymmetric swirling flows for another paper. Non-zero swirl can lead to non-axisymmetric instability, which will add considerably to the technical difficulties of obtaining exact analytical solutions. Axisymmetric flows in the case of zero swirl are relevant due to the ubiquitous role of round jets in engineering applications. Although the equations of motion for inviscid incompressible (ideal) axisymmetric flows are considerably simpler than the full 3-D Euler equations, they are nontrivial problems.

2.1. Vorticity formulation

The basic formulation is that due to Szeri and Holmes [29]; we will follow their notation in this paper. We will take the equations of motion to be given by

$$\frac{Dq}{Dt} \equiv q_t + J(\Psi, q) = 0 \tag{2.1}$$

$$\Delta_a \Psi \equiv \Psi_{rr} + \frac{1}{r}\Psi_r + \Psi_{xx} = -q, \tag{2.2}$$

where r is the undressed radial coordinate, $q = \frac{\omega}{r}$ is the total azimuthal vorticity density, and the Jacobian is

$$J(\Psi, q) \equiv \frac{1}{r}\left(\Psi_r q_x - \Psi_x q_r\right).$$

The stream function is denoted by $\Psi(x, y)$. In this formulation,

$$u = \frac{1}{r}\Psi_r;\ v = -\frac{1}{r}\Psi_x$$

where u and v are respectively the axial and radial components of velocity.

The main parameters in the axially uniform axial mean flow are the mean flow rate Q and the radial shear $K = \Psi_{rr}$. A high Q together with a low K corresponds to a high energy, low enstrophy jet. On the other hand, a low Q and high K state corresponds to a low energy, high enstrophy jet. As we shall see below, these two regimes in Q and K are qualitatively different. We will characterize them as different phases in a axisymmetric version of an energy-enstrophy theory for fully developed 2D turbulence.

Using the expression (1.1) with $S \approx 0.094$ from the self-similar solution of high Reynolds number turbulent jet [4], [5], we deduce that the rate of linear growth of jet width is less than 10 percent. We are thus partly justified in drawing the conclusion that for the purpose of modeling the steady-state flows of a high Re axisymmetric jet, it is sufficient to consider a cylindrical flow domain $\{0 \leq r < a,$ $x \in [d, d+2l]\ \}$ with periodic boundary conditions, which is not too close to the jet nozzle. This domain is also applicable to the axially-periodic axisymmetric vortex ring formation problem.

We will first introduce the axially-periodic finite domain

$$D'' = \{(x,r) \mid x \in [-l,l],\ r \in (-a,a)\},$$

where we impose the axisymmetry condition that for all x,

$$\Psi(x,r) = \Psi(x,-r).$$

This is equivalent to working in the domain

$$D' = \{(x,r) \mid x \in (-l,l),\ r \in [0,a)\}.$$

When l and a are large, D' will be well approximated by the semi-infinite domain

$$D = \{(x, r) \mid x \in (-\infty, \infty),\ r \in [0, \infty)\}.$$

We will work primarily with an infinite domain in the standard thermodynamic limit which will be discussed in the next section.

A natural formulation due to Kelvin and Benjamin [31] is given below for this problem in terms of a Hamiltonian function

$$-H = \int_D q\Psi r dx dr, \tag{2.3}$$

which can be written in Green's function form as

$$-H = \int_D \int_D G(x, r; x', r') q(x, r) q(x', r') r r' dx dr dx' dr'. \tag{2.4}$$

Given the nature of the domain D, the method of images gives the Green's function

$$G(x-x', r, r') = -\frac{1}{2}\left\{\ln\left[(x-x')^2 + rr'(r-r')^2\right] + \ln\left[(x-x')^2 + rr'(r+r')^2\right]\right\}. \tag{2.5}$$

Clearly this Green's function G is the solution of the problem

$$\Delta_s G = -\delta(x', r'),$$

and it can be expanded in usual fashion

$$G(x - x', r, r') = \sum_{\vec{k}} \frac{1}{(k_2^2 + k_1^2)} e^{ik_1(x-x')} \Phi_{k_2}(r) \Phi^*_{k_2}(r')$$

in terms of the eigenfunctions

$$\Psi_{(k_1,k_2)} = \Phi_{k_2}(r) e^{ik_1 x} = c J_o(k_2 r) e^{ik_1 x}$$

of the problem

$$\Delta_s \Psi + \lambda^2 \Psi = 0$$

on the semi-infinite domain D. Here the wave-vector $\vec{k} = (k_1, k_2)$; $J_o(k_2 r)$ is the zeroth order Bessel's function which is finite at $r = 0$; and $\lambda^2 = k_2^2 + k_1^2$.

It follows directly from the form of its dependence on x and r that the Hamiltonian is translationally invariant along the axial direction but is dependent on the actual locations in the radial direction. Besides the energy H, these equations have an infinite family of conserved quantities given by

$$\Omega_f = \int_D f(q) r dx dr$$

where f is any continuous function. Clearly, the enstrophy or the L_2 norm of the azimuthal vorticity density q is such a conserved quantity

$$\Omega \equiv \int_D q^2 r dx dr = K. \tag{2.6}$$

2.2. Boundary conditions

When working with the finite periodic domain D'' we will fix the boundary condition

$$\Psi(x, a) = 0$$

which is suitable for the study of axially-periodic axisymmetric vortex rings when the fixed radius a is sufficiently large; we get that $J_o(k_2 a) = 0$. This implies that the eigenvalues k_2 are given by the nodes λ_{0s} of the Bessel's function J_o, i.e.

$$k_2 = \frac{\lambda_{0s}}{a}, \text{ for } s = 1, \ldots.$$

The axisymmetry and smoothness of the flow about the $r = 0$ axis implies the following for the domain D'' :

(A) The derivatives Ψ_{rr} and Ψ_{xr} are necessarily zero at $r = 0$;

(B) The fluid velocity at $r = a$ is identical to that at $r = -a$.

As a consequence of the periodic boundary condition in the x direction and (B) for the finite domain D'', we have that the total circulation

$$\Gamma \equiv \int_{D''} q \; r dx dr = 0 \tag{2.7}$$

by Stokes theorem.

By the natural condition that the flow vanishes as $|r| \to \infty$, and assuming the same entry and exit flow at the ends $|x| \to \infty$, we get the following for domain D, which is applicable to the study of steady-state axisymmetric but non-periodic flows:

(C) The azimuthal vorticity q tends to zero as $|r| \to \infty$;

(D) The total circulation $\Gamma \equiv \int_D q \; r dx dr = 0$ again by an application of Stokes theorem.

A slight modification of the previous conditions on D obtained by relaxing the condition that the flow are identical at the ends $|x| \to \infty$ to the new condition that the entry and exit flows have zero radial velocity component, yields the following which is a better model for the study of turbulent axisymmetric free jets than the periodic domain D'' :

(E) The total circulation $\Gamma \equiv \int_D q \, r dx dr = 0$ again by an application of Stokes theorem.

We will defer discussion of applications of the last conditions on D to a future paper.

3. Kraichnan's Energy-Enstrophy Theory

The main object of study in Kraichnan's Energy - Enstrophy theory is the Gibbs doubly-canonical probability measure

$$\begin{aligned} P(\vec{s};\beta,\mu) &= \frac{1}{Z}\exp\{-\beta E(\vec{s}) - \mu\Omega(\vec{s})\} \\ &= \frac{1}{Z}\exp\left\{-\mu\left(\frac{\beta}{\mu}E(\vec{s}) + \Omega(\vec{s})\right)\right\} \end{aligned} \tag{3.1}$$

where Z is the partition function, $\vec{s}$ is the vorticity distribution

$$\vec{s} = (s_1, s_2, ..., s_M) \in (-\infty, \infty)^M$$

at M fixed lattice sites within a fixed bounded flow domain D; $E(\vec{s})$ and β are the logarithmic interaction energy and inverse temperature, $\Omega(\vec{s})$ is the enstrophy and μ its Lagrange multiplier. This theory is called the Energy-Enstrophy theory because of the explicit presence of the energy and enstrophy constraints, and was first formulated in terms of a spectral or Fourier representation for the case of ideal flows on the plane by Kraichnan [18], [20].

The form (3.1) of the above Gibbs factor can be derived from a Maximum Entropy Principle based on the two explicit constraints

$$E(\vec{s}) = E_0;\ \Omega(\vec{s}) = \Omega_0$$

or equivalently a least biased information-theoretic formulation based on one-point statistics $\rho(x_j, s)$ with the additional probability constraint $\int \rho(x_j, s)\ ds = 1$ for each x_j in the lattice. This theory is also known as the empirical statistical mechanics theory [12], [13], [17]. Where it is valid, this theory has a rigorous foundation in terms of large deviation results in probability theory. It is characteristic in this theory to have a factorization of the joint probability distribution of the lattice into a product of Gaussians, thus the label of one-point statistics. The original form of this factorization was given by Joyce and Montgomery [14] and Lundgren and Pointin [15] in the particle or point vortex formulation of the statistics of the 2D Euler equation in a nonextensive continuum limit. The explicit spatial formulation of the Euler equations in terms of one-point statistics was originally due to Miller [12] and Robert [13], again in the nonextensive thermodynamic limit.

Several properties of this class of theories are worth making explicit. The first property is:

Proposition 3.1. *The doubly-canonical probability measure*

$$P(\vec{s};\beta,\mu)=\frac{1}{Z}\exp\{-\beta E(\vec{s})-\mu\Omega(\vec{s})\}$$

is equivalent to that in the Gaussian Model, that is, Gaussian i.i.d. statistics holds for the Energy-Enstrophy Gibbs factor in (3.1).

Proof. With energy given by

$$E(\vec{s})=-\frac{1}{2\pi}\sum_{j<k}s_js_k\ln l_{jk} \tag{3.2}$$

where l_{jk} denotes the fixed separation between the $j-th$ and $k-th$ lattice sites, and entrophy by

$$\Omega(\vec{s}) = \sum_{j=1}^{M} s_j^2$$

the expression above for the Gibbs factor is easily seen to be the canonical probability measure of lattice Gaussian i.i.d. variables with common variance $1/\mu$ and zero mean, by noting that

$$\begin{aligned} \exp\{-\mu\Omega(\vec{s})\}\ d\vec{s} &= \exp\left\{-\mu\sum_{j=1}^{M} s_j^2\right\}\ \prod_{j=1}^{M} ds_j \\ &= \prod_{j=1}^{M} \exp\left\{-\mu s_j^2\right\}\ ds_j. \end{aligned} \tag{3.3}$$

By canonical probability measure, we mean that there is a $\exp\{\ -\beta E(\vec{s})\}$ in the Gibbs factor. ■

It is very important to note that the zero mean i.i.d. in (3.3) does not imply that the only large-scale coherent structure in this Energy-Enstrophy theory is the trivial or spatially uniform one. The second observation, well-known in condensed matter and particle physics, is that the Gaussian Model has a low temperature defect. We will prove this property in the specific case of low positive and negative temperatures.

Proposition 3.2. *The Gaussian Model in (3.1) has the following specific defect: at sufficiently low positive temperatures or $\beta > 0$ large enough with respect to $\mu > 0$, and negative temperatures or $\beta < 0$, the Gibbs factor* $\exp\{-\beta E(\vec{s})-\mu\Omega(\vec{s})\}$ *is unbounded on the phase space $\vec{s} \in (-\infty,\infty)^M$.*

Proof. We note that under the vorticity scaling $\vec{s} \to \lambda\vec{s}$, the energy E scales like

$$E(\lambda\vec{s}) = \lambda^2 E(\vec{s})$$

and the enstrophy like

$$\Omega(\lambda\vec{s}) = \lambda^2\Omega(\vec{s}).$$

Thus, the Gibbs factor for fixed M, β and μ scales like

$$\begin{aligned} & \exp\{-\beta E(\lambda\vec{s}) - \mu\Omega(\lambda\vec{s})\} \\ = \ & \exp\{-\lambda^2[\beta E(\vec{s}) + \mu\Omega(\vec{s})]\}. \end{aligned} \tag{3.4}$$

The exact solution of the Gaussian model (3.1) for $\beta < 0$ implies that the expected value of the energy $E(\vec{s})$ is positive due to the negative value of the logarithm: neighboring lattice sites have spins s_j of the same sign when $\beta < 0$. Equation (3.4) then implies that the Gibbs factor tends to ∞ as $\lambda \to \infty$ when $\beta < 0$.

When $\beta > 0$, the exact solution of the Gaussian model (3.1) implies that neighboring lattice sites have spins s_j of opposite signs. Thus, the expected value of the energy $E(\vec{s})$ is negative when $\beta > 0$. Again, equation (3.4) implies that the Gibbs factor tends to ∞ as $\lambda \to \infty$ when $\beta > 0$ *is* sufficiently large compared to $\mu > 0$. ■

In numerous Monte-Carlo simulations of (3.1) which are reported in Nebus' PhD thesis [23], we showed that for the values of β and μ in the above proposition, there are no lower bound for the energy E nor upper bound for the enstrophy Ω. Instead we observed that generically over a large number of runs, $E(\vec{s}) \to \infty$ and $\Omega(\vec{s}) \to \infty$ quadratically as a function of the number of Monte-Carlo sweeps in a run for $\beta < 0$. These observations from Monte-Carlo simulations in [23] are consistent with the statement of Proposition 2. As discussed above, the Energy-Enstrophy theory is equivalent to a one-point statistics mean field theory [27] through the Maximum Entropy Principle specialized to a lattice with M nodes.

3.1. Standard Thermodynamic Limit

Proposition 2 is a statement for fixed M and is distinct from the fact that as M tends to ∞ in the nonextensive limit, the vortex strength has to be scaled by $1/\sqrt{M}$ so that the interaction energy remains bounded in this limit for a fixed finite domain. The situation is different when we attempt to model 2D or axisymmetric flows in unbounded domains where the standard thermodynamic limit is the physically relevant one to use. In the standard or extensive limit, the above mean field assumption of Gaussian i.i.d. statistics has not been proven to be valid in the problem of a neutral vortex or Coulomb gas in an unbounded domain. Thus, by Proposition 1, the usual energy-enstrophy theory based on a canonical enstrophy constraint may not be valid in the case of unbounded flow domains. In any case, according to Occam's Principle of making the least possible apriori assumptions in the construction of any physical models, we should not assume the validity of the mean field.

The easiest thing to do for axisymmetric flows in unbounded domains is therefore, to abandon the canonical enstrophy constraint (which has been shown in Proposition 1 to be equivalent to Gaussian i.i.d. statistics) in favor of a microcanonical enstrophy constraint. We will show in the next section that the microcanonical constraint is mathematically equivalent to Kac's spherical constraint in the famous Spherical Model which means that we can take the long range Spherical Model to be the correct reformulation of the Energy-Enstrophy theory for unbounded flow domains in the standard thermodynamic limit.

The standard thermodynamic limit for 2D and axisymmetric turbulence in unbounded flow domains can be reformulated in the usual form for a sequence of increasing, finite flow domains D_M on the boundaries ∂D_M of which one prescribe free boundary conditions. The area A_M of D_M is proportional to the number of lattice sites M in the domains. Next, one scales the strength of vortex interaction $\mu \sim \mu^*/\sqrt{M}$ so that the total interaction energy per lattice site is held fixed as M tends to ∞. This means that the temperature is not scaled in this formulation of the standard limit. Note that in this limit, total enstrophy is held fixed as M tends to ∞. Later we will see that in terms of the lattice spins $x_j \in (-\infty, \infty)$, this conservation of total enstrophy is given by the well-known spherical constraint, i.e.,

$$\sum_{j=1}^{M} s_j^2 = M.$$

Next, we show that the above formulation of the standard limit in terms of a sequence of increasing domains and scaling $\mu \sim \mu^*/\sqrt{M}$ is indeed equivalent to the problem in an unbounded flow domain where we consider a sequence of meshes L_M of M lattice sites and no scaling of vortex strength. One immediate problem in showing this equivalence is the technical matter of what kinds of mesh L_M do we use to fill out the infinite domain. It turns out that the problem of a vortex gas in an unbounded domain and an exact solution due to Ginibre provides a vital ingredient, namely, the M lattice sites should be placed uniformly in a disk of radius $\sqrt{M}$ centered at the origin [19]. Since the vortex interaction $\mu = \mu^*$ is fixed at the outset, the total energy increases proportionally with M^2 but the energy density (per unit flow area) increases linearly in M. The enstrophy density (per unit flow area) is now held fixed as $M \rightarrow \infty$. In view of the increasing energy density with M, the temperature has to be scaled by M in the current version of the standard limit. We will use both of these equivalent versions but will emphasize the former where $\mu \sim \mu^*/\sqrt{M}$.

While the above standard limit allows us to study the infinite domain problem as the number M of lattice sites tends to ∞ in such a way as to keep the density σ of lattice site fixed, it does not address the important computational issue of refining the mesh so that we locally approach the original continuum problem of axisymmetric Euler statistical mechanics. This issue must therefore be addressed now: to treat the infinite domain problem, we first pick a density σ for the mesh and generate a mesh with M uniformly distributed sites according to the rule discussed above, and we increase M until we have reached a large enough radius $\sqrt{M}$ of the domain D_M to cover all the physical phenomena that we are investigating; next, we change the mesh to a higher fixed density σ and then go through the $M-indexed$ sequence again; this increase of mesh density is carried out until the mesh is fine enough to effectively compute the physical phenomena we are studying. There is however no attempt to take the mesh density σ to infinity. The only limiting process considered in this paper is the standard thermodynamic limit as $M \to \infty$.

4. Spherical Model

4.1. Lattice Hamiltonians

Underlying this formulation is a family of lattice Hamiltonians defined by

$$H(M) = -\frac{1}{2\pi}\sum_{j<k} J_{jk}\ s_j s_k \tag{4.1}$$

with

$$J_{jk} = \mu^2 \left(\ln\left[(x_j - x_k)^2 + r_j r_k\,(r_j - r_k)^2\right] + \ln\left[(x_j - x_k)^2 + r_j r_k\,(r_j + r_k)^2\right]\right); \tag{4.2}$$

$s_j \in (-\infty, \infty)$ is the value of the local vorticity or spin and $z_j = (x_j, r_j)$ gives the position of lattice site j in cylindrical coordinates. Now we fix a square lattice with a total of M sites and diameter $L = M^{1/2}$ within the domain D.

In order for the above family of lattice Hamiltonians to be a good model of the infinite flow domain problem, we will take the limit $M \to \infty$ in a definite manner suitable for the statistical mechanics of unbounded axisymmetric fluid turbulence, that is,

$$
\begin{aligned}
M &= 4^m M_* \to \infty \\
L &= M^{1/2} \to \infty, \\
\mu &= 2^{-m} \mu_* \to 0 \\
\text{as } m &\to \infty,
\end{aligned}
\tag{4.3}
$$

where the starred quantities are the initial values of the number of lattice sites M and vorticity scale μ. The proof that the family given by (4.1) and (4.2) has standard thermodynamic limit as $m \to \infty$ will follow from the existence and construction of closed form analytic solutions to this problem.

4.2. Key Lemma

Here we prove a key result justifying the use of Kac's spherical models to study the axisymmetric flow problem.

Lemma 4.1. *The microcanonical enstrophy constraint $\Omega = K$ is mathematically equivalent to Kac's spherical constraint $\sum_{i=1}^{M} s_i^2 = M$ for all values of M.*

Proof. The lattice form of the enstrophy constraint is given by

$$\mu_*^2 \sum_{i=1}^{M_*} s_i^2 = K$$

in terms of the initial values of relevant quantities. Moreover, we are free to choose the initial values μ_* and M_* so that

$$\sum_{i=1}^{M_*} s_i^2 = M_* = \frac{K}{\mu_*^2}. \tag{4.4}$$

Since the initial enstrophy K must be preserved as m increases (where the unstarred quantities scale as in (4.3)), we have

$$\mu^2 \sum_{i=1}^{M} s_i^2 = 4^{-m} \mu_*^2 \sum_{i=1}^{4^m M_*} s_i^2 = K.$$

Then it follows immediately that

$$4^{-m} \sum_{i=1}^{M} s_i^2 = M_*$$

which in view of (4.3) implies

$$\sum_{i=1}^{M} s_i^2 = M \tag{4.5}$$

as $m \to \infty$. ■

5. Exact solutions via the Spherical model

In this section we present an exact solution of the equilibrium statistical mechanics problem given by (4.1), (4.2) and (4.5), which according to Lemma 3.1, is a lattice form of the conservation of total enstrophy $\Omega = K$ (2.6). The partition function for the family of Spherical models parametrized by the index m in equation (4.3) is

$$Z_S = \int_{-\infty}^{\infty} ds_1 ... \int_{-\infty}^{\infty} ds_M \exp\left[\frac{\beta}{2\pi}\sum_{i<j=1}^{M} s_i J_{ij} s_j + \beta\gamma\sum_{i=1}^{M} s_i\right] \delta\left(\sum_{j=1}^{M} s_j^2 - M\right) \quad (5.1)$$

$$= \int_{-\infty}^{\infty} ds_1 ... \int_{-\infty}^{\infty} ds_M \exp\left[\frac{\beta}{2}\sum_{i<j=1}^{M} s_i J_{ij} s_j + \beta\gamma\sum_{i=1}^{M} s_i\right] \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp' \exp\left[p'\left(M - \sum_{j=1}^{M} s_j^2\right)\right], \quad (5.2)$$

where we have used the augmented Hamiltonian expression

$$H = -\frac{1}{2\pi}\sum_{i<j=1}^{M} s_i J_{ij} s_j - \gamma\sum_{i=1}^{M} s_i$$

to incorporate the total circulation constraint. We will focus on the case where total circulation $\Gamma \equiv \sum_{i=1}^{M} s_i = 0$.

The next step involves a diagonalization of the interaction J_{ij} by discrete Fourier transforms, i.e.,

$$\frac{\beta}{2\pi}\sum_{i<j=1}^{M} s_i J_{ij} s_j = \frac{\beta\mu_*^2}{8\pi M}\sum_{\vec{q}\neq\vec{0}} a(\vec{q})|z_{\vec{q}}|^2, \tag{5.3}$$

where $z_{\vec{q}}$ are the Fourier transform of $s_{\vec{j}}$, and $a(\vec{q})$ are the eigenvalues of J_{ij}. Since $\Gamma \equiv \sum_{i=1}^{M} s_i = 0$, we have

$$z_{\vec{0}} = \sum_{i=1}^{M} s_i = 0.$$

Since

$$a(\vec{q}) = -|\vec{\phi}|^{-2},$$

the above partition function is therefore given by

$$\begin{aligned} Z_S &= \frac{e^{M\alpha}}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} dp\ e^{pM}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\sum_{\vec{q}\neq\vec{0}}|z_{\vec{q}}|^2\left(p-\tilde{\beta}a(\vec{q})\right)\right]\prod_{\vec{q}\neq\vec{0}} dz_{\vec{q}} \\ &= \frac{\pi^{\frac{M}{2}}e^{M\alpha}}{\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} dp\left(\frac{e^{pM}}{\prod_{\vec{q}\neq\vec{0}}(p+\tilde{\beta}|\vec{\phi}|^{-2})^{1/2}}\right). \end{aligned} \tag{5.4}$$

provided

$$p + \tilde{\beta}|\vec{\phi}|^{-2} > 0. \tag{5.5}$$

Here

$$\tilde{\beta} = \frac{\beta \mu_*^2}{4\pi M}$$

and $\vec{\phi} = (\phi_1, \phi_2)$ where

$$\begin{aligned} \phi_1 &= \frac{2\pi q_1}{L}, \; q_1 \in \{0, 1, ..., L-1\} \\ \phi_2 &= \frac{\lambda_{0s}}{L}, \; s = 1, 2, ..., L. \end{aligned} \tag{5.6}$$

We have used, in equality (5.4), the elementary integral

$$\int_{-\infty}^{\infty} \exp(-cx^2)\; dx = \left(\frac{\pi}{c}\right)^{1/2},$$

which holds for any $c > 0$, with

$$c = \frac{1}{2}\left(p - \tilde{\beta} a(\vec{q})\right);$$

whence the condition for well-definition, equation (5.5).

The partition function Z_S for fixed M is given by

$$Z_S = \frac{\pi^{\frac{M}{2}} e^{M\alpha}}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} dp e^{pM} \exp\left(-\frac{1}{2} \sum_{(q_1,s)\neq(0,0)}^{(L,L)} \ln\left[p + \frac{\beta\mu_*^2}{4\pi}(q_1^2 + \lambda_{0s}^2)^{-1}\right]\right)$$

$$
\begin{aligned}
&= \frac{\pi^{\frac{M}{2}} e^{M\alpha}}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} dp e^{pM} \exp\left(-\frac{1}{2} \sum_{(q_1,s)\neq(0,0)}^{(L,L)} \ln\left[\frac{\beta\mu_*^2}{4\pi} \left(\xi + (q_1^2 + \lambda_{0s}^2)^{-1} \right) \right] \right) \\
&= \frac{\pi^{\frac{M}{2}} e^{M\alpha}}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} dp e^{pM} e^{-\frac{M}{2}\ln\left(\frac{\beta\mu_*^2}{4\pi}\right)} \exp\left(-\frac{1}{2} \sum_{(q_1,s)\neq(0,0)}^{(L,L)} \left[\ln\left(\xi + (q_1^2 + \lambda_{0s}^2)^{-1} \right) \right] \right) \\
&= \frac{\pi^{\frac{M}{2}} e^{M\alpha}}{2\pi i} \left(\frac{\beta\mu_*^2}{4\pi} \right)^{-\frac{M}{2}} \int_{\alpha-i\infty}^{\alpha+i\infty} dp e^{pM} \exp\left(-\frac{1}{2} \sum_{(q_1,s)\neq(0,0)}^{(L,L)} \left[\ln\left(\xi + (q_1^2 + \lambda_{0s}^2)^{-1} \right) \right] \right) \\
&= \frac{\pi^{\frac{M}{2}} e^{M\alpha}}{2\pi i} \left(\frac{\beta\mu_*^2}{4\pi} \right)^{1-\frac{M}{2}} \int_{\alpha-i\infty}^{\alpha+i\infty} d\xi \exp\left(\xi \frac{\beta\mu_*^2}{4\pi} - \frac{1}{2M} \sum_{(q_1,s)\neq(0,0)}^{(L,L)} \left[\ln\left(\xi + (q_1^2 + \lambda_{0s}^2)^{-1} \right) \right] \right)^M \\
&= \frac{\pi^{\frac{M}{2}} e^{M\alpha}}{2\pi i} \left(\frac{\beta\mu_*^2}{4\pi} \right)^{1-\frac{M}{2}} \int_{\alpha-i\infty}^{\alpha+i\infty} d\xi \exp g(M,\xi)
\end{aligned}
$$

where

$$
g(M,\xi) = M\left(\xi \frac{\beta\mu_*^2}{4\pi} - \frac{1}{2M} \sum_{(q_1,s)\neq(0,0)}^{(L,L)} \left[\ln\left(\xi + (q_1^2 + \lambda_{0s}^2)^{-1} \right) \right] \right)
$$

and

$$
\xi = 4\pi p/\beta\mu_*^2
$$

By the saddle point method,

$$
Z_S(M) \simeq \frac{\pi^{\frac{M}{2}} e^{M\alpha}}{2\pi i} \left(\frac{\beta\mu_*^2}{4\pi} \right)^{1-\frac{M}{2}} \frac{e^{g(M,\xi_S)}}{\sqrt{2\pi g''(M,\xi_S)}} \tag{5.7}
$$

where ξ_S satisfies the saddle point condition which for fixed value of M is given by

$$\frac{\beta\mu_*^2}{4\pi} = \frac{1}{2M} \sum_{(q_1,s)\neq(0,0)}^{(L,L)} \frac{1}{\xi_S + (q_1^2 + \lambda_{0s}^2)^{-1}} \tag{5.8}$$

As $M \to \infty$, the accuracy of the saddle point expression on the right hand side of (5.7) improves, whence the following limit exists

$$\lim_{M\to\infty} Z_S(M) \left[\frac{\pi^{\frac{M}{2}} e^{M\alpha}}{2\pi i} \left(\frac{\beta\mu_*^2}{4\pi} \right)^{1-\frac{M}{2}} \right]^{-1} = \frac{e^{g(\xi_S)}}{\sqrt{2\pi g''(\xi_S)}}$$

with

$$g(\xi) = \lim_{M\to\infty} g(M, \xi)$$

independent of M. Next we analyze the solvability of the saddle point condition (5.8) and the well-definition condition (5.5) in a range of temperatures.

5.1. Positive temperatures

Expression (5.5) is automatically satisfied for $\beta > 0$. The largest value of the right hand side of (5.8) is achieved when $\xi_S = 0$. In that case, the saddle point condition takes the critical form

$$\frac{\beta_c\mu_*^2}{4\pi} = \frac{1}{2M} \sum_{(q_1,s)\neq(0,0)}^{(L,L)} (q_1^2 + \lambda_{0s}^2) \to \infty$$

as $M \;\; \to \;\; \infty$

whence the conclusion that there is no positive temperature phase transition since $T_c = 1/\beta_c = 0$. Furthermore, we observe that the saddle point condition (5.8) can be solved for $\xi_S = \xi_S(\beta) > 0$ for any $\beta \in [0, \infty)$ in the standard limit of $M \to \infty$. And for $\beta > 0$, the dorminant term in the sum in (5.8) corresponds to the largest wavenumbers which satisfy

$$(q_1^2 + \lambda_{0s}^2) = 2M. \tag{5.9}$$

5.2. Negative temperatures

An equivalent expression for the well-definition of the partition function (5.5) for any given value of M is

$$\frac{\beta \mu_*^2}{4\pi} \left[\xi + (q_1^2 + \lambda_{0s}^2)^{-1} \right] > 0$$

which implies that for negative temperatures, i.e., $\beta < 0$,

$$\left[\xi + (q_1^2 + \lambda_{0s}^2)^{-1} \right] < 0. \tag{5.10}$$

Since it is necessary for $p \geq 0$, and thence $\xi = 4\pi p/\beta\mu_*^2$ to be non-positive when $\beta < 0$, the most critical wave numbers in (5.10) are those with

$$(q_1^2 + \lambda_{0s}^2) = 1. \tag{5.11}$$

In other words, for $\beta < 0$, the most significant terms in the analysis of the saddle point condition are those of largest wavelength, which for a given mesh with M lattice sites, is equal to $L = \sqrt{M}$. Thus, from (5.10) we conclude that

$$\xi \leq \xi_c = -1. \tag{5.12}$$

Moreover, substituting the critical value ξ_c from (5.12) into the saddle point condition (5.8) yields for any M,

$$\frac{\beta_c \mu_*^2}{4\pi} = \frac{1}{2M} \sum_{(q_1,s)\neq(0,0)}^{(L,L)} \frac{1}{\xi_c + (q_1^2 + \lambda_{0s}^2)^{-1}} = -\infty,$$

whence, the nonexistence of a negative temperature phase transition. Indeed, the saddle point condition (5.8) can be solved for $\xi_S = \xi_S(\beta) < -1$ for any negative $\beta \in (-\infty, 0)$.

6. Correlations

The two-point vorticity correlations for different temperature regimes will now be calculated. We will obtain correlations for two distinct temperature regimes: (i) temperatures $\infty > T \geq 0$, and (ii) negative temperatures $-\infty < T < 0$. The expectation operator $\langle \cdot \rangle$ is defined by the probability

$$P_M = \frac{1}{Z_S} \exp(-\beta H) \delta(\Omega - M)$$

since $\Gamma = 0$.

6.1. Positive temperatures $\infty > T > 0$

For finite values of M, the correlations are given by

$$\begin{aligned} \langle s_i s_j \rangle &= \frac{1}{\beta M} \frac{\partial \ln Z_S}{\partial J_{ij}} \qquad (6.1) \\ &\simeq \frac{1}{2\beta M} \sum_{q_1=1}^{L} \sum_{s=1}^{L} \Big(\xi_S + (q_1^2 + \lambda_{0s}^2)^{-1} \Big)^{-1} e^{i \frac{2\pi q_1}{L}(x_i - x_j)} J_o(\frac{\lambda_{0s}}{L} r_i) J_o(\frac{\lambda_{0s}}{L} r_j), \end{aligned}$$

Since for given $\beta > 0$, $\xi_S(\beta) > 0$ is fixed, the dominant term in the sum in (6.1) corresponds to the largest wavenumbers given by (5.9). It follows that the two-point vorticity correlations are given by

$$\langle s_i s_j \rangle \sim \frac{1}{2M\beta}\left(\xi_S + \frac{1}{2M}\right)^{-1} e^{i2\pi(x_i - x_j)} J_o(\frac{\lambda_{0L}}{L} r_i) J_o(\frac{\lambda_{0L}}{L} r_j)$$

where $\xi_S(\beta)$ is fixed by the saddle point condition in terms of $\beta > 0$ and has a well-defined limit as $M \to \infty$. Therefore, $p_S(\beta)$ too has a well-defined limit and since $\frac{\lambda_{0L}}{L} \to \pi$ as $M \to \infty$,

$$\langle s_i s_j \rangle \to \frac{1}{2Mp_S(\beta)} e^{i2\pi(x_i - x_j)} J_o(\pi r_i) J_o(\pi r_j) = 0. \tag{6.2}$$

6.2. Negative temperatures $-\infty < T < 0$

For finite M, the correlations $\langle s_i s_j \rangle$ are again given by equation (6.1) but since all the denominators $(\xi_S + (q_1^2 + \lambda_{0s}^2)^{-1})$ in (6.1) are negative and finite when $\infty < \beta < 0$, with the smallest in absolute value corresponding to the critical wavenumber $(q_1 = 1, s = 0)$, the dominant term in (6.1) is the first term in the following expression:

$$\langle s_i s_j \rangle \sim \frac{1}{2\beta M}\left\{\frac{e^{i\frac{2\pi}{L}(x_i - x_j)}}{(\xi_S + 1)}\right\} + O(\beta^{-1}). \tag{6.3}$$

The accuracy of the dominant term in (6.3) clearly improves as β tends to $-\infty$. Since there are no negative temperature phase transitions in the range $-\infty < \beta < 0$, the structure of the two-point correlations must remain the same except for a proportionality constant which depends on β. Thence, the $O(\beta^{-1})$ terms in (6.3) must be near zero for all $\beta < 0$. Therefore, the vorticity correlation is given by

$$\langle s_i s_j \rangle = \frac{1}{2\beta M} \left\{ \frac{e^{i\frac{2\pi}{L}(x_i - x_j)}}{(\xi_S + 1)} \right\}$$

The limit

$$\lim_{M \to \infty} \langle s_i s_j \rangle = e^{i\frac{2\pi}{L}(x_i - x_j)} \tag{6.4}$$

exists and is independent of $\beta < 0$.

7. Physical applications and comparisons with experiments

The main physical consequences of the theoretical analysis in this paper are: (i) there are no phase transitions in the traditional sense in the standard thermodynamic limit, (ii) nonetheless, there are qualitative differences in the equilibrium vorticity patterns: the positive temperature phase (with relatively high energy and intermediate enstrophy) is a largely random mix of azimuthal vorticity with no clear coherent structures (see expression (6.2)), and (iii) the negative temperature (or very high energy, low enstrophy) phase is an axial mean flow (round jet) which is nearly uniform in the axial direction (see expression (6.4)).

The lowest theoretical value of the Dirichlet Quotient (or ratio of enstrophy to energy) [33] is attained at the negative temperature phase. Infact, this minimum ratio is a universal constant related to the primary eigenvalue of an associated Laplacian in a Poincare's inequality [34]. A dynamical theory for high Re numbers axisymmetric flows can be constructed to show that the physical principle of Selective Decay or Minimum Enstrophy is valid, that is, the Dirichlet Quotient tends to a theoretical minimum at a globally stable state. The equilibrium statistical mechanics theory described in this paper is nevertheless connected to the dynamical Selective Decay principle: the negative temperature phase here is unique in the sense that it is the family of flow states for which the Dirichlet Quotient is minimum. Thus, for a fixed value of the enstrophy, this phase has highest possible energy. Likewise, for a fixed value of the kinetic energy, this phase has lowest possible enstrophy. It is important to note that the negative temperature phase is not a single flow state, but rather a one parameter family of flow states which is parametrised by the axial velocity U at the centerline of the jet.

The high enstrophy and relatively low energy of the state in (ii) correspond to a low flow rate, high shear axisymmetric in which the flow rate is too weak to prevent the azimuthal vorticity from becoming unstable to a chaotic state. The low enstrophy and relatively high energy state in (iii) is the relaxed end-state of a round jet in which the high flow rate Q overcomes the tendency of the azimuthal vorticity to decay into chaos. This almost axially uniform jet has a velocity profile which is predominantly axial, with a superimposed sinusoidal perturbation at the wavelength prescribed by the length l of the domain (see expression (6.4)). This coherent structure can be viewed as a pair of opposite signed vortex rings per axial period superimposed on a uniform axial flow [26], [9].

Our approach is restricted to strictly axisymmetric phenomena and cannot be used to study the important three-dimensional properties of some high Reynolds number turbulent jets and mixing layers. This is a definite shortcoming in the spherical model technique which we hope is compensated by the fact that it can be solved exactly. In a future paper, we plan to report on numerical work to check the above theoretical predictions of two different phases or turbulent flow states in the flow rate Q and shear K parameter space.

7.1. Rayleigh's stability criterion

Rayleigh [35] showed that for inviscid incompressible Poiseuille flow in a circular pipe, a necessary and sometimes sufficient condition for instability of the uniformly axial flow of zero swirl is that the axial velocity profile must have a inflection point. This criterion can be translated into an equivalent statement about the azimuthal vorticity of the basic flow: *a necessary and sometimes sufficient condition for the basic axial flow to be unstable is for its azimuthal vorticity to change sign somewhere in the flow domain (cf.* [35] page 361-366).

Applying Rayleigh's result to the correlation structures in our analysis, we find that the above negative temperature state of very high energy and low enstrophy has a averaged azimuthal vorticity distribution which satisfies Rayleigh's criterion for stability. In other words, this unique macro-state of the energy-enstrophy statistical mechanics theory for inviscid incompressible axisymmetric flows, which has a correlation structure associated with a nearly uniform axial flow, is stable.

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