Exactly-solvable Ising-Heisenberg model for the coupled barotropic fluid - rotating solid sphere system - condensation of super and sub-rotating barotropic flow states

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Abstract

Exact solutions of a family of Heisenberg-Ising spin-lattice models for a coupled barotropic flow - massive rotating sphere system under microcanonical constraint on relative enstrophy is obtained by the method of spherical constraint. Phase transitions representative of Bose-Einstein condensation in which highly ordered super and sub-rotating states self-organize from random initial vorticity states are calculated exactly and related to three key parameters - spin of sphere, kinetic energy of the barotropic flow which is specified by the inverse temperature and amount of relative enstrophy which is held fixed. Angular momentum of the barotropic fluid relative to the rotating frame of the infinitely massive sphere is the main order parameter in this statistical mechanics problem − it is not constrained either canonically nor microcanonically as coupling between the fluid and the rotating sphere by a complex torque is responsible for its change. This coupling and exchange of angular momentum is a necessary condition for condensation in this spin-lattice system. There is no low temperature defects in this model - the partition function is calculated in closed form for all positive and negative temperatures. Also note-worthy is the fact that this statistical equilibrium model is not a mean field model and can be extended to treat fluctuations if required in more complex coupled flows.

1 Introduction

Consider the system consisting of a rotating high density rigid sphere of radius R, enveloped by a thin shell of barotropic (non-divergent) fluid. The barotropic flow is assumed to be inviscid, apart from an ability to exchange angular momentum and energy with the heavy solid sphere. In addition we assume that the fluid is in radiation balance and there is no net energy gain or loss from insolation. This provides a crude model of the complex planet - atmosphere interactions, including the enigmatic torque mechanism responsible for the phenomenon of atmospheric super-rotation.

We will build an equilibrium statistical mechanics model to study the relaxation of this complex phenomenon, possibly in terms of phase transitions that are dependent on a few key parameters in the problem. For a problem concerning super-rotation on a spherical surface there is little doubt that one of the key parameters is angular momentum of the fluid. The total angular momentum of the fluid and solid sphere is a conserved quantity. We could either model the problem as such, that is, the angular momentum of the sphere is large but finite and thus can vary, or consider the sphere to have infinite angular momentum, in which case, it serves as a reservoir of angular momentum and the active part of the model is just the fluid.

It is clear that a quasi - 2d geophysical relaxation problem will involve energy and enstrophy. The total energy of the fluid and sphere is conserved in any frame, both rotating and inertial ones. It consists entirely of the kinetic energy of barotropic flow plus that of the solid sphere because we have assumed the sphere to be a rigid solid that does not deform, and there is no gravitational potential energy in the fluid since it has uniform thickness and density, and its upper surface is a rigid lid. Conservation of relative enstrophy is treated here as a microcanonical constraint, modifying the classical energy-enstrophy theories [14] in substantial ways, chief amongst them being removal of the Gaussian low temperature defect while retaining the exact solvability of the model.

Higher vorticity moments are considered to be less significant than enstrophy in statistical equilibrium models of quasi-2d geophysical flows [14]. A detailed variational analysis of this topic is available in [8].

The results in this paper is motivated by the variational analysis reported in [12] and the careful and detailed simulation results first reported in the paper [7] and later in the book by Lim and Nebus, [6]. They agree in large part also with the mean field theories reported in [3] and [4].

2 Energy and angular momentum in the fluid-sphere system

The time dependent kinetic energy in the inertial frame ($\Omega = 0$) of the barotropic fluid component of the above two components system is given by

$$H_0[q] = \frac{1}{2}\int_{S^2} dx \ \left[u^2 + v^2\right]$$

where u and v are the zonal and meridional components of the fluid velocity in the inertial frame.

This kinetic energy of the fluid component in the inertial frame can also be written in terms of an arbitrary rotating frame as

$$\begin{aligned} H[q] &= \frac{1}{2}\int_{S^2} dx \left[(u_r + u_p)^2 + v_r^2\right] \\ &= \frac{1}{2}\int_{S^2} dx \left[(u_r^2 + v_r^2) + 2u_r u_p\right] + \frac{1}{2}\int_{S^2} dx \ u_p^2 \\ &= -\frac{1}{2}\int_{S^2} dx \ \psi q + \frac{1}{2}\int_{S^2} dx \ u_p^2 \end{aligned}$$

where $u_r = u - u_p$ is the relative zonal velocity, ψ is the stream function for the relative flow and

$$\begin{aligned} q &= \omega + 2\Omega\cos\theta \\ &= \Delta\psi + 2\Omega\cos\theta \end{aligned}$$

is the vorticity in the rest frame in terms of the relative vorticity ω and the planetary vorticity $2\Omega\cos\theta$; here θ denotes co-latitude on the unit sphere S^2. Clearly, the value of

$$H[q] = H_0[q]$$

does not depend on the choice of Ω, except that, as will be shown next, by choosing $\Omega > 0$, we can conveniently measure the varying amount of angular momentum in the fluid. However, unlike the situation for the standard BVE (which will be discussed next), there is clearly a difference between the $\Omega > 0$ and $\Omega = 0$ expressions for the rest frame kinetic energy of the same generalized barotropic flow.

Dropping the last term which is a constant and rewriting the rest frame kinetic energy of the fluid (in terms of relative zonal velocity u_r and meridional $v = v_r$),

$$H[q] = \frac{1}{2}\int_{S^2} dx\ (u_r^2 + v_r^2) + \int_{S^2} dx\ u_r u_p$$

we observe that the second term is the projection of relative velocity onto the velocity of a spherical shell rotating at angular velocity Ω, which is proportional to the net angular momentum of the relative flow. This term cannot be zero for all time since there is no distinguished value for the spin rate of the solid sphere when angular momentum is exchanged between the fluid and the sphere. In other words, after choosing some convenient fixed spin rate $\Omega > 0$ to index a rotating frame, the fluid could gain or loose angular momentum to the sphere and this shows up in the time-varying inner product

$$M_\Omega = \int_{S^2} dx\ u_r u_p.$$

To continue with the calculation of energy and momentum in the two components system, we note that the rest frame kinetic energy of the rigid sphere is easily calculated to be

$$A\Omega_s$$

where A is a constant that depends on the radius R and the density of the sphere and Ω_s is its changeable angular velocity. Angular momentum of the sphere in the rest frame is directly related to its kinetic energy and given by the linear expression

$$B\Omega_s.$$

By conservation of total kinetic energy (sum of fluid and solid sphere energy) in the rest frame, the sum

$$H[q] + A\Omega_s = const$$

even as both terms in the sum changes as Ω_s changes. Similarly by conservation of total angular momentum, the sum

$$\int_{S^2} dx\ u_r u_p + B\Omega_s = const$$

even as both its terms may change over time with Ω_s.

2.1 The standard BVE

For pedagogical purposes, we now compare the kinetic energy and angular momentum expressions for the standard barotropic vorticity model in which there is neither exchange of energy nor momentum with the sphere (the fluid component is energetically and torque-wise isolated). This is not the subject of this paper. The kinetic energy of the fluid in the rest (inertial) frame is given by

$$H_0^{'} = \frac{1}{2}\int_{S^2} dx \ (u^2 + v^2)$$

which is superficially the same expression as H_0 above but now $H_0^{'}$ is a constant in time. In a frame that is rotating at angular velocity Ω, the kinetic energy of the fluid is given by

$$H'[q] = \frac{1}{2}\int_{S^2} dx \ (u_r^2 + v_r^2) + \int_{S^2} dx \ u_r u_p + \frac{1}{2}\int_{S^2} dx \ u_p^2$$

which is again superficially similar to the expression $H[q]$ but differs from that earlier expression because it is now a constant. Unlike the previous situation, the net angular momentum of the fluid relative to this rotating frame is fixed in time and given by

$$M_\Omega' = \int_{S^2} dx \ u_r u_p = const.$$

Moreover, there is now a special choice of frame angular velocity Ω' (or gauge) for which

$$M_\Omega' = \int_{S^2} dx \ u_r u_p = 0$$

for all time, namely the angular velocity Ω' of a rigidly rotating spherical shell whose angular momentum equals that of the fluid. With this choice Ω', the expression $H'[q]$ becomes

$$H'[q] = \frac{1}{2}\int_{S^2} dx \ (u_r^2 + v_r^2) + \frac{1}{2}\int_{S^2} dx \ u_p^2$$

which apart from the constant term $\frac{1}{2}\int_{S^2} dx \ u_p^2$ has the same form as the rest frame kinetic energy H_0'. This is the often stated line that the kinetic energy expression of the standard barotropic vorticity model (BVE which has constant energy and momentum) has the same form in any frame, both inertial and rotating ones. Although the kinetic energies $H'[q] = \frac{1}{2}\int_{S^2} dx$

$(u_r^2 + v_r^2)$ and $H_0' = \frac{1}{2}\int_{S^2} dx\ (u^2 + v^2)$ of the same fluid flow, have the same form, the relative velocity (u_r, v_r) in the special frame labeled by Ω' has zero net angular momentum $M_\Omega' = 0$ but the same flow's absolute velocity (u, v) in the inertial frame has fixed nonzero angular momentum. This fact is properly reflected in the statistical equilibrium models for the standard BVE by Frederiksen et al [14] where the conservation of fluid angular momentum is imposed as an additional microcanonical constraint $M_\Omega' = 0$.

3 Statistical mechanics and enstrophy

To construct a statistical equilibrium model for this first system, we should use a formulation that is microcanonical in both the total rest frame kinetic energy and the total rest frame angular momentum. This formulation allows kinetic energy and angular momentum to be exchanged between the two subsystems in the relaxation process towards statistical equilibrium. However, such a doubly microcanonical statistical ensemble is very cumbersome to solve.

We therefore assume that the sphere has infinite mass and a fixed angular velocity Ω, and thus, acts as two related infinite reservoirs of rest frame kinetic energy $H[q]$ and angular momentum M_Ω for the fluid. This simple step is justified in the study of most planetary atmospheres by the relatively massive planetary spheres in the problem. It leads to a significant reduction of technical difficulties because the equilibrium statistical mechanics is now based on a doubly canonical ensemble in kinetic energy $H[q]$ and angular momentum M_Ω with corresponding Lagrange Multipliers or chemical potentials β and α.

We observe that the key expression $H[q]$ in this formulation for generalized barotropic flows (that exchanges energy and momentum with an infinite reservoir) is independent of the choice of $\Omega > 0$, precisely because it is the rest frame kinetic energy of the fluid. Thus, we should choose $\Omega > 0$ to be the fixed angular velocity of the massive solid sphere. For this choice of $\Omega > 0$, the fluctuations of M_Ω measure the amount of super (resp sub-) rotation in the fluid relative to the frame in which the solid sphere is fixed. Unlike the standard BVE, there is really no special choice Ω' for which the net angular momentum term $M_{\Omega'}$ vanish for all time.

This problem is still not well-posed because without fixing or limiting the size of the relative flow in some suitable norm (such as relative enstrophy in the frame rotating), the energy $H[q]$ and angular momentum M_Ω can in principle become unbounded at fixed reservoir temperatures. So we impose the condition of fixed relative enstrophy since it is the square of the $L_2(S^2)$ norm of the relative vorticity ω. The classical models based on subjecting enstrophy to a canonical constraint leads to Gaussian models which are not defined at small absolute values of the temperature [11].

We note that fixing the relative enstrophy by a microcanonical constraint means two things: (1) it does not mean that the energy $H[q]$ and angular momentum M_Ω are fixed, and (2) it does not mean that the resulting mixed ensemble is intractable although in general microcanonical constraints give rise to great analytical difficulties. With one more minor adjustment, we show in this paper that the resulting models are exactly solvable spherical Ising and Heisenberg models that can be solved in closed form by the method of steepest descent.

The last adjustment we make is to couple the two remaining reservoirs into one, namely fix the statistical temperature $T = \beta^{-1}$ of a single energy reservoir instead of having two separate inverse temperatures β and α for the energy and angular momentum respectively. This simple step is justified in the following argument. Expression $H[q]$ for the rest frame kinetic energy shows that net angular momentum is essentially the second and independent part of $H[q]$; the first part of $H[q]$ is the relative kinetic energy in the rotating frame. That is, even as the whole $H[q]$ fluctuates in relaxation with respect to the infinite energy reservoir, the two active parts of $H[q]$, namely the relative kinetic energy term (1st term) and the angular momentum part (2nd term) exchange energy constantly even after equilibrium is reached. Thus, the collapse of two reservoirs into a single energy reservoir in this particular problem, retains the physically important and statistically independent mechanism of angular momentum fluctuations.

To summarize this elementary but important material, we note that the kinetic energy expression used in the derivation of the spin-lattice models in this paper is just the changeable rest frame kinetic energy of the fluid (written with respect to a frame rotating at fixed $\Omega > 0$) minus the constant term $\frac{1}{2}\int_{S^2} dx\ u_p^2$,

$$H[q] = \frac{1}{2}\int_{S^2} dx\ (u_r^2 + v_r^2) + \int_{S^2} dx\ u_r u_p.$$

In general both terms fluctuate independently as the sum $H[q]$ changes in time due to energy and angular momentum exchanges with the coupled infinite reservoir. By mapping spins to local vorticity, the first term $\frac{1}{2}\int_{S^2} dx$ $(u_r^2 + v_r^2)$ by itself gives rise to long range Ising type spin-lattice models without an external field. Using moreover, the analogy between magnetic moments and angular momentum, it is easy to see that the second term $M_\Omega = \int_{S^2} dx\ u_r u_p$ which represents the changeable net angular momentum of the fluid (relative to the frame rotating at fixed $\Omega > 0$), becomes a standard external field term in a Heisenberg model. It is more convenient to use the Heisenberg models which are natural vectorial reformulations of the Ising models for generalized barotropic flow, as shown below.

The principle of angular momentum conservation and the analogy between angular momentum and magnetic moments are both beautifully illustrated in the famous Einstein-de Haas experiment where a ferromagnetic rod is suspended by a thread inside a coil. Upon turning on the current, the rod is magnetized, that is, its microscopic magnetic spins are aligned. What is surprising is the fact that the rod rotates inside the coil because the macroscopic alignment of magnetic moments results in a nonzero net angular momentum inside the rod due to large numbers of aligned orbiting electrons, and since no torque is applied to the system, the rod reacts by rotating the opposite way to conserve angular momentum.

A related process, albeit one that involves phase transitions, is shown in this paper by obtaining exact solutions for the partition functions of the spherical Heisenberg models which are the above Heisenberg models plus the spherical constraint from fixing the relative enstrophy of the flow. At sufficiently hot negative statistical temperatures (with small absolute values), we show that the spherical Heisenberg model for generalized barotropic flows goes through a second order phase transition between disordered states of local spins (vorticity) at low energy and a global ordered state at very high energy where the local spins sum to a total magnetization (net angular momentum) aligned with the rotation axis of the solid sphere.

Reverting to the actual situation where the solid sphere has finite mass, this phase transition means that the two component fluid -sphere system undergoes a process very similar to the Einstein-de Haas phenomenon: in the high energy ordered state the barotropic fluid layer acquires positive net angular momentum (super-rotates relative to the solid sphere) due to macroscopic alignment of local vorticity which compares with the orbiting electrons in the ferromagnetic rod acquiring net angular momentum from alignment of magnetic moments; the solid sphere slows its rotation rate to conserve total angular momentum, just as the magnetized rod rotates the opposite way. Sattinger summarized the phenomenon discussed here succintly in the phrase "...many little spins turn into a big spin .." [13]

4 Heisenberg Model for Barotropic Statistics

Recall that in the spherical Ising model for barotropic flow, given N fixed mesh points x_k on S^2 and the voronoi cells based on this mesh [?], we approximate the relative vorticity by discretizing the vorticity field as a piecewise constant function,

$$\omega(x) = \sum_{j=1}^{N} s_j H_j(x),$$

where $s_j = \omega(x_j)$ and $H_j(x)$ is the characteristic function for the domain D_j, that is

$$H_j(x) = \begin{cases} 1 & x \in D_j \\ 0 & otherwise. \end{cases}$$

There is however a more natural vectorial formulation that leads to a Heisenberg model for barotropic flows on a massive sphere. Instead of representing the local relative vorticity $\omega(x_j)$ at lattice site x_j by a scalar s_j, it is natural to represent it by the vector

$$\vec{s}_j = s_j \vec{n}_j$$

where $\vec{n}_j$ denotes the outward unit normal to the sphere S^2 at x_j. Similarly, we represent the spin $\Omega > 0$ of the rotating frame by the vector

$$\vec{h} = \frac{2\pi}{N} \Omega \vec{n}$$

where $\vec{n}$ is the outward unit normal at the north pole of S^2. Denoting by γ_{jk} the angle subtended at the center of S^2 by the lattice sites x_j and x_k, we obtain the following Heisenberg model for the total (fixed frame) kinetic energy of a barotropic flow in terms of a rotating frame at spin rate Ω,

$$H_H^N = -\frac{1}{2} \sum_{j \neq k}^{N} J_{jk} \vec{s}_j \cdot \vec{s}_k + \vec{h} \cdot \sum_{j=1}^{N} \vec{s}_j \tag{1}$$

where the interaction matrix is now given by the infinite range

$$J_{jk} = \frac{16\pi^2}{N^2} \frac{\ln(1 - \cos\gamma_{jk})}{\cos\gamma_{jk}},$$

the dot denotes the inner product in R^3 and $\vec{h}$ denotes a fixed external field.

The Kac-Berlin method [10] can be modified [2] to treat the spherical Heisenberg model which consists of H_H^N and the spherical or relative enstrophy constraint,

$$\frac{4\pi}{N}\sum_{j=1}^{N}\vec{s}_j\cdot\vec{s}_j = Q.$$

In addition, Stokes theorem implies that it is natural to treat only the case of zero circulation,

$$\frac{4\pi}{N}\sum_{j=1}^{N}\vec{s}_j\cdot\vec{n}_j = 0.$$

Looking ahead, we note the important fact that the following vectorial sum or magnetization

$$\Gamma = \frac{4\pi}{N}\sum_{j=1}^{N}\vec{s}_j$$

will turn out to be a natural order parameter for the statistics of barotropic flows on a rotating sphere.

Can the Heisenberg model H_H^N on S^2 support phase transitions? More precisely we check what the Mermin-Wagner theorem has to say about H_H^N : (1) it has spatial dimension $d = 2$, (2) it has a continuous symmetry group, namely for each element $g \in SO(3)$,

$$H_H^N(g\vec{s}) = H_H^N(\vec{s}),$$

and (3) it has infinite range interaction, that is, for any sequence of uniform lattices of N sites on S^2,

$$\lim_{N\to\infty}\frac{4\pi}{N}\sum_{j=1}^{N}J_{jk} = -\infty.$$

Properties (1) and (2) by themselves would have implied via the Mermin-Wagner theorem, that H_H^N does not support phase transitions, since all $d \leq 2$, finite range models with a continuous symmetry group cannot have them. However, property (3) violates the finite range condition of this theorem. Hence, H_H^N on S^2 can in principle have phase transitions in the thermodynamic limit.

It is interesting to compare this Heisenberg model H_H^N on S^2 with the Ising type model H_N for the same barotropic flow in the last section. The

Mermin-Wagner theorem there allows H_N to have phase transitions in the thermodynamic limit for a different reason: the Ising type interaction $J_{jk} = \frac{16\pi^2}{N^2}\ln(1-\cos\gamma_{jk})$ has finite range instead of infinite range but H_N does not have a continuous symmetry group, only the discrete symmetry Z_2.

Careful monte-Carlo simulations of this model show that there is one negative critical temperature in this model, $T_c < 0$ for all values of the spin rate Ω [7]. An extension of the Kac-Berlin method to the spherical Heisenberg model for Barotropic flows on a rotating sphere will show that BEC transitions through a symmetry-breaking Goldstone mode to the single ground state ψ_{10}, occurs for sufficiently high kinetic energies or small negative values of the temperature T.

5 Solution of the spherical Heisenberg model for $\Omega > 0$

The family of Heisenberg models H_H^N derived above for the barotropic fluid - solid sphere system in a frame rotating at angular velocity $\Omega > 0$ have external fields $\vec{h}(N) = \frac{2\pi}{N}\Omega\vec{n}$ and infinite range interactions

$$J_{jk} = \frac{16\pi^2}{N^2}\frac{\ln(1-\cos\gamma_{jk})}{\cos\gamma_{jk}}.$$

Combining it with the vectorial spherical constraint,

$$\frac{4\pi}{N}\sum_{j=1}^{N}\vec{s}_j\cdot\vec{s}_j = Q,$$

we obtain an extension of Kac's spherical model [10] to a one-parameter family of spherical Heisenberg models which is parametrized by the size N of the Voronoi lattice on S^2.

This family of spherical Heisenberg models for barotropic vortex statistics allows us to model the thermal interactions between local relative vorticity $\omega(x)$ and a kinetic energy reservoir at any fixed temperature T. The spherical constraint enforces the microcanonically fixed relative enstrophy $Q > 0$ but allows angular momentum in each of the three principal directions to change. Similar to the equilibrium condensation process found in the case $\Omega = 0$ for the spherical Ising model [Lim06a], kinetic energy of barotropic flow settles into a Goldstone symmetry-breaking ground state at very small negative temperatures $T_c < T < 0$ (associated with extremely large energies). Unlike the $\Omega = 0$, there is no 3-fold degeneracy in the Goldstone modes and only the mode ψ_{10} which carries angular momentum that is aligned with the rotation axis $\Omega\vec{n}$, has a large amplitude.

The exact solution of the spherical Heisenberg models H_H^N proceeds along similar lines to the Kac-Berlin method for the spherical Ising model. In the thermodynamic or continuum limit as $N \to \infty$, the partition function is calculated using Laplace's integral form,

$$\begin{aligned}
Z_H^N &\propto \int D(\vec{s}) \exp\left(-\beta H_H^N(\vec{s})\right) \delta\left(Q\frac{N}{4\pi} - \sum_{j=1}^N \vec{s}_j \cdot \vec{s}_j\right) \\
&= \int D(\vec{s}) \exp\left(-\beta H_H^N(\vec{s})\right) \left(\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} d\eta \exp\left(\eta\left(Q\frac{N}{4\pi} - \sum_{j=1}^N \vec{s}_j \cdot \vec{s}_j\right)\right)\right) \\
&= \int D(\vec{s}) \exp\left(\frac{\beta}{2}\sum_{j\neq k}^N J_{jk}\vec{s}_j \cdot \vec{s}_k - \beta\vec{h}\cdot\sum_{j=1}^N \vec{s}_j\right) \left(\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} d\eta \exp\left(\eta\left(N - \frac{4\pi}{Q}\sum_{j=1}^N \vec{s}_j \cdot \vec{s}_j\right)\right)\right)
\end{aligned}$$

where moreover, the microstate

$$\vec{s} = \{\vec{s}_1, ..., \vec{s}_N\} \in R^{3N}$$

satisfies the zero circulation condition

$$\sum_{j=1}^N \vec{s}_j \cdot \vec{n}_j = 0.$$

Thus,

$$\begin{aligned}
Z_H^N &\propto \left(\int D(\vec{s}) \exp\left(\frac{\beta}{2}\sum_{j\neq k}^N J_{jk}\vec{s}_j \cdot \vec{s}_k - \beta\vec{h}\cdot\sum_{j=1}^N \vec{s}_j\right) \int_{a-i\infty}^{a+i\infty} \frac{d\eta}{2\pi i} \exp\left(\eta\left(N - \frac{4\pi}{Q}\sum_{j=1}^N \vec{s}_j \cdot \vec{s}_j\right)\right)\right) \\
&= \int D(\vec{s}) \int_{a-i\infty}^{a+i\infty} \frac{d\eta}{2\pi i} \exp\left(N\left(\begin{array}{c} \eta - \frac{4\pi}{QN}\eta\sum_{j=1}^N \vec{s}_j \cdot \vec{s}_j \\ +\frac{\beta}{2N}\sum_{j\neq k}^N J_{jk}\vec{s}_j \cdot \vec{s}_k - \frac{\beta}{N}\vec{h}\cdot\sum_{j=1}^N \vec{s}_j \end{array}\right)\right) \\
&= \int D(\vec{s}) \int_{a-i\infty}^{a+i\infty} \frac{d\eta}{2\pi i} \exp\left(N\left(\eta - \frac{1}{N}\sum_{j\neq k}^N K_{jk}(Q,\beta,\eta)\ \vec{s}_j \cdot \vec{s}_k - \frac{\beta}{N}\vec{h}\cdot\sum_{j=1}^N \vec{s}_j\right)\right)
\end{aligned}$$

where

$$K_{jk}(Q,\beta,\eta) = \left\{\begin{array}{cc} \frac{4\pi}{Q}\eta & j = k \\ -\frac{\beta}{2}J_{jk} & j \neq k \end{array}\right\}.$$

To evaluate the Gaussian integrals in Z_H^N, we expand the relative vorticity vectorfield again, this time, in terms of the spherical harmonics,

$$\vec{\omega}(x) = \sum_{l=1,m=-l}^{\infty,l} \alpha_{lm}\psi_{lm}(x)\vec{n}(x)$$

where $\vec{n}(x)$ is the outward unit normal to S^2 at x. We stress that this expansion need not include $\psi_{00}(x) = c$ because of the zero circulation condition on microstates $\vec{s}$.

Solution of the Gaussian integrals requires diagonalizing the interaction in H_H^N in terms of the spherical harmonics $\{\psi_{lm}\}_{l=1,}^{\infty}$ which are natural Fourier modes for Laplacian eigenvalue problems on S^2 with zero circulation:

$$\begin{aligned} \vec{s}_j &= \vec{n}_j \sum_{l=1}^{\infty}\sum_{m=-l}^{l} \alpha_{lm}\psi_{lm}(x_j) \\ -\frac{1}{2}\sum_{j\neq k}^{N} & J_{jk}\ \vec{s}_j\cdot\vec{s}_k = \frac{1}{2}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\lambda_{lm}\alpha_{lm}^2 \\ \vec{h}\cdot\sum_{j=1}^{N} & \vec{s}_j = \frac{1}{2}\Omega C\alpha_{10} \end{aligned}$$

where the eigenvalues of the Green's function for the Laplace-Beltrami operator on S^2 are

$$\lambda_{lm} = \frac{1}{l(l+1)},\ \ l = 1, ..., \sqrt{N},\ \ m = -l, ..., 0, ..., l$$

and α_{lm} are the corresponding amplitudes. Thus,

$$\frac{1}{N}\sum_{j\neq k}^{N} K_{jk}(Q,\beta,\eta)\ \vec{s}_j\cdot\vec{s}_k = \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left(\frac{\beta}{2N}\lambda_{lm} + \frac{\eta}{Q}\right)\alpha_{lm}^2$$

and

$$\begin{aligned} Z_H^N &\propto \int D(\vec{s})\int_{a-i\infty}^{a+i\infty}\frac{d\eta}{2\pi i}\exp\left(N\left(\eta - \frac{1}{N}\sum_{j\neq k}^{N}K_{jk}(Q,\beta,\eta)\ \vec{s}_j\cdot\vec{s}_k - \frac{\beta}{N}\vec{h}\cdot\sum_{j=1}^{N}\vec{s}_j\right)\right) \\ &= \int\prod_{m=-1}^{1}d\alpha_{1m}\int D_{l\geq 2}(\alpha)\int_{a-i\infty}^{a+i\infty}\frac{d\eta}{2\pi i}\exp\left\{N\left[\begin{array}{c} \eta - \sum_{l=2}^{\infty}\sum_{m=-l}^{l}\left(\frac{\beta}{2N}\lambda_{lm} + \frac{\eta}{Q}\right)\alpha_{lm}^2 \\ -\left(\frac{\beta}{4N} + \frac{\eta}{Q}\right)\sum_{m=-1}^{1}\alpha_{1m}^2 - \frac{\beta}{2N}\Omega C\alpha_{10} \end{array}\right]\right\} \end{aligned}$$

$$= \int \prod_{m=-1}^{1} d\alpha_{1m} \int_{a-i\infty}^{a+i\infty} \frac{d\eta}{2\pi i} \exp\left\{ N \left[\eta - \left(\frac{\beta}{4N} + \frac{\eta}{Q} \right) \sum_{m=-1}^{1} \alpha_{1m}^2 - \frac{\beta}{2N} \Omega C \alpha_{10} \right] \right\}$$

$$\int D_{l\geq 2}(\alpha) \exp\left(-N \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \left(\frac{\beta}{2N} \lambda_{lm} + \frac{\eta}{Q} \right) \alpha_{lm}^2 \right)$$

where the order of integration of the term $\exp\left(-N \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \frac{\eta}{Q} \alpha_{lm}^2\right)$ can be interchanged by choosing $\mathrm{Re}(\eta) = a > 0$ large enough.

5.1 Restricted partition function and non-ergodic modes

Next we write the problem in terms of the restricted partition function $Z_H^N(\alpha_{10}, \alpha_{1,\pm 1}; \beta, Q, \Omega)$, that is,

$$\begin{aligned} Z_H^N(\beta, Q, \Omega) &\propto \int \prod_{m=-1}^{1} d\alpha_{1m} \; Z_H^N(\alpha_{10}, \alpha_{1,\pm 1}; \beta, Q, \Omega) \\ &= \int \prod_{m=-1}^{1} d\alpha_{1m} \int_{a-i\infty}^{a+i\infty} \frac{d\eta}{2\pi i} \; \exp\left\{ N \left[\eta - \left(\frac{\beta}{4N} + \frac{\eta}{Q} \right) \sum_{m=-1}^{1} \alpha_{1m}^2 - \frac{\beta}{2N} \Omega C \alpha_{10} \right] \right\} \end{aligned}$$

$$\int D_{l\geq 2}(\alpha) \exp\left(-N \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \left(\frac{\beta}{2N} \lambda_{lm} + \frac{\eta}{Q} \right) \alpha_{lm}^2 \right).$$

Because of non-ergodicity of the condensed modes, we should not integrate over the ordered modes in this problem, namely α_{1m}, which are the amplitudes of the 3-fold degenerate ground modes ψ_{1m} that carry global angular momentum. This often used physical argument in the condensed matter literature will for the first time be turned into a rigorous proof here. A pertinent and important question arises at this point [9]: how many and what are the condensed modes in any given spherical model? We will show later that only one single class of modes can have nonzero amplitudes in the condensed phase of this problem, namely those belonging to the meridional wave number $l = 1$.

The statistics of the problem are therefore completely determined by the restricted partition function $Z_H^N(\alpha_{10}, \alpha_{1,\pm 1}; \beta, Q, \Omega)$. Amplitudes $\alpha_{10}, \alpha_{1,\pm 1}$ of the ordered modes appear as parameters in this restricted partition function, and will have to be evaluated separately.

Standard Gaussian integration is used to evaluate the last integral, which yields, after scaling $\beta' N = \beta$,

$$\int_{l\geq 2} D(\alpha) \exp\left(-\sum_{l=2}\sum_{m=-l}^{l}\left(\frac{\beta' N\lambda_{lm}}{2}+\frac{N\eta}{Q}\right)\alpha_{lm}^2\right) = \prod_{l=2}^{\sqrt{N}}\prod_{m=-l}^{l}\left(\frac{\pi}{\frac{N\eta}{Q}+\frac{\beta' N}{2}\lambda_{lm}}\right)^{1/2},$$

provided the physically significant Gaussian conditions hold: for $l \geq 2$,

$$\frac{\beta'\lambda_{lm}}{2}+\frac{\eta}{Q}=\frac{\beta'}{2l(l+1)}+\frac{\eta}{Q}>0. \tag{2}$$

Then the partition function takes the form

$$Z_H^N(\alpha_{10},\alpha_{1,\pm 1};\beta,Q,\Omega) \propto \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} d\eta\ \exp\left\{N\left[\begin{array}{c}\eta-\left(\frac{\beta'}{4}+\frac{\eta}{Q}\right)\sum_{m=-1}^{1}\alpha_{1m}^2-\frac{\beta'}{2}\Omega C\alpha_{10}\\ -\frac{1}{2N}\sum_{l=2}\sum_m \ln\left(\frac{N\eta}{Q}+\frac{\beta' N}{2}\lambda_{lm}\right)\end{array}\right]\right\}$$

where the free energy per site evaluated at the most probable macrostate is $-\frac{1}{\beta'}\ F(\eta(\beta'),Q,\beta')$ with

$$F(\eta(\beta'),Q,\beta') = \eta(\beta')\left[1-\frac{1}{Q}\sum_{m=-1}^{1}\alpha_{1m}^2\right]-\frac{\beta'}{4}\sum_{m=-1}^{1}\alpha_{1m}^2-\frac{\beta'}{2}\Omega C\alpha_{10}$$
$$-\frac{1}{2N}\sum_{l=2}\sum_{m}\ln\left(\frac{N\eta}{Q}+\frac{\beta' N}{2}\lambda_{lm}\right).$$

5.2 Planck's theorem, Saddle points and the Themodynamic limit

Provided that the saddle point $\eta(\beta')$ can be determined at given inverse temperature β', Planck's theorem states that the thermodynamically stable (most probable) macrostate is given by the maximum of the expression $F(\eta(\beta'),Q,\beta')$. At positive temperatures, the structure of this expression where it concerns the ground modes α_{1m}, namely,

$$\chi(\alpha_{10},\alpha_{1,\pm 1};\beta,Q,\Omega) = \eta(\beta')\left[1-\frac{1}{Q}\sum_{m=-1}^{1}\alpha_{1m}^2\right]-\left[\frac{\beta'}{4}\sum_{m=-1}^{1}\alpha_{1m}^2+\frac{\beta'}{2}\Omega C\alpha_{10}\right],$$

and the fact that the saddle point $\eta(\beta')$ must be positive, suggests that for any positive value of the saddle point, the expression χ and therefore

$F(\eta(\beta'), Q, \beta')$ is maximized by $\sum_{m=-1}^{1} \alpha_{1m}^2 = 0$ for all $\beta' > 0$ when planetary spin Ω is small, and by $\alpha_{10} < 0$ for large $\beta' > 0$ when planetary spin Ω is large. At negative temperatures, we expect to find a finite critical point where the two opposing parts of χ are balanced. In order to prove that these heuristic expectations are valid, we will solve the restricted partition function in closed form by the method of steepest descent.

The saddle point condition gives one equation for the determination of four variables η, α_{1m} in terms of inverse temperature β' and relative enstrophy Q,

$$0 = \frac{\partial F}{\partial \eta} = \left(1 - \frac{1}{Q}\sum_{m=-1}^{1} \alpha_{1m}^2\right) - \frac{1}{2NQ}\sum_{l=2}\sum_{m}\left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2}\lambda_{lm}\right)^{-1} \tag{3}$$

where $\eta = \eta(\beta')$ is taken to be the value of the saddle point. Note that it does not depend on the planetary spin rate $\Omega > 0$. We note in passing that the same equation holds in the $\Omega = 0$ case. There are two natural subcases for the saddle point condition, namely, (A) the disordered phase (for $|T'| \gg 1$) where equation (3) has finite solution $\eta(\beta') > 0$, and $\alpha_{1m} = 0$ for $m = -1, 0, 1$; and (B) the ordered or condensed phase (for $|T'| \ll 1$) where equation (3) has finite solution $\eta(\beta') > 0$ only when $\alpha_{1m} \neq 0$ for some m. In case (A) which will be solved below, there is no need to invoke additional equations of state as the amplitudes $\alpha_{1m} = 0$ for $m = -1, 0, 1$.

Case (B) requires three more conditions to determine the three amplitudes α_{1m} and the saddle point $\eta(\beta') > 0$. They are provided by equations of state (or Planck's theorem) for the condensed phase (which do not hold in the disordered phase):

$$0 = \frac{\partial F}{\partial \alpha_{10}} = -\left(\frac{2\eta(\beta')}{Q} + \frac{\beta'}{2}\right)\alpha_{10} - \frac{\beta'}{2}\Omega C \tag{4}$$

$$0 = \frac{\partial F}{\partial \alpha_{1,\pm 1}} = -\left(\frac{2\eta(\beta')}{Q} + \frac{\beta'}{2}\right)\alpha_{1,\pm 1}. \tag{5}$$

Thus, a coupled system of four algebraic equations (3), (4), (5) determines four unknowns in terms of the planetary spin $\Omega > 0$, the relative enstrophy $Q > 0$ and the scaled inverse temperature β'. The last two equations of state for $\alpha_{1,\pm 1}$ implies that either

$$\alpha_{1,\pm 1} = 0 \text{ or } \left(\frac{2\eta(\beta')}{Q} + \frac{\beta'}{2}\right) = 0.$$

The first equation of state differs from the other two; this represents reduction of the $SO(3)$ symmetry that existed in the $\Omega = 0$ case to S^1 symmetry in the case of nonzero planetary spin. Together these three equations of state imply that when $\Omega > 0$, the only possible solution is without tilt,

$$\begin{aligned} \alpha_{10} &= -\frac{\beta'\Omega C}{2}\left(\frac{2\eta(\beta')}{Q} + \frac{\beta'}{2}\right)^{-1} \neq 0, \\ \alpha_{1,\pm 1} &= 0. \end{aligned} \tag{6}$$

These values of α_{lm} will be substituted back into the saddle point condition (3) to yield a single equation that will be solved below.

The Gaussian conditions (2) imply that for $l > 1$,

$$\frac{\beta'}{2l(l+1)} + \frac{\eta(\beta')}{Q} > 0.$$

The critical temperature can be obtained from the saddle point condition: (A) in the disordered phase at large $|T|$,

$$1 = \lim_{N\to\infty} \frac{1}{2NQ} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^{l} \left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2l(l+1)}\right)^{-1} \tag{7}$$

where the large N limit on the RHS is well-defined and finite for any finite $|\beta'|$ provided

$$\eta(\beta') \geq \eta^* = \frac{|\beta'|Q}{4} > 0, \tag{8}$$

because then, each term ($l \geq 2$) in the sum is majorized: for negative temperatures,

$$\left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2l(l+1)}\right)^{-1} \leq \left(-\frac{\beta'}{4} + \frac{\beta'}{2l(l+1)}\right)^{-1} \leq \left(-\frac{\beta'}{6}\right)^{-1},$$

and for positive temperatures,

$$\left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2l(l+1)}\right)^{-1} \leq \left(\frac{\beta'}{4} - \frac{\beta'}{2l(l+1)}\right)^{-1} \leq \left(\frac{\beta'}{6}\right)^{-1};$$

and the corresponding expressions have well-defined positive limits, i.e., for all negative and finite β',

$$\lim_{N\to\infty} \frac{1}{2NQ} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^{l} \left(-\frac{\beta'}{4} + \frac{\beta'}{2l(l+1)}\right)^{-1} < \infty, \tag{9}$$

and for all positive and finite β',

$$\lim_{N\to\infty} \frac{1}{2NQ} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^{l} \left(\frac{\beta'}{4} - \frac{\beta'}{2l(l+1)} \right)^{-1} < \infty.$$

And (B) in the ordered phase at small $|T|$,

$$\left(1 - \frac{1}{Q}\alpha_{10}^2\right) = \lim_{N\to\infty} \frac{1}{2NQ} \sum_{l=2} \sum_{m} \left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2l(l+1)} \right)^{-1} \tag{10}$$

where a similar argument proves that the RHS is well-defined and finite provided $\eta(\beta') \geq \eta^*$.

This proves that the thermodynamic or continuum limit of the spherical Heisenberg model H_H^N is well-defined for all negative temperatures because it turns out (and is shown below) that the saddle point satisfies (8) for the disordered as well as the ordered phases. Later we will show that this thermodynamic limit exists for all positive temperatures as well.

The large $|T|$ or small $|\beta'|$ saddle point condition in case (A),

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^{l} \left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2l(l+1)} \right)^{-1} = Q, \tag{11}$$

can be solved and has the property that $\eta(\beta') \searrow 1$ as $|\beta'| \to 0$.

In case (B), when $|\beta'|$ is large, we discuss (i) $\beta' < 0$ and (ii) $\beta' > 0$ separately.

5.3 Negative critical temperature

For case (i) $\beta' < 0$, a point is reached at β'_c where

$$-\infty < \beta'_c(Q) = \lim_{N\to\infty} \frac{1}{QN} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^{l} \left(\lambda_{lm} - \frac{1}{2} \right)^{-1} < 0,$$

such that for $\beta' < \beta'_c(Q) < 0$,

$$\lim_{N\to\infty} \frac{1}{\beta' NQ} \sum_{l=2} \sum_{m} \left(-\frac{1}{2} + \lambda_{lm} \right)^{-1} < 1.$$

(We note the significant fact that $T_c'(Q)$ depends linearly on the relative enstrophy Q but does not depend on Ω.) In other words, the extreme saddle point

$$\eta^* = -\frac{\beta' Q}{4}$$

is no longer adequate to solve (7) for $\beta' < \beta_c'(Q) < 0$; any larger value $\eta > \eta^*$ does not work either because then

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^{l} \left(\frac{\eta}{Q} + \frac{\beta'}{2l(l+1)} \right)^{-1} < Q$$

for $\beta' < \beta_c'(Q) < 0$. It remains to check that $\eta < \eta^*$ cannot be used. This is due to the fact that

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^{l} \left(\frac{\eta^*}{Q} + \frac{\beta_c'}{2l(l+1)} \right)^{-1} = Q$$

which implies

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^{l} \left(\frac{\eta}{Q} + \frac{\beta'}{2l(l+1)} \right)^{-1} > Q$$

if $\eta < \eta^*$ and $\beta' < \beta_c' < 0$.

From this discussion of (i) $\beta' \leq \beta_c' < 0$, and after substituting the nonzero solution (6) of the equations of state back into the saddle point equation,

$$\left(1 - \frac{1}{Q}\alpha_{10}^2\right) = \lim_{N\to\infty} \frac{1}{2NQ} \sum_{l=2} \sum_{m} \left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2l(l+1)} \right)^{-1}, \tag{12}$$

we derive a single equation,

$$\left(1 - \frac{(\beta')^2\Omega^2 C^2}{16Q}\left(\frac{\eta(\beta',\Omega,Q)}{Q} + \frac{\beta'}{4}\right)^{-2}\right) = \lim_{N\to\infty} \frac{1}{2NQ} \sum_{l=2} \sum_{m} \left(\frac{\eta(\beta',\Omega,Q)}{Q} + \frac{\beta'}{2l(l+1)} \right)^{-1} \tag{13}$$

for the saddle point $\eta(\beta',\Omega,Q) \geq \eta^*$ when $\beta' \leq \beta_c' < 0$.

The RHS of this equation can be made larger (resp. smaller) than 1 by choosing $\eta(\beta') < \eta^*$ ($> \eta^*$ resp.) and since $Q = \sum_{l=1}\sum_m \alpha_{lm}^2$ is the relative enstrophy, we must have

$$0 \leq \left(1 - \frac{1}{Q}\alpha_{10}^2\right) \leq 1$$

which means that its LHS lie between 0 and 1. Thus, by choosing a suitable $\eta(\beta') \geq \eta^*$ we should be able to satisfy (13) for $\beta' \leq \beta_c' < 0$. It remains to check that this is consistent with the property $\alpha_{10}^2 \leq Q$ of the ordered solution (6), that is, for all Ω,

$$\frac{(\beta')^2\Omega^2 C^2}{16Q}\left(\frac{\eta(\beta',\Omega,Q)}{Q}+\frac{\beta'}{4}\right)^{-2} \leq 1. \tag{14}$$

Thus, to prove that the saddle point condition (13) has solutions $\eta(\beta',\Omega,Q) \geq \eta^*$ for all $\Omega > 0$, $Q > 0$, and for all $\beta' \leq \beta_c'(Q) < 0$, it is sufficient to note that for fixed $\beta' \leq \beta_c'(Q)$, its RHS($\eta(\beta')$) < 1, decreases as $\eta > \eta^*$ increases, while its LHS($\eta(\beta')$) < 1, increases as $\eta > \eta^*$ increases; and in such a way that RHS($\eta(\beta')$) is surjective on the interval $(0,1)$ with $\lim_{\eta(\beta')\nearrow\infty} RHS(\eta(\beta')) = 0$ for any fixed $\beta' < \beta_c'$, and $RHS(\eta^*(\beta_c')) = 1$, and LHS($\eta(\beta')$) is surjective on $(0,1)$ with $\lim_{\eta(\beta')\nearrow\infty} LHS(\eta(\beta')) = 1$ for any fixed $\beta' < \beta_c'$,and $LHS(\bar{\eta}(\beta')) = 0$ for the solution

$$\bar{\eta}(\beta') = -\frac{\beta'\Omega C\sqrt{Q}}{4} - \frac{\beta' Q}{4} > \eta^*$$

of

$$\frac{(\beta')^2\Omega^2 C^2}{16Q}\left(\frac{\bar{\eta}(\beta',\Omega,Q)}{Q}+\frac{\beta'}{4}\right)^{-2} = 1.$$

Condition (14) then implies that for all $\Omega > 0$, $Q > 0$, and for all $\beta' \leq \beta_c'(Q) < 0$,the saddle point $\eta(\beta',\Omega,Q)$ satisfies

$$\eta(\beta',\Omega,Q) \geq -\frac{\beta'\Omega C\sqrt{Q}}{4} - \frac{\beta' Q}{4} > \eta^*$$

which proves the reflection property in the following remark.

Remark 1: *Since the extreme saddle point*

$$\eta^* = -\frac{\beta' Q}{4}$$

satisfies the saddle point conditions (7) and (10) only at the single value of the temperature $T_c' < 0$ that separates the disordered phase from the condensed phase, but not at other $T < 0$, we have shown that the usual phenomenon known as, sticking of the saddle point in the ordered phase, does not hold here. A more appropriate label for this new saddle point behaviour seen in *the spherical Heisenberg models for barotropic flows on a rotating sphere, is jumping and reflection of the saddle point at the negative critical point. Indeed the proof above shows that, for all* $\Omega > 0$ *and* $Q > 0$, *and for all* $\beta' < \beta'_c(Q) < 0$, *the saddle point* $\eta(\beta') \geq -\frac{\beta'\Omega C\sqrt{Q}}{4} - \frac{\beta' Q}{4} > \eta^*$.

We summarize the above results in the physical theorem:

Theorem 1: (A) *For all* $\Omega > 0$ *and* $Q > 0$, *the quantity*

$$\beta'_c(Q, N) = \frac{1}{QN} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^{l} \left(\lambda_{lm} - \frac{1}{2} \right)^{-1} < 0$$

has a well-defined and finite limit, called the critical inverse temperature,

$$\beta'_c(Q) = \lim_{N \to \infty} \beta'_c(Q, N) > -\infty,$$

that is independent of the rate of spin Ω.

(B) Moreover, the thermodynamic limit exists for the spherical Heisenberg models H_H^N *in the sense that for any* $Q > 0$ *and* $\Omega > 0$, *the saddle point conditions,*

$$\begin{aligned} 1 &= \lim_{N \to \infty} \frac{1}{2NQ} \sum_{l=2} \sum_{m} \left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2l(l+1)} \right)^{-1} \\ \left(1 - \frac{1}{Q}\alpha_{10}^2 \right) &= \lim_{N \to \infty} \frac{1}{2NQ} \sum_{l=2} \sum_{m} \left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2l(l+1)} \right)^{-1}, \end{aligned}$$

are well-defined and finite, and the saddle point satisfies the condition

$$\eta(\beta') \geq \eta^* = -\frac{\beta' Q}{4} > 0$$

for all $\beta' < 0$.

(C) For all $\Omega > 0$ *and* $Q > 0$, *and for all* $\beta' < \beta'_c(Q) < 0$, *the ordered phase takes the form of the tiltless* ($\alpha_{1,\pm 1} = 0$) *ground mode* $\alpha_{10}(\beta', \Omega, Q)\psi_{10}$ *with amplitude*

$$\alpha_{10} = -\frac{\beta'\Omega C}{2} \left(\frac{2\eta(\beta')}{Q} + \frac{\beta'}{2} \right)^{-1} > 0,$$

which implies that it is aligned with the rotation $\Omega > 0$ *(super-rotating) and is linear in* Ω.

5.4 Positive temperature

For case (ii) $\beta' > 0$, we note that the Gaussian conditions (2) are automatically satisfied since $\eta(\beta', Q)/Q > 0$ is required of the saddle point of the equation

$$\left(1 - \frac{1}{Q}\alpha_{10}^2\right) = \lim_{N\to\infty} \frac{1}{2NQ} \sum_{l=2} \sum_{m} \left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2l(l+1)}\right)^{-1}, \tag{15}$$

We deduce from the saddle point condition (15), that for any $Q > 0$, and any positive $\beta'(Q) < \infty$, there is a saddle point $\eta(\beta', Q) > 0$ associated with the disordered phase $\alpha_{10} = 0$. Otherwise, there is a finite critical point $\beta'_{cc}(Q) > 0$ that satisfies the equation,

$$\beta'_{cc} = \lim_{N\to\infty} \frac{1}{2NQ} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^{l} 2l(l+1) < \infty,$$

which is a contradiction since the sum on the RHS does not converge. The extreme case, namely $\eta(\beta') = 0$, for the saddle point condition holds at precisely one point, that is, for $\beta' = \infty$.

We will show next that (15) has more than one saddle points at all positive temperatures. In addition to the disordered phase solution found above, the pure ground mode phase is a saddle point $\eta', \alpha_{10} < 0$ of (15). Using the solution (6) of the equation of state for amplitude α_{10} in (15) gives us the final form of the saddle point condition,

$$\left(1 - \frac{(\beta')^2\Omega^2C^2}{16Q}\left(\frac{\eta'(\beta',\Omega,Q)}{Q} + \frac{\beta'}{4}\right)^{-2}\right) = \lim_{N\to\infty} \frac{1}{2NQ} \sum_{l=2} \sum_{m} \left(\frac{\eta'(\beta',\Omega,Q)}{Q} + \frac{\beta'}{2l(l+1)}\right)^{-1} \tag{16}$$

The alternate saddle point - if it exists - satisfies

$$\eta'(\beta',\Omega,Q) > \eta(\beta',Q) \tag{17}$$

because the LHS (16) must satisfy the pair of inequalities,

$$0 < \left(1 - \frac{(\beta')^2\Omega^2C^2}{16Q}\left(\frac{\eta'(\beta',\Omega,Q)}{Q} + \frac{\beta'}{4}\right)^{-2}\right) < 1. \tag{18}$$

A useful condition that is equivalent to the upper bound is

$$\left(\frac{\beta'}{4}\right)^2 \frac{\Omega^2 C^2}{Q}\left(\frac{\eta'(\beta',\Omega,Q)}{Q}+\frac{\beta'}{4}\right)^{-1} < \left(\frac{\eta'(\beta',\Omega,Q)}{Q}+\frac{\beta'}{4}\right). \tag{19}$$

From this we deduce that when the planetary spin $\Omega > \Omega_c \equiv \sqrt{Q/C^2}$, the LHS(16) can be made to satisfy the lower bound in (18) by choosing $\eta' > \eta'_c(\beta',\Omega,Q) > 0$ where LHS(16) equals zero at $\eta'_c(\beta',\Omega,Q) < \infty$. LHS(16) equals one at $\eta' = \infty$. If $\Omega = \Omega_c$, then clearly, $\eta'_c(\beta',\Omega,Q) = 0$. But for $\Omega < \Omega_c$, any $\eta' > 0$ will satisfy the bounds in (18).

$\Omega > \Omega_c$: RHS(16) equals one at the disordered phase saddle point $\eta(\beta',Q)$ which is independent of Ω but equals $R(\eta'_c(\beta',\Omega,Q))$ at $\eta'_c(\beta',\Omega,Q) > 0$. The RHS(16) decreases to zero from the value $R(\eta'_c(\beta',\Omega,Q))$ while LHS(16) increases to one from zero as $\eta' > \eta'_c(\beta',\Omega,Q)$ increases towards ∞. There is an alternate (pure ground mode) saddle point solution of (16) since LHS(16) equals RHS(16) for some $\eta' \in (\eta'_c(\beta',\Omega,Q),\, \infty)$ by continuity.

On the other hand, if the planetary spin Ω is smaller than the critical value $\Omega_c = \sqrt{Q/C^2}$, then at

$$\eta'(\beta',\Omega,Q) = \eta(\beta',Q),$$

the RHS(16) equals one and the LHS(16) equals $L_c(\eta(\beta',Q),\Omega) \in (0,1)$. RHS(16) decreases from one to zero as η' increases from $\eta(\beta',Q)$ to ∞. LHS(16) increases from $L_c(\eta(\beta',Q),\Omega)$ towards one as η' increases from $\eta(\beta',Q)$ to ∞. There is again an alternate (pure ground mode) saddle point solution of (16) since LHS(16) equals RHS(16) for some $\eta' \in (\eta(\beta',Q),Q)$, $\infty)$ by continuity.

This also shows that the thermodynamic limit exists for the spherical Heisenberg models H_H^N for all positive temperatures, in the sense that for any $Q > 0$, $\Omega > 0$ and all $\beta' > 0$, the saddle point condition (15) is well-defined and finite along the saddle points $\eta(\beta',Q)$ and $\eta'(\beta',\Omega,Q)$.

It remains to show that the disordered phase is preferred at high positive temperatures and the pure ground mode phase with counter-rotation $\alpha_{10} < 0$ is preferred at low positive temperatures. To compare the free energy per site of these two phases we ignore for the moment the infinite sum of logarithmic terms in F and focus on the part which depends on α_{10} :

$$\chi(\alpha_{10};\beta,Q,\Omega) \;\; = \;\; \eta'(\beta')\left(1-\frac{\alpha_{10}^2}{Q}\right) - \frac{\beta'}{4}\left(\alpha_{10}^2 + 2\Omega C\alpha_{10}\right)$$

$$= \eta'(\beta')\left(1 - \frac{(\beta')^2\Omega^2 C^2}{16Q}\left(\frac{\eta'(\beta',\Omega,Q)}{Q} + \frac{\beta'}{4}\right)^{-2}\right)$$
$$-\frac{\beta'}{4}\left[\frac{(\beta')^2\Omega^2 C^2}{16}\left(\frac{\eta'(\beta',\Omega,Q)}{Q} + \frac{\beta'}{4}\right)^{-2} - \frac{\beta'\Omega^2 C^2}{2}\left(\frac{\eta'(\beta',\Omega,Q)}{Q} + \frac{\beta'}{4}\right)^{-1}\right]$$

$$= -\frac{(\beta')^2\Omega^2 C^2}{16}\left(\frac{\eta'(\beta',\Omega,Q)}{Q} + \frac{\beta'}{4}\right)^{-2}\left(\frac{\eta'}{Q} + \frac{\beta'}{4}\right) + \frac{(\beta')^2\Omega^2 C^2}{8}\left(\frac{\eta'(\beta',\Omega,Q)}{Q} + \frac{\beta'}{4}\right)^{-1}$$
$$= \frac{(\beta')^2\Omega^2 C^2}{16}\left(\frac{\eta'(\beta',\Omega,Q)}{Q} + \frac{\beta'}{4}\right)^{-1}.$$

The same quantity for the disordered phase is given by

$$\chi(\alpha_{10} = 0; \beta, Q, \Omega) = \eta(\beta').$$

Comparing them we get the following inequality which implies that the pure ground mode phase is preferred:

$$\frac{(\beta')^2\Omega^2 C^2}{16Q} > \frac{\eta(\beta',Q)}{Q}\left(\frac{\eta'(\beta',\Omega,Q)}{Q} + \frac{\beta'}{4}\right). \tag{20}$$

Fixing $\frac{\Omega^2 C^2}{Q} > 0$, we deduce from (17) that (20) holds only if $\beta' > \beta'_{cc}(\Omega, Q)$ where

$$\frac{(\beta'_{cc})^2\Omega^2 C^2}{16Q} = \frac{\eta(\beta'_{cc},Q)}{Q}\left(\frac{\eta'(\beta'_{cc},\Omega,Q)}{Q} + \frac{\beta'_{cc}}{4}\right)$$

where such a positive value $\beta'_{cc} < \infty$ exists by virtue of the mean value theorem because for β' near zero,

$$\frac{(\beta')^2\Omega^2 C^2}{16Q} < \frac{\eta(\beta',Q)}{Q}\left(\frac{\eta'(\beta',\Omega,Q)}{Q} + \frac{\beta'}{4}\right)$$

since $\eta(\beta', Q)$ increases as β' decreases, and for β' very large,

$$\frac{(\beta')^2\Omega^2 C^2}{16Q} > \frac{\eta(\beta',Q)}{Q}\left(\frac{\eta'(\beta',\Omega,Q)}{Q} + \frac{\beta'}{4}\right)$$

since $\eta(\beta', Q)$ decreases down to zero as β' increases to ∞. We used the fact that both saddle points $\eta(\beta', Q)$ and $\eta'(\beta', \Omega, Q)$ are smooth functions of β' in the range $(0, \infty)$.

Returning to the infinite sum of logarithmic terms in F,

$$- \lim_{N \to \infty} \frac{1}{2N} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^{l} \ln \left(\frac{N\eta}{Q} + \frac{\beta' N}{2} \lambda_{lm} \right),$$

we note that the value of this convergent sum for saddle point $\eta(\beta', Q)$ is bigger than that for $\eta'(\beta', \Omega, Q)$ in view of (17) but this difference is logarithmic in the difference $\eta'(\beta', \Omega, Q) - \eta(\beta', Q) > 0$, and is therefore dominated by the algebraic difference

$$\chi(\alpha_{10}; \beta, Q, \Omega) - \chi(\alpha_{10} = 0; \beta, Q)$$

discussed above.

This completes the proof that the disordered phase is preferred at high positive temperatures but the ordered phase is preferred at low enough temperatures where, unlike the negative critical point, the threshold value $\beta'_{cc}(\Omega, Q)$ depends on both relative enstrophy Q and planetary spin Ω. From (19) we deduce that $\eta'(\beta', \Omega, Q)$ is linear in Ω. Since $\eta(\beta', Q)$ does not depend on Ω, this implies $\beta'_{cc}(\Omega, Q)$ decreases as planetary spin Ω increases. Thus, as Ω decreases to zero, the threshold value $\beta'_{cc}(\Omega, Q)$ tends to ∞, and the disordered phase is preferred at all positive temperatures in the case of a non-rotating massive sphere.

Unlike the critical phenomenology of the barotropic fluid - sphere system at very high energies (negative temperatures) which we have shown arises from the reflection of the saddle point at the extreme value η^* (the disordered phase does not satisfy the saddle point condition at negative T' when $|T'| \ll 1$), its critical phenomenology at positive temperature is not so much based on the breakdown of the saddle points as on the system's preference for a smaller free energy. Transitions between these positive temperature phases for $\Omega > 0$ are characterized by a greater degree of smoothness than its negative temperature counterpart since the free energy is automatically continuous at $\beta'_{cc}(\Omega, Q) > 0$.

6 Proof of gapless and unique ground mode condensation

An important result needed in the above exact solution of the spherical Heisenberg model is the number and type of modes in the condensed phase. We will prove that there is only one thermodynamically stable class of modes ($l = 1$) that has nonzero energy in the condensed phase and thence, exactly these three modes are non-ergodic in this problem. The proof is based on the existence of multiple saddle points $\eta'(\beta') \neq \eta(\beta')$ and an application of Planck's theorem.

The first two parts of the proof are relatively short. The complete proof will be given in the order: (1) at positive temperatures, the only nontrivial condensed mode is associated with $\alpha_{10} < 0$ signifying a counter rotating solid body flow, (2) at negative temperatures, there cannot be gapped nontrivial condensed modes where gapped means that there are ergodic modes l in between condensed modes l', and (3) at negative temperatures, the only relevant nontrivial condensed mode is the ground mode $\alpha_{10} > 0$ which is associated with super-rotating solid-body flows. Part (3) is longer because of the important property that there is more than one saddle points at some negative T, namely (i) the pure ground mode saddle point $\alpha_{10} > 0$ and (ii) the condensed phase $\alpha_{2m} \neq 0$, $\alpha_{10} < 0$. We will show that saddle point (i) has higher free energy per site than saddle point (ii) for all values of $T < T_c < 0$ where the latter is condensed. By the extension of Planck's theorem to negative temperatures, the preferred macrostate is the one with highest free energy, namely the pure ground mode.

6.1 (1) condensed modes at positive T cannot have wavenumber $l > 1$

Assuming that there is a set of condensed modes with single $l > 1$ at positive T, the finite part of the per site free energy expression F - after dropping the infinite sum - has the form

$$\begin{aligned}\chi(\eta(\beta'), Q, \beta') &= \eta(\beta')\left[1 - \frac{1}{Q}\left\{\sum_{m=-l}^{l} \alpha_{lm}^2 + \sum_{m=-1}^{1} \alpha_{1m}^2\right\}\right] \\ &\quad - \frac{\beta'}{4}\sum_{m=-1}^{1} \alpha_{1m}^2 - \frac{\beta'}{2}\Omega C \alpha_{10} - \left(\frac{\beta' \lambda_{lm}}{2}\right)\sum_{m=-l}^{l} \alpha_{lm}^2.\end{aligned}$$

Similar to the approach in the previous section, the amplitudes α_{lm} of these modes are fixed by additional equations of state

$$\frac{\partial F}{\partial \alpha_{lm}} = \frac{\partial \chi}{\partial \alpha_{lm}} = -\left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{l(l+1)}\right)\alpha_{lm} = 0.$$

In order for at least one α_{lm} to be nonzero,

$$\left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{l(l+1)}\right) = 0$$

which contradicts the positivity of β' and $\frac{\eta(\beta')}{Q}$. The distinguished ground mode α_{10} can be nonzero because of the inhomogeneous term $-\frac{\beta'}{2}\Omega C$ in the corresponding equation of state for the amplitude.

6.2 (2) condensed modes must be gapless at negative T

We will prove next that at negative temperatures, there cannot be any nonzero condensed modes $l' \geq 2$ that is separated by ergodic modes in between. The situation is clear from the simplest case of a single additional nonzero condensed mode with $l' > 2$. Then, $l = 2$ corresponds to ergodic modes which have to be integrated in the Gaussian integrals, but similar to the above analysis of the equations of state, a nonzero amplitude $\alpha_{l'm} \neq 0$ implies that $\frac{\beta'\lambda_{l'm}}{2} + \frac{\eta(\beta')}{Q} = 0$ which in turn means that $\frac{\beta'\lambda_{2m}}{2} + \frac{\eta(\beta')}{Q} < 0$, violating the Gaussian integrability condition.

6.3 (3) the only condensed mode at negative T is $\alpha_{10} > 0$

Assume (for reductio ad absurdum) that more than one class of modes have nonzero amplitude in the condensed phase of this problem, namely those belonging to wavenumbers l_1, l_2 and also $l = 1$ (this being the distinguished ground mode of the problem, has to be part of the condensed phase) where $l_2 = l_1 + 1 = 3$. Other cases where l_2, l_1 and $l = 1$ are not consecutive, cannot appear in the condensed phase of a solvable spherical model, because the Gaussian solvability conditions

$$\frac{\beta'\lambda_{lm}}{2} + \frac{\eta}{Q} = \frac{\beta'}{2l(l+1)} + \frac{\eta}{Q} > 0$$

are violated as shown above.

Then, the restricted partition function is given by

$$Z_H^N(\beta,Q,\Omega) \propto \int \prod_{m=-1}^{1} d\alpha_{1m} \prod_{m=-l_1}^{l_1} d\alpha_{l_1 m} \prod_{m=-l_2}^{l_2} d\alpha_{l_2 m} Z_H^N(\alpha_{1m},\alpha_{l_1 m},\alpha_{l_2 m};\beta,Q,\Omega)$$

$$= \int \prod_{m=-1}^{1} d\alpha_{1m} \prod_{m=-l_1}^{l_1} d\alpha_{l_1 m} \prod_{m=-l_2}^{l_2} d\alpha_{l_2 m} \int_{a-i\infty}^{a+i\infty} \frac{d\eta}{2\pi i} \exp\left\{ N \left[\begin{array}{c} \eta - \left(\frac{\beta\lambda_{l_1 m}}{2N} + \frac{\eta}{Q}\right)\sum_{m=-1}^{1} \alpha_{l_1 m}^2 \\ - \left(\frac{\beta\lambda_{l_2 m}}{2N} + \frac{\eta}{Q}\right)\sum_{m=-1}^{1} \alpha_{l_2 m}^2 \\ - \left(\frac{\beta}{4N} + \frac{\eta}{Q}\right)\sum_{m=-1}^{1} \alpha_{1m}^2 \\ -\frac{\beta}{2N}\Omega C\alpha_{10} \end{array} \right] \right\}$$

$$\int D_{l\neq 1,l_1,l_2}(\alpha) \exp\left(-N \sum_{l\neq 1,l_1,l_2}^{\infty} \sum_{m=-l}^{l} \left(\frac{\beta}{2N}\lambda_{lm} + \frac{\eta}{Q}\right)\alpha_{lm}^2 \right).$$

Standard Gaussian integration is used to evaluate the last integral, which yields, after scaling $\beta' N = \beta$,

$$\int_{l\neq 1,l_1,l_2} D(\alpha) \exp\left(- \sum_{l\neq 1,l_1,l_2} \sum_{m=-l}^{l} \left(\frac{\beta' N\lambda_{lm}}{2} + \frac{N\eta}{Q}\right)\alpha_{lm}^2 \right) = \prod_{l\neq 1,l_1,l_2}^{\sqrt{N}} \prod_{m=-l}^{l} \left(\frac{\pi}{\frac{N\eta}{Q} + \frac{\beta' N}{2}\lambda_{lm}} \right)^{1/2},$$

provided the physically significant Gaussian conditions hold: for $l \neq 1, l_1, l_2$,

$$\frac{\beta'\lambda_{lm}}{2} + \frac{\eta}{Q} = \frac{\beta'}{2l(l+1)} + \frac{\eta}{Q} > 0.$$

Then the restricted partition function takes the form

$$Z_H^N(\alpha_{1m},\alpha_{l_1 m},\alpha_{l_2 m};\beta,Q,\Omega) \propto \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} d\eta \; \exp\left\{ N \left[\begin{array}{c} \eta - \left\{ \begin{array}{c} \left(\frac{\beta'\lambda_{l_1 m}}{2} + \frac{\eta}{Q}\right)\sum_{m=-1}^{1} \alpha_{l_1 m}^2 \\ + \left(\frac{\beta'\lambda_{l_2 m}}{2} + \frac{\eta}{Q}\right)\sum_{m=-1}^{1} \alpha_{l_2 m}^2 \end{array} \right\} \\ - \left(\frac{\beta'}{4} + \frac{\eta}{Q}\right)\sum_{m=-1}^{1} \alpha_{1m}^2 - \frac{\beta'}{2}\Omega C\alpha_{10} \\ -\frac{1}{2N}\sum_{l\neq 1,l_1,l_2}\sum_{m=-l}^{l} \ln\left(\frac{N\eta}{Q} + \frac{\beta' N}{2}\lambda_{lm}\right) \end{array} \right] \right\}$$

where the free energy per site evaluated at the most probable macrostate is $-\frac{1}{\beta'}\ F(\eta(\beta'),Q,\beta')$ with

$$\begin{aligned} F(\eta(\beta'),Q,\beta') \;&=\; \eta(\beta')\left[1 - \frac{1}{Q}\left\{ \sum_{m=-l_1}^{l_1} \alpha_{l_1 m}^2 + \sum_{m=-l_2}^{l_2} \alpha_{l_2 m}^2 + \sum_{m=-1}^{1} \alpha_{1m}^2 \right\} \right] \\ -\frac{\beta'}{4}\sum_{m=-1}^{1} \quad & \alpha_{1m}^2 - \frac{\beta'}{2}\Omega C\alpha_{10} - \left\{ \left(\frac{\beta'\lambda_{l_1 m}}{2}\right)\sum_{m=-l_1}^{l_1} \alpha_{l_1 m}^2 + \left(\frac{\beta'\lambda_{l_2 m}}{2}\right)\sum_{m=-l_1}^{l_2} \alpha_{l_2 m}^2 \right\} \end{aligned}$$

$$-\frac{1}{2N}\sum_{l\neq 1,l_1,l_2}\sum_m \ln\left(\frac{N\eta}{Q}+\frac{\beta' N}{2}\lambda_{lm}\right).$$

The saddle point condition gives one equation for the determination of the variables η, α_{1m}, α_{l_1m}, α_{l_1m} in terms of inverse temperature β', relative enstrophy Q, and the fixed rate of spin $\Omega > 0$ of the planetary frame,

$$0 = \frac{\partial F}{\partial \eta} = 1 - \frac{1}{Q}\left\{\sum_{m=-l_1}^{l_1}\alpha_{l_1m}^2 + \sum_{m=-l_2}^{l_2}\alpha_{l_2m}^2 + \sum_{m=-1}^{1}\alpha_{1m}^2\right\} - \frac{1}{2NQ}\sum_{l\neq 1,l_1,l_2}\sum_m\left(\frac{\eta(\beta')}{Q}+\frac{\beta'}{2}\lambda_{lm}\right)^{-1}$$

where $\eta = \eta(\beta')$ is taken to be the value of the saddle point. Equations to close the system are provided by equations of state (or Planck's theorem) for the condensed phase: for $l = 1$,

$$\begin{aligned} 0 &= \frac{\partial F}{\partial \alpha_{10}} = -\left(\frac{2\eta(\beta')}{Q}+\frac{\beta'}{2}\right)\alpha_{10} - \frac{\beta'}{2}\Omega C \\ 0 &= \frac{\partial F}{\partial \alpha_{1,\pm 1}} = -\left(\frac{2\eta(\beta')}{Q}+\frac{\beta'}{2}\right)\alpha_{1,\pm 1}; \end{aligned}$$

and for $l = l_1, l_2$, $m = -l, ..., 0, ..., l$,

$$\begin{aligned} 0 &= \frac{\partial F}{\partial \alpha_{l_1m}} = -\left(\frac{2\eta(\beta')}{Q}+\beta'\lambda_{l_1m}\right)\alpha_{l_1m} \\ 0 &= \frac{\partial F}{\partial \alpha_{l_2m}} = -\left(\frac{2\eta(\beta')}{Q}+\beta'\lambda_{l_2m}\right)\alpha_{l_2m}. \end{aligned}$$

We assume that $\alpha_{1m} \neq 0$ for some $m = -1, 0, 1$.

The proof by contradiction continues by further assuming that at least one of the $l' = l_1, l_2$ equations of state are satisfied by nonzero amplitudes, say $\alpha_{l_1m} \neq 0$ for some $m = -l_1, ..., 0, ..., l_1$. Then

$$\left(\frac{2\eta(\beta')}{Q}+\beta'\lambda_{l_1m}\right) = 0$$

which in turn implies

$$\begin{aligned} \left(\frac{2\eta(\beta')}{Q}+\beta'\lambda_{l_2m}\right) &> 0, \\ \left(\frac{2\eta(\beta')}{Q}+\frac{\beta'}{2}\right) &< 0, \end{aligned}$$

and thence,

$$\begin{aligned} \alpha_{l_2 m} &= 0, \\ \alpha_{1,\pm 1} &= 0, \\ \alpha_{10} &\neq 0. \end{aligned}$$

The first of the $l = 1$ equations of state then implies

$$\alpha_{10} = -\frac{\beta'}{2}\Omega C \left(\frac{2\eta(\beta')}{Q} + \frac{\beta'}{2}\right)^{-1} < 0 \text{ for } \beta' < 0 \tag{21}$$

since $\Omega C > 0$; and vice-versa for $\beta' > 0$.

The contradiction at negative temperatures is obtained first by showing that although the saddle point equation

$$1 - \frac{1}{Q}\left\{\sum_{m=-2}^{2} \alpha_{2m}^2 + \alpha_{10}^2\right\} = \lim_{N\to\infty} \frac{1}{2NQ} \sum_{l\neq 1,2,3} \sum_{m} \left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2}\lambda_{lm}\right)^{-1} \tag{22}$$

has solution $\eta'(\beta')$, $\alpha_{10} < 0$, $\alpha_{2m} \neq 0$ in addition to the pure ground mode solution $\eta(\beta')$, $\alpha_{10} > 0$, $\alpha_{2m} = 0$ at large values of the scaled inverse temperature $\beta' < \beta'_c < 0$, these counter-rotating solutions have lower values of $F' = F(\eta'(\beta'), Q, \beta')$ than the same expression F for the pure ground mode solution. Here $\beta'_c < 0$ is the most negative inverse temperature for which (22) has unique saddle point, namely the disordered phase, $\eta, \alpha_{10} = 0, \alpha_{2m} = 0$. This critical point has the same value critical inverse temperature as obtained in the last section because the RHS (22) is the same as its counterpart in the pure ground mode case in the thermodynamic limit. For all $\beta' < \beta'_c$, an argument similar to that used in the previous section to prove the existence of the pure ground mode saddle point, can be used here to prove the alternative saddle point equation (22) has two saddle points, namely the sticking one,

$$\eta'(\beta') = -\frac{\beta' Q}{12}, \alpha_{10} < 0, \alpha_{2m} \neq 0 \tag{23}$$

and reflected pure ground mode solution

$$\eta(\beta') > \eta^* = -\frac{\beta' Q}{4}, \alpha_{10} > 0, \alpha_{2m} = 0. \tag{24}$$

By the extension of Planck's theorem to negative temperatures, solutions $\alpha_{10} < 0$, $\alpha_{2m} \neq 0$ at $\beta' < 0$ are not the most probable and statistically stable macrostate because they have lower per site free energy.

Using the sticking property, we deduce that the free energy expression $F(\eta'(\beta'), Q, \beta')$ is given by

$$\begin{aligned} F' &= \eta'(\beta') - \frac{\beta'}{2}\Omega C \alpha_{10} - \left(\frac{\eta'(\beta')}{Q} + \frac{\beta'}{4}\right)\alpha_{10}^2 \\ - \lim_{N\to\infty} & \frac{1}{2N}\sum_{l\neq 1,2,3}\sum \ln\left(\frac{N\eta'}{Q} + \frac{\beta' N}{2}\lambda_{lm}\right), \end{aligned}$$

which apart from the first terms in the infinite sum that vanish like $N^{-1}\ln N$ for large N, and the new value $\eta'(\beta') < \eta(\beta')$, has the same form as $F(\eta(\beta'), Q, \beta')$ in the pure ground mode case. Equation (23) and $\alpha_{10} < 0$ imply that the first two terms of F' is smaller than those in F and the third term is bigger. So for $\Omega > 0$ large enough, since $\eta'(\beta')$ is independent of Ω, we can make $F' < F$.

We can do more by using the sticking property of $\eta'(\beta')$. Substituting $\eta'(\beta') = -\frac{\beta' Q}{12}$ and $\alpha_{10} < 0$ from (21) into the first three terms in F', and using the reflection property of the saddle point $\eta(\beta')$, namely, $\left(\frac{2\eta(\beta')}{Q} + \frac{\beta'}{2}\right) > 0$, we obtain the required comparison at $\beta' < 0$, that is,

$$\begin{aligned} F_3'(\eta'(\beta')) &= -\frac{\beta' Q}{12} + \frac{1}{2}\left(\frac{\beta'}{2}\right)^2 \Omega^2 C^2 \left(\frac{\beta'}{3}\right)^{-1} \\ &= -\frac{\beta' Q}{12} + \frac{3\beta'}{8}\Omega^2 C^2 < -\frac{\beta' Q}{4} \\ &< -\frac{\beta' Q}{4} + \frac{1}{2}\left(\frac{\beta'}{2}\right)^2 \Omega^2 C^2 \left(\frac{2\eta(\beta')}{Q} + \frac{\beta'}{2}\right)^{-1} \\ &< F_3(\eta(\beta')). \end{aligned}$$

In other words, for all $\beta' < \beta_c'$ the per site free energy expression $F(\eta(\beta')) > F'(\eta'(\beta'))$. Thus, the pure ground mode condensed phase is preferred in the thermodynamic limit.

This completes the proof of the result that for all values of planetary spin at negative temperatures, there is exactly one thermodynamically stable saddle point, that is the disordered phase, $\eta, \alpha_{10} = 0$ at negative $\beta' > \beta_c'$, and the pure ground mode condensed phase, $\eta, \alpha_{10} > 0$ at negative $\beta' < \beta_c'$.

An important consequence of this result is that the alternative condensed phase, $\eta'(\beta')$, $\alpha_{10} < 0$, $\alpha_{2m} \neq 0$ can in principle be thermodynamically stable in other geophysical flow problems such as the generalized Shallow Water Equations (GSWE) on a massive rotating sphere.

7 Solution of the spherical Ising model for $\Omega = 0$

We briefly review the exact solution of the spherical Ising model in the special case of a fixed frame and refer the reader to the literature for details [1]. In view of the sections on the role of energy and angular momentum in the formulation of a statistical mechanics for the barotropic fluid - solid sphere system, one might ask why we need to solve this case when we have already solved the general $\Omega > 0$ case in the previous section. This case applies to the situation of a non-rotating infinitely massive solid sphere which behaves like three infinite reservoirs of angular momentum for each of the three principle directions. As will be shown below this case differs from the $\Omega > 0$ case because without a distinguished axis of rotation, the ground or ordered modes in the problem has $SO(3)$ degeneracy (due to the 3-fold degeneracy of the $l = 1$ spherical harmonics). Symmetry breaking in the phase transition for this case is therefore more explicit than in the $\Omega > 0$ case. In fact it is clearly an instance of so-called spontaneous symmetry breaking (SSB) in the general Ginsburg-Landau formulation of second order phase transitions. Similar to the Heisenberg models, the negative critical temperature here is due to the anti-ferromagnetic nature of the logarithmic interaction in the energy.

The partition function for the spherical Ising model has the form

$$Z_N \propto \int D(\vec{s}) \exp\left(-\beta H_N(\vec{s})\right) \delta\left(Q\frac{N}{4\pi} - \sum_{j=1}^{N} \vec{s}_j \cdot \vec{s}_j\right)$$

where the path-integral is taken over all microstates $\vec{s}$ with zero circulation. In the thermodynamic or continuum limit as $N \to \infty$, the partition function is calculated using Laplace's integral form,

$$\begin{aligned} Z_N &\propto \int D(\vec{s}) \exp\left(-\beta H_N(\vec{s})\right) \delta\left(Q\frac{N}{4\pi} - \sum_{j=1}^{N} \vec{s}_j \cdot \vec{s}_j\right) \\ &= \int D(\vec{s}) \exp\left(-\beta H_N(\vec{s})\right) \left(\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} d\eta \exp\left(\eta\left(Q\frac{N}{4\pi} - \sum_{j=1}^{N} \vec{s}_j \cdot \vec{s}_j\right)\right)\right). \end{aligned}$$

Solution of the Gaussian integrals require diagonalizing the interaction in H_N in terms of the spherical harmonics $\{\psi_{lm}\}_{l=1,}^{\infty}$ which are natural Fourier modes for Laplacian eigenvalue problems on S^2 with zero circulation. Since the ordered modes are associated with $l = 1$, we do not need to integrate over α_{1m} and henceforth discuss only the restricted partition function

$$Z_N(\beta, Q; \alpha_{10}, \alpha_{1,\pm 1}) \propto \int D_{l\geq 2}(\alpha) \exp\left(-\frac{\beta}{2}\sum_{l=1}\sum_{m=-l}^{l} \lambda_{lm}\alpha_{lm}^2\right)$$

$$\left(\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} d\eta \exp\left(\eta N\left(1-\frac{4\pi}{Q}\sum_{l=1}\sum_{m=-l}^{l}\alpha_{lm}^2\right)\right)\right)$$

where the eigenvalues of the Green's function for the Laplace-Beltrami operator on S^2 are

$$\lambda_{lm} = \frac{1}{l(l+1)},\ l = 1, ..., \sqrt{N},\ m = -l, ..., 0, ..., l$$

and α_{lm} are the corresponding amplitudes.

Next we exchange the order of integration, which is allowed provided $a > 0$ is chosen large enough so that the integrand is absolutely convergent, and rescale $\beta' N = \beta$,

$$Z_N(\beta, Q; \alpha_{10}, \alpha_{1,\pm 1}) \propto \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} d\eta \exp\left(\eta N\left(1-\frac{4\pi}{Q}\sum_{m=-l}^{l}\alpha_{1m}^2\right)-\frac{\beta' N}{2}\sum_{m=-1}^{1}\lambda_{1m}\alpha_{1m}^2\right)$$

$$\int_{l\geq 2} D(\alpha)\exp\left(-\sum_{l=2}\sum_{m=-l}^{l}\left(\frac{\beta' N\lambda_{lm}}{2}+N\eta\frac{4\pi}{Q}\right)\alpha_{lm}^2\right).$$

We stress that

$$\exp\left(-\frac{\beta' N}{2}\sum_{m=-1}^{1}\lambda_{1m}\alpha_{1m}^2\right)$$

is not part of the integrand since the ground modes are not integrated. Moreover, by standard procedures for a Gaussian, we explicitly solve the inner integral,

$$\int_{l\geq 2} D(\alpha)\exp\left(-\sum_{l=2}\sum_{m=-l}^{l}\left(\frac{\beta' N\lambda_{lm}}{2}+N\eta\frac{4\pi}{Q}\right)\alpha_{lm}^2\right) = \prod_{l=2}^{\sqrt{N}}\prod_{m=-l}^{l}\left(\frac{\pi}{N\eta\frac{4\pi}{Q}+\frac{\beta' N}{2}\lambda_{lm}}\right)^{1/2},$$

provided the following physically important conditions hold

$$\frac{\beta'\lambda_{lm}}{2}+\eta\frac{4\pi}{Q} > 0,\ l = 2, ..., \sqrt{N},\ m = -l, ..., 0, ..., l. \tag{25}$$

Thus,

$$Z_N(\beta, Q; \alpha_{10}, \alpha_{1,\pm1}) \propto \int_{a-i\infty}^{a+i\infty} d\eta \exp N \left[\begin{array}{c} \eta\left(1 - \frac{4\pi}{Q}\sum_{m=-1}^{1}\alpha_{1m}^2\right) - \frac{\beta'}{2}\sum_{m=-1}^{1}\lambda_{1m}\alpha_{1m}^2 \\ -\frac{1}{2N}\sum_{l=2}\sum_m \ln\left(N\eta\frac{4\pi}{Q} + \frac{\beta' N}{2}\lambda_{lm}\right) \end{array} \right],$$

which can be written in steepest descent form,

$$\begin{aligned} Z &\propto \lim_{N\to\infty}\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} d\eta \exp\left(NF(\eta, Q, \beta')\right) \\ &= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} d\eta \exp\left(-\beta' g(\eta, Q, \beta')\right) \\ &= \exp\left(-\beta' g(\eta(\beta'), Q, \beta')\right) \end{aligned}$$

in the thermodynamic limit as $N \to \infty$,where the free energy per site, after separating out the 3-fold degenerate ground states ψ_{10},$\psi_{l,\pm1}$, is given by

$$-\frac{1}{\beta'}\ F(\eta(\beta'), Q, \beta') \ \text{ with}$$

$$\begin{aligned} F(\eta(\beta'), Q, \beta') &= \eta(\beta')\left(1 - \frac{4\pi}{Q}\sum_{m=-1}^{1}\alpha_{1m}^2\right) - \frac{\beta'}{2}\sum_{m=-1}^{1}\lambda_{1m}\alpha_{1m}^2 \\ -\frac{1}{2N}\sum_{l=2} & \sum_m \ln\left(N\eta(\beta')\frac{4\pi}{Q} + \frac{\beta' N}{2}\lambda_{lm}\right) \end{aligned}$$

and the total free energy

$$g(\eta(\beta'), Q, \beta') = \lim_{N\to\infty}\left(-\frac{N}{\beta'}\ F(\eta(\beta'), Q, \beta')\right).$$

The saddle point parameter $\eta = \eta(\beta')$ is determined by solving the following set of four equations for (η, α_{1m}) in terms of given values of inverse temperature β' and relative enstrophy Q.

The saddle point condition is

$$0 = \frac{\partial F}{\partial \eta} = \left(1 - \frac{4\pi}{Q}\sum_{m=-1}^{1}\alpha_{1m}^2\right) - \frac{2\pi}{Q}\sum_{l=2}^{\sqrt{N}}\sum_{m=-l}^{l}\left(N\eta\frac{4\pi}{Q} + \frac{\beta' N}{2}\lambda_{lm}\right)^{-1}. \quad (26)$$

A set of three additional conditions to close the system is given by the equations of state for $m = -1, 0, 1$,

$$0 = \frac{\partial F}{\partial \alpha_{1m}} = \left(\frac{8\pi\eta}{Q} + \beta'\lambda_{1m}\right)\alpha_{1m}. \quad (27)$$

The last 3 equations have solutions

$$\alpha_{1m} \ = 0 \text{ or } \frac{8\pi\eta}{Q} + \beta'\lambda_{1m} = 0, \text{ for each } m.$$

This means that in order to have nonzero amplitudes in at least one of the ground / condensed states (which are the only ones to have angular momentum), $\frac{4\pi\eta^*}{Q} = -\frac{\beta'}{4}$,which implies that the inverse temperature must be negative, $\beta' < 0$.

The Gaussian condition (25) on the modes with $l = 2$,

$$\frac{\beta'}{12} - \frac{\beta'}{2} > 0,$$

can only be satisfied by $\beta' < 0$ when there is any energy in the angular momentum containing ground modes.

Substituting this nonzero solution into the saddle point equation yields

$$\begin{aligned} 0 &= \left(1 - \frac{4\pi}{Q}\sum_{m=-1}^{1}\alpha_{1m}^2\right) - \frac{4\pi}{Q}\frac{T'}{N}\sum_{l=2}^{\sqrt{N}}\sum_{m=-l}^{l}\left(\lambda_{lm} - \frac{1}{2}\right)^{-1} \\ &= \left(1 - \frac{4\pi}{Q}\sum_{m=-1}^{1}\alpha_{1m}^2\right) - \frac{T'}{T_c'} \end{aligned}$$

where the critical inverse temperature is negative, finite, and inversely proportional to the relative enstrophy Q,

$$-\infty < \beta_c' = \frac{4\pi}{QN}\sum_{l=2}^{\sqrt{N}}\sum_{m=-l}^{l}\left(\lambda_{lm} - \frac{1}{2}\right)^{-1} < 0.$$

The saddle point equation provides a way to compute the equilibrium amplitudes of the ground modes for temperatures hotter than $T_c' < 0$, that is,

$$\text{for } T_c' < T' < 0, \ \sum_{m=-1}^{1}\alpha_{1m}^2(T') = \frac{Q}{4\pi}\left(1 - \frac{T'}{T_c'}\right).$$

The above argument shows that at positive temperatures (low barotropic energy), there cannot be any energy in the solid-body rotating modes. In other words, there is no phase transition at positive temperatures when $\Omega = 0$. This is the spin-lattice representation of the self-organization of barotropic energy into a large-scale coherent flow at very high energies in the form of symmetry-breaking Goldstone modes. The reader should compare the predictions of the spherical model for barotropic vortex statistics contained in these formulae with the results of Monte-Carlo simulations in the next chapter. In particular, the linear dependence of the negative critical temperature $T_c = T_c(Q) < 0$ on the relative enstrophy should be noted in the spherical model solution as well as the Monte-Carlo simulations [**?**].

The free energy per site in the thermodynamic limit ($N \to \infty$) has the form

$$f(\eta, Q, \beta') = \lim_{N\to\infty} \left(-\frac{1}{\beta'} \; F(\eta(\beta'), Q, \beta') \right) = u - Ts$$

$$= \frac{1}{4} \sum_{m=-1}^{1} \alpha_{1m}^2 + \frac{Q}{16\pi} \left(1 - \frac{4\pi}{Q} \sum_{m=-1}^{1} \alpha_{1m}^2 \right) + \lim_{N\to\infty} \frac{T'}{2N} \sum_{l=2} \sum_{m} \ln \frac{N}{2T'} \left(-\frac{1}{2} + \lambda_{lm} \right)$$

$$= \frac{Q}{16\pi} + \lim_{N\to\infty} \frac{T'}{2N} \sum_{l=2} \sum_{m} \ln \frac{N}{2T'} \left(-\frac{1}{2} + \lambda_{lm} \right)$$

and

$$\begin{aligned}
\frac{d}{dT'} f(\eta^*, Q, T) &= \lim_{N\to\infty} \frac{1}{2N} \sum_{l=2} \sum_{m} \ln \frac{N}{2T'} \left(-\frac{1}{2} + \lambda_{lm} \right) \\
- \lim_{N\to\infty} & \quad \frac{T'}{2N} \sum_{l=2} \sum_{m} \frac{2T'}{N} \left(-\frac{1}{2} + \lambda_{lm} \right)^{-1} \frac{N}{2T'^2} \left(-\frac{1}{2} + \lambda_{lm} \right) \\
&= \lim_{N\to\infty} \frac{1}{2N} \sum_{l=2} \sum_{m} \ln \frac{N}{2T'} \left(-\frac{1}{2} + \lambda_{lm} \right) - \lim_{N\to\infty} \frac{1}{2N} \sum_{l=2} \sum_{m} 1 \\
&= -1 + \lim_{N\to\infty} \frac{1}{2N} \sum_{l=2} \sum_{m} \ln \frac{N}{2T'} \left(-\frac{1}{2} + \lambda_{lm} \right)
\end{aligned}$$

when $\sum_{m=-1}^{1} \alpha_{1m}^2 > 0$ and $T_c' < T' < 0$. We used

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{l=2} \sum_{m} 1 = 1$$

as well as the sticking value of the saddle point

$$\frac{4\pi\eta^*}{Q} = -\frac{\beta'}{4}.$$

When $T' < T_c' < 0$, the saddle point equation implies that

$$\lim_{N\to\infty} \frac{2\pi}{QN} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^{l} \left(\eta \frac{4\pi}{Q} + \frac{1}{2T'} \lambda_{lm} \right)^{-1} = 1 \tag{28}$$

giving

$$\begin{aligned} f(\eta(\beta'), Q, \beta') &= \lim_{N\to\infty} \left(-\frac{1}{\beta'} \, F(\eta(\beta'), Q, \beta') \right) \\ &= -\eta(T')T' + \lim \frac{T'}{2N} \sum_{l=2} \sum_{m} \ln \left(N\eta(T') \frac{4\pi}{Q} + \frac{N}{2T'} \lambda_{lm} \right) \end{aligned}$$

because $\sum_{m=-1}^{1} \alpha_{1m}^2 = 0$ and

$$\begin{aligned} \frac{d}{dT'} f(\eta(T'), Q, T) &= -\eta(T') - \eta'(T')T' + \lim \frac{1}{2N} \sum_{l=2} \sum_{m} \ln \left(N\eta(T') \frac{4\pi}{Q} + \frac{N}{2T'} \lambda_{lm} \right) \\ &+ \lim \frac{T'}{2N} \sum_{l=2} \sum_{m} \left(\eta(T') \frac{4\pi}{Q} + \frac{1}{2T'} \lambda_{lm} \right)^{-1} \left(-\frac{\lambda_{lm}}{2(T')^2} + \eta'(T') \frac{4\pi}{Q} \right) \\ &= -\eta(T') - \eta'(T')T' + \lim \frac{1}{2N} \sum_{l=2} \sum_{m} \ln \left(N\eta(T') \frac{4\pi}{Q} + \frac{N}{2T'} \lambda_{lm} \right) \\ &+ \lim \eta'(T')T' \frac{2\pi}{QN} \sum_{l=2} \sum_{m} \left(\eta(T') \frac{4\pi}{Q} + \frac{1}{2T'} \lambda_{lm} \right)^{-1} \\ &- \lim \frac{1}{4NT'} \sum_{l=2} \sum_{m} \lambda_{lm} \left(\eta(T') \frac{4\pi}{Q} + \frac{1}{2T'} \lambda_{lm} \right)^{-1} \\ &= -\eta(T') - \eta'(T')T' + \lim \frac{1}{2N} \sum_{l=2} \sum_{m} \ln \left(N\eta(T') \frac{4\pi}{Q} + \frac{N}{2T'} \lambda_{lm} \right) \\ &+ \eta'(T')T' - \lim \frac{1}{4NT'} \sum_{l=2} \sum_{m} \lambda_{lm} \left(\eta(T') \frac{4\pi}{Q} + \frac{1}{2T'} \lambda_{lm} \right)^{-1} \\ &= -\eta(T') + \lim \frac{1}{2N} \sum_{l=2} \sum_{m} \ln \left(N\eta(T') \frac{4\pi}{Q} + \frac{N}{2T'} \lambda_{lm} \right) \\ &- \lim \frac{1}{4NT'} \sum_{l=2} \sum_{m} \lambda_{lm} \left(\eta(T') \frac{4\pi}{Q} + \frac{1}{2T'} \lambda_{lm} \right)^{-1}. \end{aligned}$$

In comparing the free energy per site on both sides of $T_c^{'}$, we use the fact that the saddle point parameter is stuck at the value η^* for all $T_c^{'} \le T' < 0$, that is,

$$\eta(T') = \eta^* \equiv -\frac{Q}{16\pi T_c^{'}},$$

to compute

$$\begin{aligned}
f(T_c^{'} &< T') = \frac{Q}{16\pi} + \lim_{N\to\infty} \frac{T'}{2N} \sum_{l=2}\sum_{m} \ln \frac{N}{2T'}\left(-\frac{1}{2} + \lambda_{lm}\right) \\
f(T_c^{+}) &= \frac{Q}{16\pi} + \lim_{N\to\infty} \frac{T_c'}{2N} \sum_{l=2}\sum_{m} \ln \frac{N}{2T_c'}\left(-\frac{1}{2} + \lambda_{lm}\right) \\
f(T^{'} &< T_c') = -\eta(T')T' + \lim \frac{T'}{2N} \sum_{l=2}\sum_{m} \ln \left(N\eta(T')\frac{4\pi}{Q} + \frac{N}{2T'}\lambda_{lm}\right) \\
f(T_c^{-}) &= \frac{Q}{16\pi} + \lim_{N\to\infty} \frac{T_c'}{2N} \sum_{l=2}\sum_{m} \ln \frac{N}{2T_c'}\left(-\frac{1}{2} + \lambda_{lm}\right)
\end{aligned}$$

where equation (28) yields the saddle point parameter in terms of temperature, $\eta = \eta(T')$. We deduce that the free energy per site is the same on both sides of the critical point. This means there is no latent heat involved in the phase transition at $T_c' < 0$ and it is therefore, a second order transition.

Since

$$\begin{aligned}
\left[\frac{d}{dT}f\right](T_c &< T') = -1 + \lim_{N\to\infty} \frac{1}{2N} \sum_{l=2}\sum_{m} \ln \frac{N}{2T'}\left(-\frac{1}{2} + \lambda_{lm}\right) \\
&= -\frac{4\pi T_c'}{QN} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^{l} \left(\lambda_{lm} - \frac{1}{2}\right)^{-1} + K(T_c' < T') \\
\left[\frac{d}{dT}f\right](T' &< T_c') = -\eta(T') + \lim \frac{1}{2N} \sum_{l=2}\sum_{m} \ln \frac{N}{2T'}\left(\eta(T')T'\frac{8\pi}{Q} + \lambda_{lm}\right) \\
-\lim_{N\to\infty} & \quad \frac{1}{4NT'} \sum_{l=2}\sum_{m} \lambda_{lm} \left(\eta(T')\frac{4\pi}{Q} + \frac{1}{2T'}\lambda_{lm}\right)^{-1} \\
&= -\eta(T') + K(T' < T_c') - L(T' < T_c^{'}),
\end{aligned}$$

and

$$T_c' = \left[\frac{4\pi}{QN} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^{l} \left(\lambda_{lm} - \frac{1}{2}\right)^{-1}\right]^{-1},$$

the above expressions for the derivative $\frac{d}{dT}f$ at both sides of the critical temperature T_c become

$$\begin{aligned}
\left[\frac{d}{dT}f\right](T_c^+) &= -\lim_{N\to\infty}\frac{16\pi^2 T_c'}{Q4\pi N}\sum_{l=2}^{\sqrt{N}}\sum_{m=-l}^{l}\left(\lambda_{lm}-\frac{1}{2}\right)^{-1}+K(T_c') \\
&= -\frac{16\pi^2 T_c'}{Q}\left[\lim_{N\to\infty}\frac{1}{4\pi N}\sum_{l=2}^{\sqrt{N}}\sum_{m=-l}^{l}\left(\lambda_{lm}-\frac{1}{2}\right)^{-1}\right]+K(T_c') \\
\left[\frac{d}{dT}f\right](T_c^-) &= \frac{Q}{16\pi T_c'}+K(T_c')-L(T_c') \\
&= \left[\lim_{N\to\infty}\frac{1}{4\pi N}\sum_{l=2}^{\sqrt{N}}\sum_{m=-l}^{l}\left(\lambda_{lm}-\frac{1}{2}\right)^{-1}\right]+K(T_c')-L(T_c')
\end{aligned}$$

where

$$0 < K(T_c') \equiv \lim_{N\to\infty}\frac{1}{2N}\sum_{l=2}\sum_{m}\ln\frac{N}{2T_c'}\left(-\frac{1}{2}+\lambda_{lm}\right) < \infty$$

$$-\infty < L(T_c') \equiv \lim_{N\to\infty}\frac{1}{2N}\sum_{l=2}\sum_{m}\left(-\frac{1}{2}+\lambda_{lm}\right)^{-1}\lambda_{lm} < 0$$

since

$$-\infty < \lim_{N\to\infty}\frac{1}{N}\sum_{l=2}^{\sqrt{N}}\sum_{m=-l}^{l}\left(\lambda_{lm}-\frac{1}{2}\right)^{-1} = \frac{Q}{4\pi T_c} < 0$$

and for all N,

$$\frac{1}{N}\sum_{l=2}^{\sqrt{N}}\sum_{m=-l}^{l}\left(\lambda_{lm}-\frac{1}{2}\right)^{-1} < \frac{1}{2N}\sum_{l=2}\sum_{m}\left(-\frac{1}{2}+\lambda_{lm}\right)^{-1}\lambda_{lm} < 0.$$

The significant point here is that the specific heat has a discontinuity at T_c' :

$$\begin{aligned}
\Delta &= \left[\frac{d}{dT}f\right](T_c^+)-\left[\frac{d}{dT}f\right](T_c^-) \\
&= L(T_c')-\left(\frac{16\pi^2 T_c'}{Q}+1\right)\left[\lim_{N\to\infty}\frac{1}{4\pi N}\sum_{l=2}^{\sqrt{N}}\sum_{m=-l}^{l}\left(\lambda_{lm}-\frac{1}{2}\right)^{-1}\right] \\
&= \lim_{N\to\infty}\frac{1}{2N}\sum_{l=2}\sum_{m}\left(-\frac{1}{2}+\lambda_{lm}\right)^{-1}\left(\lambda_{lm}-\left(\frac{8\pi T_c'}{Q}+\frac{1}{2\pi}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
\frac{Q}{16\pi T_c'} &= \lim_{N\to\infty} \frac{1}{4\pi N} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^{l} \left(-\frac{1}{2}+\lambda_{lm}\right)^{-1} \simeq \frac{1}{2\pi} \lim_{N\to\infty} \frac{1}{2N} \sum_{l=2}^{\sqrt{N}} \frac{(2l+1)}{-1/2+1/l(l+1)} \\
&= -\frac{1}{2\pi} \lim_{N\to\infty} \frac{1}{N} \sum_{l=2}^{\sqrt{N}} \frac{(2l+1)l(l+1)}{l(l+1)-2} > -\frac{1}{2\pi} \lim_{N\to\infty} \frac{1}{N} \sum_{l=2}^{\sqrt{N}} (2l+1) > -\infty
\end{aligned}
$$

which is independent of enstrophy Q since $L(T_c')$ is and T_c' is proportional to Q, and which we expect to be positive.

From the expression

$$
f(T_c^{'} < T') = \frac{Q}{16\pi} + K(T')T'
$$

we deduce that $f(T_c^{'} < T')$ increases as T' increases away from $T_c^{'}$ and also that $K(T')T'$ consists of the sum

$$
-T's(T') + \left(u(T') - \frac{Q}{16\pi}\right)
$$

where $s(T')$ is the entropy per site and $u(T')$ is the internal energy per site. At $T' = 0$, the internal energy per site $u(T') = \frac{Q}{16\pi}$ consists entirely of energy in the ground modes. At $T_c^{'} < T' < 0$, this represents the fact that $u(T') - \frac{Q}{16\pi} > 0$ is that part of the internal energy in the ergodic modes.

From (28), we deduce that $\eta = \eta(T')$ decreases as T' becomes more negative than $T_c^{'}$. Then, from

$$
f(T^{'} < T_c') = -\eta T' + \lim \frac{T'}{2N} \sum_{l=2} \sum_{m} \ln \left(N\eta \frac{4\pi}{Q} + \frac{N}{2T'} \lambda_{lm} \right)
$$

we deduce that $f(T^{'} < T_c')$ decreases as T' decreases away from $T_c^{'}$. Here, all the internal energy is in the ergodic modes and none in the ground modes, and the entropic term $-T's(T')$ becomes more dominant as $T' < 0$ decreases.

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