The Super and Sub-Rotation Of Barotropic Atmospheres On A Rotating Planet

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A statistical mechanics canonical spherical energy-enstrophy theory of the super-rotation phenomenon in a quasi-2D barotropic fluid coupled by inviscid topographic torque to a rotating solid body, is solved in closed form in Fourier space, with inputs on the value of the energy to enstrophy quotient of the fluid, and two planetary parameters - radius of the planet and its rate and axis of spin. This allows calculations that predict the following physical consequences:(A) two critical points associated with the condensation of high and low energy (resp.) states in the form of distinct super-rotating and sub-rotating (resp.) solid-body flows, (B) only solid-body flows having wavenumbers $l = 1$, $m = 0$ - *tiltless* rotations - are excited in the ordered phases, (C) the asymmetry between the super-rotating and sub-rotating ordered phases where the sub-rotation phase transition requires moreover that the planetary spin is sufficiently large, and thus, less commonly observed than the super-rotating phase, (D) non-excitation of spherical modes with wavenumber $l > 1$ in barotropic fluids. Comparisons to other canonical, microcanonical and dynamical theories suggests that this theory complements and completes older theories by predicting the above specific outcomes.

PACS numbers:

Keywords: Energy-Enstrophy Theory, long range Spherical model, phase transition, rotating atmospheres

I. INTRODUCTION

In this paper, we put forward and analyze the hypothesis that a statistical mechanics theory can be constructed and solved exactly to predict the enigmatic super-rotations of the Venusian atmosphere [1] and Titan's atmosphere, which seek to explain this phenomena from the point of view of phase transitions to self-organized domain-scale coherent structures in forced-damped rotating quasi-2D barotropic flows. Associated with this enigma, is a list of unexplained phenomena for which we seek specific answers from this theory: (1) What are the physical quantities in a rational theory that can predict the outcomes of super-rotation? (2) If such a theory can be constructed, are super- and sub-rotations equally likely - in other words is the theory symmetrical when the planetary spin is nonzero? (3) And if solid-body rotations are predictable, does theory further predict that they are always aligned with the planets' axis of rotation, that is, are they tiltless? (4) If predictable, does non-divergent barotropic theory predict that the self-organized phases are composed purely of solid-body rotations, that is, the lowest spherical harmonics? (5) Why are higher harmonics not allowed in the condensed phases of this theory? Answers to these questions are provided in the concluding section of this paper, summarizing the technical details of the proofs which are in the appendices.

In previous work on a special case of the super-rotation problem which does not adequately address the Venusian enigma [17], a spherical energy-enstrophy theory for a barotropic fluid was used to predict phase transitions to triply-degenerate purely rotating flows on the surface of a non-rotating solid body. It allows exact calculations of the canonical partition functions for the special case of the general super-rotation problem, where the angular momentum L of the barotropic fluid is allowed to change - constrained only by the fixed enstrophy of the fluid- but the frame of reference remains an inertial or non-rotating one. This special case is important however, because it closely model the situation in some cryogenic experiments. Using a thin film of liquid He that undergoes a superfluid transition on small glass beads, G. Williams demonstrated the reality of this special case of the super-rotation problem.

Nonetheless, in view of its non-rotating frame of reference, this special case [17] in which the solid body is clearly not a perfect sphere in order to transfer angular momentum to the fluid, does not fit naturally into the available statistical mechanics framework. In contrast to the special case of a non-rotating planet [11], [17], we surmise that the following sub-cases of the general problem have been analyzed in the literature: (i) angular momentum of fluid $L > 0$ is fixed and the coordinate frame is rotating at rate $\Omega > 0$ - this holds when the geometry of the solid surface is perfectly spherical; (ii) angular momentum of fluid $L > 0$ changes and the coordinate frame or solid sphere is rotating at rate $\Omega > 0$ - this holds when the geometry of the solid surface is **NOT** perfectly spherical and topographical torques transfer angular momentum between the fluid and the heavier solid body; (iii) angular momentum of fluid is zero at all times, $L = 0$, and the coordinate frame or solid sphere is non-rotating, $\Omega = 0$ - this holds when the geometry of the solid surface is perfectly spherical. Because the fluid's angular momentum $L > 0$ changes but the solid body or the coordinate frame is non-rotating, the special case does not fit naturally into the standard augmented energy framework, $E = KE - \Omega L$, where in addition to either the microcanonical or canonical constraint on the kinetic energy of the fluid, it also requires an angular momentum reservoir with fixed $\Omega > 0$ to be in contact with the barotropic fluid. In other familiar applications of this approach, the fixed pressure $P > 0$ of a reservoir is conjugate to the variable volume of a gas $V > 0$, or the fixed external magnetic field $B > 0$ is conjugate to the variable magnetization $M > 0$ of the crystal).

To further distinguish the general super-rotation problem - where L changes and $\Omega > 0$ - from the special case in [11], [17], we remark on how the symmetry and spectral properties of the $\Omega > 0$ case differ from the $\Omega = 0$ case when the fluid's angular momentum L is variable. The energy functional in the general case where $\Omega > 0$ and L changes, has smaller symmetry group due to the distinguished axis of planetary spin, than the special case. These differences have physically significant consequences: in the $\Omega > 0$ case of the spherical energy-enstrophy theory, (I) only the untilted super (sub)-rotating mode that is aligned with the rotation axis can be excited, versus high energy excitation of all three solid-body rotation modes with lowest wavenumber $l = 1$ when $\Omega = 0$; (II) higher modes with wavenumber $l > 1$ cannot be excited, versus the possible excitations of these modes when $\Omega = 0$ case; (III) in addition to the high energy phase transition to untilted super-rotation, there is another low energy phase transition to the sub-rotating untilted solid-body mode, versus a single high energy phase transition when $\Omega = 0$. These contrasting physical predictions of the spherical energy-enstrophy theory for the general super-rotation problem in this paper, versus its precursor in [17], arises purely because the saddle points in its steepest descent solution *do not stick*, versus the *usual sticking* of saddle points when $\Omega = 0$ where the phrase *stick* refers to the lost of analyticity at phase transitions [21].

In line with the majority of works on the applications of statistical physics, we will work only with physical quantities such as kinetic energy of flow and enstrophy and circulation constraints in the partition function formulation. This approach avoids working with the dynamical equations that govern the time evolution of these physical quantities and their associated continuum fields such as relative vorticity. Such a method is designed as a viable theoretical alternative to the difficult and often expensive calculations needed to predict future properties of atmospheric systems from initial data. Moreover, in the field of planetary atmospheres, there is the prevalent problem of a lack of data on early atmospheres despite recent advances in earth paleontology. A more complete study of the dynamics of planetary atmospheres will depend on new advances being made in nonequilibrium statistical physics [34], [35] and [32].

This paper is organized as follows: Section 2 gives a brief resume of the experimental and observational facts on Venus' and Titan's super-rotating middle atmospheres, including the values of the physical quantities such as the objects' radii, density of atmospheres and rates of spin that are relevant to the application of the spherical energy-enstrophy model. It also enumerates several basic assumptions used in the modeling of these planetary atmospheres by a non-divergent barotropic vorticity model on an imperfectly-spherical solid surface in which the total angular momentum of the coupled atmosphere and the massive solid body are conserved, but the atmospheric angular momentum is variable, in contrast to the case studied in [11].

Section 3 gives a brief critical review of the equilibrium statistical models for large-scale coherent structures in quasi-2D rotating flows, including a comparison of the microcanonical (maximum entropy) and constrained canonical approaches, with emphasis on the justification of our employment of an energy reservoir in this class of problems by the Fourier space formulation in which the energy interactions are diagonalized and hence of short or zero range.

Section 4 gives the steepest descent or saddle point solution in the thermodynamic limit of the spherical energy-enstrophy model in the general rotating case, which allows the calculations, in terms of the fixed atmospheric enstrophy and planetary rate of spin (with planetary radius and density of barotropic fluid normalized to one), of two distinct critical temperatures for phase transitions to respectively super-rotating and sub-rotating tiltless nearly solid-body flows and the amplitudes of these self-organized modes as a function of the mean barotropic kinetic energy. Details of proofs of distinctive properties of the saddle points in this general rotating case, including the non-sticking of the saddle points and the existence of the thermodynamic limit of the spherical energy-enstrophy model, are given in 3 appendices (which may be submitted as supplementary materials). The key appendix gives the proof of the Lowest Harmonics Result, which justifies in the rotating planet case, the IR cutoff (at wavenumber $l = 1$) used in the steepest descent solution of the spherical energy-enstrophy theory since, any other choice of this cutoff $l > 1$ results in zero amplitudes of the higher spherical harmonics except the solid-body rotation modes.

Section 5 gives physical consequences of the exact results in this paper in applications to planetary atmospheres and concluding remarks on the specific differences between the results of the spherical energy-enstrophy model and microcanonical (maximum entropy) models. In particular, we explain why the main physical predictions below are direct consequences of the non-sticking of the saddle points in the organized phases for the general rotating case (versus the sticking of the saddle points in the special non-rotating case in [17]: (i) the pure ground modes of solid-body rotations exist without higher-order harmonics (ii) the lack of tilt in the exact solutions, and (iii) super-rotators are more common than sub-rotators in barotropic atmospheres where sub-rotators are allowed only when the planetary spin is large enough compared to the enstrophy of the atmosphere.

II. VENUS AND TITAN - OBSERVATIONS AND MODEL

Venus' super-rotation is first observed by the Venera and Magellan missions and remains the objective of more recent projects such as JAXA's Venus Climate Orbiter. Titan's super-rotating atmosphere was discovered more recently in the Huygens-Cassini mission to the gas giants [2]. Venus' upper troposphere, which is about 20 km thick and reaches to 65 km from the surface [1], rotates like a solid-body or top once in 4 earth days with cloud-top wind speed of 100 meters per second while Venus the planet spins clockwise very slowly once every 243 earth days [1] - hence super-rotation. Observations of Venus' planetary spin rate show a slight change in the length of the Venusian day to compensate for the super-rotation of its heavy carbon dioxide atmosphere. This enigmatic super-rotation is also the subjects in several numerical studies cited in [1].

For the construction of the spherical energy-enstrophy theory, we assume that the mechanical system consisting of the solid planet and its enveloping atmosphere does not experience any significant external torques, its atmosphere is relatively massive but the mass of the solid planet is much larger than that of its atmosphere - all of these assumptions are very close to the reality of the planetary atmospheric systems to which we expect to apply the results given here. While the planet Venus has a mass that is about 80 percent the earth's mass, the Venusian atmosphere has a mass of 4 x $10^{20}kg$ which is about 90 times the mass of the earth's total atmosphere; the surface density of the Venusian atmosphere is $67kg/m^3$, about 7 percent that of liquid water. The significant energy of super-rotation comes from torque-less solar radiation which acts via gravitational instability at small scales, and transfers energy to large scales through the inverse cascade. In this process, the kinetic energy accumulates into the largest planetary scales including the solid-body rotation modes associated with spherical harmonics of azimuthal wavenumber $l = 1$. The dissipation in this damped-driven system occurs both at the viscous very-small scales and the largest scales through the planetary boundary layer (which are not modeled in this paper). However, the most likely mechanism for transferring the substantial angular momentum from the planet itself into the super-rotating atmosphere, is the topographical stress or mountain torques between a non-perfectly spherical planet and its atmosphere. This mechanism does not require viscosity and acts rather through normal or pressure forces at oblique points of the solid planet [27].

The second observation in our solar system, to which may be compared the results in this paper, is the large moon Titan (almost half the diameter of the earth and larger than the planet Mercury) which has a synchronous spin rate of 15 earth days (faster than Venus) and a heavy and thick Nitrogen-based atmosphere (comparable to Earth's atmospheric density) that is known to super-rotate with winds of 34 m/s or 75 mph (fast enough to qualify as hurricanes and typhoons on Earth) [2]. According to our theory, Titan's spin rate is not fast enough relative to its atmospheric enstrophy and kinetic energy to support an atmosphere that sub-rotates.

While the theoretical predictions of the spherical energy-enstrophy theory [21] in this paper - that super-rotation occurs for all planetary spins provided the initial energy-to-enstrophy ratio in the barotropic atmosphere is large enough, and sub-rotation occurs only if the planetary spin is sufficiently large - can be compared to the Venusian and Titan super-rotation, it does not explain why many of the other planetary atmospheres have neither super- nor sub- rotating modes but are instead dominated by alternating zonal jets. Clearly, an extension of the current energy-enstrophy theory to the shallow-water model or the two-layer models is called for in order to account for the contributions of divergent geophysical fluid mechanisms to the condensation of kinetic energy, enstrophy and angular momentum into large-scale coherent structures. This paper therefore represents a first step in the direction where an exact-solvable model is considered desirable. Another direction to expand this work is to take into account the conservation of higher moments of vorticity such as in [35], [16].

III. STATISTICAL MODELS - CANONICAL AND MICROCANONICAL.

In the real world there are no truly equilibrium statistical phenomena. Nonetheless, for many situations where the relaxation occurs at a faster time scale than the dissipation, equilibrium statistical mechanics have provided accurate predictions of forced-damped systems out of thermodynamic equilibrium [4], [29]. Within the equilibrium statistical mechanics framework, following the classical works of Onsager [20], T.D. Lee, Kraichnan [13], and Leith [19], several theories have been proposed to address quasi-2D turbulent relaxation phenomena in macroscopic fluids in a wide variety of geometries. The earliest successful class of theories after Onsager, Lee and Kraichnan are based on the point vortex model [28]. The successful Miller-Robert-Sommeria theory [24], [25] and the Majda-Turkington theory [30] which is based additionally on prior statistics of the forcing or small scales [29] are examples of the microcanonical approach. They are based on maximal entropy rearrangements of non-overlapping vortex parcels where some or all moments of vorticity are conserved. In order to derive useful relations between the macroscopic quantities, the (microcanonical) statistical ensembles in these theories are often approached from the mean-field approximation. Large Deviation Principles have been used to prove that the mean field PDE are exact [25], [30]. Subtly different predictions for the most probable states were found in certain 2D turbulent flows [29], [25], especially in the case of

2D turbulent flows that are forced at small scales, partly because the different approaches assume distinct apriori statistics and also conserve different number of vorticity moments. A recent paper showed that the inclusion of the quartic vorticity moment into the equilibrium statistical mechanics formulation has subtle consequences at the level of the microstate [35].

The canonical approach [20], [13] substitutes the canonical Gibbs ensemble for the microcanonical one, which changes the corresponding partition function into a path integral that is frequently more amenable to theoretical analysis than the microcanonical partition functions. In the applications of thermodynamics to Black-Holes, for instance, the canonical approach based on path-integrals is the standard and accepted approach. Within the canonical approach, there are two main categories- heat baths in (i) purely isothermal cases, where only the temperature is fixed, (ii) generalized cases where besides the temperature, other thermodynamic variables such as pressure, external magnetic field or rotation rate are also fixed.

The most probable and thermodynamically-stable states in the first (i) and second (ii) categories respectively, correspond to extremal values of the free energy, (i) $F = U - TS$ where U is the internal energy and S is the entropy, and respectively, (ii) $F = U - TS - pV$ where p is the pressure that is fixed by the heat bath in addition to T,or $F = U - TS - \Omega L$ where Ω is the spin rate in rotating systems that is fixed as part of the generalized heat bath and L is the angular momentum. The canonical approach (i) is used to derive the canonical partition function of barotropic flows on a non-rotating planet [17] and is justified and applied here to the general rotating case, in which the enstrophy is constrained microcanonically but the angular momentum of the fluid is free.

In contrast to the microcanonical approach [24], [25], [12], [31], [10], [11], [29] the extremal free energy in the canonical approach [13], [17], [23] are attained by balancing the internal energy U and the entropy S at any given heat bath temperature T (positive or negative in which case, we want the maximum free energy). Thus, at critical values T_c which generically have small absolute values, phase transitions occurs in which the extremal free energy is attained, not by the maximum entropy state, but rather by a lower entropy, extremal internal energy state.

Equivalence between the canonical and microcanonical ensembles is known, however, to break down under certain conditions such as long range Coulombic and logarithmic interactions [30], except in special cases with specific constraints and spectral degeneracy [10]. By a Fourier transform to spherical harmonics, we show that the logarithmic energy interactions of the barotropic flow on the surface of a sphere (which is a closed oriented manifold) can be expressed in diagonalized form with zero range. Hence, the equivalence of the microcanonical and canonical ensembles in this case, in agreement with [10]. Thus, the predictions of these two approaches are largely in agreement - at high energy, they predict the condensation of large-scale coherent structures [10], [11].

However, non-equivalence of these ensembles, which arises in many situations such as hot plasmas in the cylindrical and toroidal geometries, leads to anomalous results such as the prediction of negative specific heat in some microcanonical ensembles but never in the canonical ones. Negative specific heats in the gravitational collapse of stellar systems were first predicted in [4].

For a review of the connections between several variational principles that have been introduced in statistical theories of 2D flows depending on the choice of the constraints, and the current discussion on the role of constraints in the context of mean field models -see [12]. Previous applications of equilibrium statistical mechanics to steady states in complex geophysical and astrophysical flows, have been reported in [5], [22], [27], [24], [25], [26], [29], [30], [10], [31], [32] [19]. With specific differences discussed below, our results complement recent works on the statistical mechanics of atmospheric flows on rotating planets and related problems connected to the energy-enstrophy model [9], [11].

IV. SPHERICAL MODEL

As explained in [17], the microcanonical enstrophy constraint in the following energy-enstrophy models is introduced to derive a version of Kac's spherical model that remains exactly solvable while the microcanonical constraint on total circulation follows from the topology of vorticity fields on a sphere. The canonical constraint here is associated with an energy reservoir that exchanges kinetic energy with intermediate and smaller scales in the flow that are bounded below by and widely separated from molecular scales.

In other words, in the Fourier space formulation which will be used throughout this paper to study the spherical energy-enstrophy model, there are two natural cutoffs - the infra-red (IR) cutoff and the ultra-violet (UV) cutoff. At each level of approximation in the spherical models, defined by the total number N of Fourier modes, the smallest scales where dissipation acts, have wavenumbers greater than the UV(N) cutoff. The largest scales (near the domain length scale) in the barotropic flow are excluded by the IR cutoff from the exchange of energy with the reservoir - these IR scales comprise the long-range order from the phase transition and are treated, in our application here of the steepest descent method, separately from the active scales with wavenumbers greater than the IR cutoff. The proof in appendix 1 below justifies the IR modes that is used below to be the modes with the lowest meridional wavenumber $l = 1$. Moreover, it will be clear below that the energy interaction terms in the partition function are diagonalized in Fourier space, hence short or zero range, which justifies the employment of an energy reservoir in the canonical statistical mechanics approach.

A. Physical quantities of the model

In summary of the physical quantities in this theory, the details of which are in [17], the rest frame total kinetic energy of the fluid is

$$\begin{aligned} H_T[q] &= \frac{1}{2}\int_{S^2} dx\,\left[(u_r+u_p)^2+v_r^2\right] \\ &= -\frac{1}{2}\int_{S^2} dx\ \psi q+\frac{1}{2}\int_{S^2} dx\ u_p^2 \end{aligned}$$

where u_r, v_r are the zonal and meridional components of the relative velocity, u_p is the zonal component of the planetary velocity (the meridional component being zero since planetary vorticity is zonal), and ψ is the stream function for the relative velocity. Since the second term $\frac{1}{2}\int_{S^2} dx\ u_p^2$ is fixed for a given spin rate Ω, it is convenient to work with the pseudo-energy as the energy functional for the model,

$$\begin{aligned} H[w] &= -\frac{1}{2}\int_{S^2} dx\ \psi q=-\frac{1}{2}\int dx\ \psi(x)\left[w(x)+2\Omega\cos\theta\right] \\ &= -\frac{1}{2}\int dx\ \psi(x)w(x)-\Omega\int dx\ \psi(x)\cos\theta \end{aligned}$$

Atmospheric angular momentum on a rotating planet is given by

$$\rho\int_{S^2} dx\ w\cos\theta=\rho\left\langle w,\cos\theta\right\rangle, \tag{1}$$

This implies that the only mode in the eigenfunction expansion of w that contributes to its net angular momentum is $\alpha_{10}\psi_{10}$ where $\psi_{10}=a\cos\theta$ is the first nontrivial spherical harmonic. It has the form of solid-body rotation vorticity.

V. STEEPEST DESCENT SOLUTION OF MODEL FOR ROTATING PLANET

To evaluate the Gaussian integrals in Z_N, it is standard practice to expand the relative vorticity vectorfield again, this time, in terms of the spherical harmonics,

$$\vec{\omega}(x)=\sum_{l=1,m=-l}^{\infty,l}\alpha_{lm}\psi_{lm}(x)\vec{n}(x)$$

where $\vec{n}(x)$ is the outward unit normal to S^2 at x. This expansion does not include $\psi_{00}(x)=c$ because of the zero circulation condition on microstates $\vec{s}$.

Solution of the Gaussian integrals requires diagonalizing the interaction in H_N in terms of the spherical harmonics $\{\psi_{lm}\}_{l=1,}^{\infty}$ which are natural Fourier modes for Laplacian eigenvalue problems on S^2 with zero circulation. The transformation between the local vorticity formulation and the Fourier modes one are given by

$$\begin{aligned} \vec{s}_j &= \vec{n}_j\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\alpha_{lm}\psi_{lm}(x_j) \\ -\frac{1}{2}\sum_{j\neq k}^{N} J_{jk}\ \vec{s}_j\cdot\vec{s}_k &= \frac{1}{2}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\lambda_{lm}\alpha_{lm}^2 \\ \vec{h}\cdot\sum_{j=1}^{N}\vec{s}_j &= \frac{1}{2}\Omega C\alpha_{10} \end{aligned}$$

$$\frac{1}{N}\sum_{j\neq k}^{N}K_{jk}(Q,\beta,\eta)\ \vec{s}_j\cdot\vec{s}_k=\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left(\frac{\beta}{2N}\lambda_{lm}+\frac{\eta}{Q}\right)\alpha_{lm}^2$$

where the eigenvalues of the Green's function for the Laplace-Beltrami operator on S^2 are

$$\lambda_{lm} = \frac{1}{l(l+1)},\ l = 1, ..., \sqrt{N},\ m = -l, ..., 0, ..., l$$

and α_{lm} are the corresponding amplitudes. A key observation here is that this energy term has zero range and thus a heat bath can easily be constructed from a ultra-violet cutoff.

The spectral gap between the $l = 1$ modes and all higher harmonics will play a key role in the proof of the **Lowest Harmonics Result** that justifies the lowest modes non-ergodic cutoff in the construction below of the reduced partition function. Thus, the partition functions in the spherical energy-enstrophy model are given in Fourier space by

$$Z_N \propto \int \prod_{m=-1}^{1} d\alpha_{1m} \int D_{l\geq 2}(\alpha) \int_{a-i\infty}^{a+i\infty} \frac{d\eta}{2\pi i} \exp\left\{ N \left[\begin{array}{c} \eta - \sum_{l=2}^{\infty}\sum_{m=-l}^{l}\left(\frac{\beta}{2N}\lambda_{lm} + \frac{\eta}{Q}\right)\alpha_{lm}^2 \\ -\left(\frac{\beta}{4N} + \frac{\eta}{Q}\right)\sum_{m=-1}^{1}\alpha_{1m}^2 - \frac{\beta}{2N}\Omega C \alpha_{10} \end{array} \right] \right\}$$

$$= \int \prod_{m=-1}^{1} d\alpha_{1m} \int_{a-i\infty}^{a+i\infty} \frac{d\eta}{2\pi i} \exp\left\{ N \left[\eta - \left(\frac{\beta}{4N} + \frac{\eta}{Q}\right) \sum_{m=-1}^{1} \alpha_{1m}^2 - \frac{\beta}{2N}\Omega C \alpha_{10} \right] \right\}$$

$$\int D_{l\geq 2}(\alpha) \exp\left(-N \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \left(\frac{\beta}{2N}\lambda_{lm} + \frac{\eta}{Q} \right) \alpha_{lm}^2 \right)$$

where the order of integration of the term $\exp\left(-N\sum_{l=2}^{\infty}\sum_{m=-l}^{l}\frac{\eta}{Q}\alpha_{lm}^2\right)$ is interchanged by choosing $Re(\eta) = a > 0$ large enough.

A. Restricted partition function and IR modes

Non-ergodicity and consequently separation of the condensed modes from the higher harmonics is a standard procedural extension to the steepest descent method for Kac's spherical models. It implies that we do not integrate over the ordered modes in this problem, namely α_{1m}, which are the amplitudes of the 3-fold degenerate ground modes ψ_{1m} that carry global angular momentum. A pertinent and important question arises at this point: what is the non-ergodic cutoff that is correct in any given spherical model? We will prove the **Lowest Harmonics Result** in a section below that only one single class of modes can have nonzero amplitudes in the (preferred or thermodynamically stable) condensed phase of this problem, namely those belonging to the meridional wave number $l = 1$.

Thus, the above problem is rewritten in terms of the restricted partition function $Z_N(\alpha_{10}, \alpha_{1,\pm 1}; \beta, Q, \Omega)$, that is,

$$\begin{aligned} Z_N(\beta, Q, \Omega) &\propto \int \prod_{m=-1}^{1} d\alpha_{1m}\ Z_N(\alpha_{10}, \alpha_{1,\pm 1}; \beta, Q, \Omega) \\ &= \int \prod_{m=-1}^{1} d\alpha_{1m} \int_{a-i\infty}^{a+i\infty} \frac{d\eta}{2\pi i}\ \exp\left\{ N \left[\eta - \left(\frac{\beta}{4N} + \frac{\eta}{Q}\right) \sum_{m=-1}^{1} \alpha_{1m}^2 - \frac{\beta}{2N}\Omega C \alpha_{10} \right] \right\} \end{aligned}$$

$$\int D_{l\geq 2}(\alpha) \exp\left(-N \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \left(\frac{\beta}{2N}\lambda_{lm} + \frac{\eta}{Q} \right) \alpha_{lm}^2 \right).$$

The statistics of the problem are now completely determined by the restricted partition function $Z_N(\alpha_{10}, \alpha_{1,\pm 1}; \beta, Q, \Omega)$. Amplitudes $\alpha_{10}, \alpha_{1,\pm 1}$ of the ordered modes appear as parameters in this restricted partition function, and will have to be evaluated separately.

Standard Gaussian integration is used to evaluate the last integral, which yields, after the non-extensive thermodynamic limit scaling, $\beta' N = \beta$,

$$\int_{l\geq 2} D(\alpha) \exp\left(-\sum_{l=2} \sum_{m=-l}^{l} \left(\frac{\beta' N \lambda_{lm}}{2} + \frac{N\eta}{Q} \right) \alpha_{lm}^2 \right) = \prod_{l=2}^{\sqrt{N}} \prod_{m=-l}^{l} \left(\frac{\pi}{\frac{N\eta}{Q} + \frac{\beta' N}{2}\lambda_{lm}} \right)^{1/2},$$

provided the physically significant Gaussian conditions hold: for $l \geq 2$,

$$\frac{\beta'\lambda_{lm}}{2} + \frac{\eta}{Q} = \frac{\beta'}{2l(l+1)} + \frac{\eta}{Q} > 0. \tag{2}$$

Then the partition function takes the form

$$Z_N(\alpha_{10}, \alpha_{1,\pm1}; \beta, Q, \Omega) \propto \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} d\eta \ \exp\left\{ N \left[\begin{array}{c} \eta - \left(\frac{\beta'}{4} + \frac{\eta}{Q}\right)\sum_{m=-1}^{1}\alpha_{1m}^2 - \frac{\beta'}{2}\Omega C\alpha_{10} \\ -\frac{1}{2N}\sum_{l=2}\sum_m \ln\left(\frac{N\eta}{Q} + \frac{\beta' N}{2}\lambda_{lm}\right) \end{array} \right] \right\}$$

$\pi p$where the free energy per site evaluated at the most probable macrostate is $-\frac{1}{\beta'}\ F(\eta(\beta'), Q, \beta')$ with

$$F(\eta(\beta'), Q, \beta') = \eta(\beta')\left[1 - \frac{1}{Q}\sum_{m=-1}^{1}\alpha_{1m}^2\right] - \frac{\beta'}{4}\sum_{m=-1}^{1}\alpha_{1m}^2 - \frac{\beta'}{2}\Omega C\alpha_{10}$$

$$-\frac{1}{2N}\sum_{l=2}\sum_m \ln\left(\frac{N\eta}{Q} + \frac{\beta' N}{2}\lambda_{lm}\right).$$

B. Saddle points and the Thermodynamic limit

Provided that the saddle point $\eta(\beta')$ can be determined at given inverse temperature β', the thermodynamically stable (most probable) macrostate is given by the maximum of the expression $F(\eta(\beta'), Q, \beta')$. At positive temperatures, the structure of this expression where it concerns the ground modes α_{1m}, namely,

$$\chi(\alpha_{10}, \alpha_{1,\pm1}; \beta, Q, \Omega) = \eta(\beta')\left[1 - \frac{1}{Q}\sum_{m=-1}^{1}\alpha_{1m}^2\right] - \left[\frac{\beta'}{4}\sum_{m=-1}^{1}\alpha_{1m}^2 + \frac{\beta'}{2}\Omega C\alpha_{10}\right],$$

and the fact that the saddle point $\eta(\beta')$ must be positive, suggests that for any positive value of the saddle point, the expression χ and therefore $F(\eta(\beta'), Q, \beta')$ is maximized by $\sum_{m=-1}^{1}\alpha_{1m}^2 = 0$ for all $\beta' > 0$ when planetary spin Ω is small, and by $\alpha_{10} < 0$ for large $\beta' > 0$ when planetary spin Ω is large. At negative temperatures, we expect to find a finite critical point where the two opposing parts of χ are balanced. In order to prove that these heuristic expectations are valid, we will solve the restricted partition function in closed form by the method of steepest descent.

The saddle point condition gives one equation for the determination of four variables η, α_{1m} in terms of inverse temperature β' and relative enstrophy Q,

$$0 = \frac{\partial F}{\partial \eta} = \left(1 - \frac{1}{Q}\sum_{m=-1}^{1}\alpha_{1m}^2\right) - \frac{1}{2NQ}\sum_{l=2}\sum_m\left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2}\lambda_{lm}\right)^{-1} \tag{3}$$

where $\eta = \eta(\beta')$ is taken to be the value of the saddle point. Note that it does not depend on the planetary spin rate $\Omega > 0$. Note that the same equation holds in the $\Omega = 0$ case [17]. There are two natural subcases for the saddle point condition, namely, (A) the disordered phase (for $|T'| \gg 1$) where equation (3) has finite solution $\eta(\beta') > 0$, and $\alpha_{1m} = 0$ for $m = -1, 0, 1$; and (B) the ordered or condensed phase (for $|T'| \ll 1$) where equation (3) has finite solution $\eta(\beta') > 0$ only when $\alpha_{1m} \neq 0$ for some m. In case (A) which will be solved below, there is no need to invoke additional equations of state as the amplitudes $\alpha_{1m} = 0$ for $m = -1, 0, 1$.

Case (B) requires three more conditions to determine the three amplitudes α_{1m} and the saddle point $\eta(\beta') > 0$. They are provided by equations of state for the condensed phase (which do not hold in the disordered phase):

$$0 = \frac{\partial F}{\partial \alpha_{10}} = -\left(\frac{2\eta(\beta')}{Q} + \frac{\beta'}{2}\right)\alpha_{10} - \frac{\beta'}{2}\Omega C \tag{4}$$

$$0 = \frac{\partial F}{\partial \alpha_{1,\pm1}} = -\left(\frac{2\eta(\beta')}{Q} + \frac{\beta'}{2}\right)\alpha_{1,\pm1}. \tag{5}$$

Thus, a coupled system of four algebraic equations (3), (4), (5) determines four unknowns in terms of the planetary spin $\Omega > 0$, the relative enstrophy $Q > 0$ and the scaled inverse temperature β'. The last two equations of state for $\alpha_{1,\pm1}$ implies that either

$$\alpha_{1,\pm1} = 0 \text{ or } \left(\frac{2\eta(\beta')}{Q} + \frac{\beta'}{2}\right) = 0.$$

The first equation of state differs from the other two; this represents reduction of the $SO(3)$ symmetry that existed in the $\Omega = 0$ case to S^1 symmetry in the case of nonzero planetary spin. Together these three equations of state imply that when $\Omega > 0$, the only possible solution is without tilt,

$$\begin{aligned} \alpha_{10} &= -\frac{\beta'\Omega C}{2}\left(\frac{2\eta(\beta')}{Q}+\frac{\beta'}{2}\right)^{-1} \neq 0, \\ \alpha_{1,\pm 1} &= 0. \end{aligned} \tag{6}$$

These values of α_{lm} will be substituted back into the saddle point condition (3) to yield a single equation that will be solved below.

The Gaussian conditions (2) imply that for $l > 1$,

$$\frac{\beta'}{2l(l+1)} + \frac{\eta(\beta')}{Q} > 0.$$

The critical temperature can be obtained from the saddle point condition: (A) in the disordered phase at large $|T|$,

$$1 = \lim_{N\to\infty} \frac{1}{2NQ} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^{l} \left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2l(l+1)}\right)^{-1} \tag{7}$$

where the large N non-extensive limit on the RHS is well-defined and finite for any finite $|\beta'|$ provided

$$\eta(\beta') \geq \eta^* = \frac{|\beta'|Q}{4} > 0, \tag{8}$$

because then, each term ($l \geq 2$) in the sum is majorized: for negative temperatures,

$$\left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2l(l+1)}\right)^{-1} \leq \left(-\frac{\beta'}{4} + \frac{\beta'}{2l(l+1)}\right)^{-1} \leq \left(-\frac{\beta'}{6}\right)^{-1},$$

and for positive temperatures,

$$\left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2l(l+1)}\right)^{-1} \leq \left(\frac{\beta'}{4} - \frac{\beta'}{2l(l+1)}\right)^{-1} \leq \left(\frac{\beta'}{6}\right)^{-1};$$

and the corresponding expressions have well-defined positive limits, i.e., for all negative and finite β',

$$\lim_{N\to\infty} \frac{1}{2NQ} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^{l} \left(-\frac{\beta'}{4} + \frac{\beta'}{2l(l+1)}\right)^{-1} < \infty, \tag{9}$$

and for all positive and finite β',

$$\lim_{N\to\infty} \frac{1}{2NQ} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^{l} \left(\frac{\beta'}{4} - \frac{\beta'}{2l(l+1)}\right)^{-1} < \infty.$$

And (B) in the ordered phase at small $|T|$,

$$\left(1 - \frac{1}{Q}\alpha_{10}^2\right) = \lim_{N\to\infty} \frac{1}{2NQ} \sum_{l=2} \sum_{m} \left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2l(l+1)}\right)^{-1} \tag{10}$$

where a similar argument proves that the RHS is well-defined and finite provided $\eta(\beta') \geq \eta^*$.

This proves that the thermodynamic or continuum limit of the spherical models H_N is well-defined for all negative temperatures because it turns out that the saddle point satisfies (8) for the disordered as well as the ordered phases.

The large $|T|$ or small $|\beta'|$ saddle point condition in case (A),

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^{l} \left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2l(l+1)}\right)^{-1} = Q, \tag{11}$$

can be solved and has the property that $\eta(\beta') \searrow 1$ as $|\beta'| \to 0$. In case (B), when $|\beta'|$ is large, we discuss (i) $\beta' < 0$ and (ii) $\beta' > 0$ separately.

C. Negative critical temperature

For case (i) $\beta' < 0$, a point is reached at β'_c where

$$-\infty < \beta'_c(Q) = \lim_{N\to\infty} \frac{1}{QN} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^{l} \left(\lambda_{lm} - \frac{1}{2}\right)^{-1} < 0,$$

such that for $\beta' < \beta'_c(Q) < 0$,

$$\lim_{N\to\infty} \frac{1}{\beta' NQ} \sum_{l=2} \sum_{m} \left(-\frac{1}{2} + \lambda_{lm}\right)^{-1} < 1.$$

(We note the significant fact that $T'_c(Q)$ depends linearly on the relative enstrophy Q but does not depend on Ω.) In other words, the extreme saddle point

$$\eta^* = -\frac{\beta' Q}{4}$$

is no longer adequate to solve (7) for $\beta' < \beta'_c(Q) < 0$. We check that $\eta < \eta^*$ cannot be used. This is due to the fact that

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^{l} \left(\frac{\eta^*}{Q} + \frac{\beta'_c}{2l(l+1)}\right)^{-1} = Q$$

which implies

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^{l} \left(\frac{\eta}{Q} + \frac{\beta'}{2l(l+1)}\right)^{-1} > Q$$

if $\eta < \eta^*$ and $\beta' < \beta'_c < 0$.

We summarize the above results in

Result 1: (A) *For all* $\Omega > 0$ *and* $Q > 0$, *the quantity*

$$\beta'_c(Q, N) = \frac{1}{QN} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^{l} \left(\lambda_{lm} - \frac{1}{2}\right)^{-1} < 0$$

has a well-defined and finite limit, called the critical inverse temperature,

$$\beta'_c(Q) = \lim_{N\to\infty} \beta'_c(Q, N) > -\infty,$$

that is independent of the rate of spin Ω.

(B) Moreover, the thermodynamic limit exists for the spherical models H_N *in the sense that for any* $Q > 0$ *and* $\Omega > 0$, *the saddle point conditions,*

$$1 = \lim_{N\to\infty} \frac{1}{2NQ} \sum_{l=2} \sum_{m} \left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2l(l+1)}\right)^{-1}$$

$$\left(1 - \frac{1}{Q}\alpha_{10}^2\right) = \lim_{N\to\infty} \frac{1}{2NQ} \sum_{l=2} \sum_{m} \left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2l(l+1)}\right)^{-1},$$

are well-defined and finite, and the saddle point satisfies the condition

$$\eta(\beta') \geq \eta^* = -\frac{\beta' Q}{4} > 0$$

for all $\beta' < 0$.

(C) For all $\Omega > 0$ *and* $Q > 0$, *and for all* $\beta' < \beta'_c(Q) < 0$, *the ordered phase takes the form of the tiltless* ($\alpha_{1,\pm1} = 0$) *ground mode* $\alpha_{10}(\beta', \Omega, Q)\psi_{10}$ *with amplitude*

$$\alpha_{10} = -\frac{\beta' \Omega C}{2} \left(\frac{2\eta(\beta')}{Q} + \frac{\beta'}{2}\right)^{-1} > 0,$$

which implies that it is aligned with the rotation $\Omega > 0$ *(super-rotating) and is linear in* Ω.

D. Positive temperature

For case (ii) $\beta' > 0$, we note that the Gaussian conditions (2) are automatically satisfied since $\eta(\beta', Q)/Q > 0$ is required of the saddle point of the equation

$$\left(1 - \frac{1}{Q}\alpha_{10}^2\right) = \lim_{N\to\infty} \frac{1}{2NQ} \sum_{l=2} \sum_{m} \left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2l(l+1)}\right)^{-1}, \tag{12}$$

We deduce from the saddle point condition (12), that for any $Q > 0$, and any positive $\beta'(Q) < \infty$, there is a saddle point $\eta(\beta', Q) > 0$ associated with the disordered phase $\alpha_{10} = 0$. Otherwise, there is a finite critical point $\beta'_{cc}(Q) > 0$ that satisfies the equation,

$$\beta'_{cc} = \lim_{N\to\infty} \frac{1}{2NQ} \sum_{l=2}^{\sqrt{N}} \sum_{m=-l}^{l} 2l(l+1) < \infty,$$

which is a contradiction since the sum on the RHS does not converge. The extreme case, namely $\eta(\beta') = 0$, for the saddle point condition holds at precisely one point, that is, for $\beta' = \infty$.

We will show next that (12) has more than one saddle points at positive temperatures. Selection of the preferred phase is then carried out by comparing their free energy in which the necessary and sufficient condition for thermodynamic stability is the extremality of the free energy values. In addition to the disordered phase solution found above, the pure ground mode phase is a saddle point $\eta', \alpha_{10} < 0$ of (12). Using the solution (6) of the equation of state for amplitude α_{10} in (12) gives us the final form of the saddle point condition,

$$\left(1 - \frac{(\beta')^2\Omega^2C^2}{16Q}\left(\frac{\eta'(\beta', \Omega, Q)}{Q} + \frac{\beta'}{4}\right)^{-2}\right) = \lim_{N\to\infty} \frac{1}{2NQ} \sum_{l=2} \sum_{m} \left(\frac{\eta'(\beta', \Omega, Q)}{Q} + \frac{\beta'}{2l(l+1)}\right)^{-1} \tag{13}$$

The alternate saddle point - if it exists - satisfies

$$\eta'(\beta', \Omega, Q) > \eta(\beta', Q) \tag{14}$$

because the LHS (13) must satisfy the pair of inequalities,

$$0 < \left(1 - \frac{(\beta')^2\Omega^2C^2}{16Q}\left(\frac{\eta'(\beta', \Omega, Q)}{Q} + \frac{\beta'}{4}\right)^{-2}\right) < 1. \tag{15}$$

A useful condition that is equivalent to the lower bound above is

$$\left(\frac{\beta'}{4}\right)^2 \frac{\Omega^2C^2}{Q}\left(\frac{\eta'(\beta', \Omega, Q)}{Q} + \frac{\beta'}{4}\right)^{-1} < \left(\frac{\eta'(\beta', \Omega, Q)}{Q} + \frac{\beta'}{4}\right). \tag{16}$$

From this, it follows that when the planetary spin $\Omega > \Omega_c \equiv \sqrt{Q/C^2}$, the LHS(13) can be made to satisfy the lower bound in (15) by choosing $\eta' > \eta'_c(\beta', \Omega, Q) > 0$ where LHS(13) equals zero at $\eta'_c(\beta', \Omega, Q) < \infty$. LHS(13) equals one at $\eta' = \infty$. If $\Omega = \Omega_c$, then clearly, $\eta'_c(\beta', \Omega, Q) = 0$. But for $\Omega < \Omega_c$, any $\eta' > 0$ will satisfy the bounds in (15).

$\Omega > \Omega_c$: RHS(13) equals one at the disordered phase saddle point $\eta(\beta', Q)$ which is independent of Ω but equals $R(\eta'_c(\beta', \Omega, Q))$ at $\eta'_c(\beta', \Omega, Q) > 0$. The RHS(13) decreases to zero from the value $R(\eta'_c(\beta', \Omega, Q))$ while LHS(13) increases to one from zero as $\eta' > \eta'_c(\beta', \Omega, Q)$ increases towards ∞. There is an alternate (pure ground mode) saddle point solution of (13) since LHS(13) equals RHS(13) for some $\eta' \in (\eta'_c(\beta', \Omega, Q), \infty)$ by continuity.

On the other hand, if the planetary spin Ω is smaller than the critical value $\Omega_c = \sqrt{Q/C^2}$, then at

$$\eta'(\beta', \Omega, Q) = \eta(\beta', Q),$$

the RHS(13) equals one and the LHS(13) equals $L_c(\eta(\beta', Q), \Omega) \in (0, 1)$. RHS(13) decreases from one to zero as η' increases from $\eta(\beta', Q)$ to ∞. LHS(13) increases from $L_c(\eta(\beta', Q), \Omega)$ towards one as η' increases from $\eta(\beta', Q)$ to ∞. There is again an alternate (pure ground mode) saddle point solution of (13) since LHS(13) equals RHS(13) for some $\eta' \in (\eta(\beta', Q), Q), \infty)$ by continuity.

This also shows that the thermodynamic limit exists for the spherical Heisenberg models H_H^N for all positive temperatures, in the sense that for any $Q > 0$, $\Omega > 0$ and all $\beta' > 0$, the saddle point condition (12) is well-defined and finite along the saddle points $\eta(\beta', Q)$ and $\eta'(\beta', \Omega, Q)$. It remains to show that the disordered phase is preferred at high positive temperatures and the pure ground mode phase with counter-rotation $\alpha_{10} < 0$ is preferred at low positive temperatures. The details are in Appendix III.

Unlike the critical phenomenology of the barotropic fluid - sphere system at very high energies (negative temperatures) which we have shown arises from the reflection of the saddle point at the extreme value η^* (the disordered phase does not satisfy the saddle point condition at negative T' when $|T'| \ll 1$), its critical phenomenology at positive temperature is not so much based on the breakdown of the saddle points as on the system's preference for a smaller free energy. Transitions between these positive temperature phases for $\Omega > 0$ are characterized by a greater degree of smoothness than its negative temperature counterpart since the free energy is automatically continuous at $\beta'_{cc}(\Omega, Q) > 0$.

We summarize the results in this section in

Result 2:

(A) The thermodynamic limit exists for the spherical models H_N in the sense that for any $Q > 0$ and $\Omega > 0$, the saddle point conditions,

$$1 = \lim_{N\to\infty} \frac{1}{2NQ} \sum_{l=2} \sum_{m} \left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2l(l+1)} \right)^{-1}$$

$$\left(1 - \frac{1}{Q}\alpha_{10}^2\right) = \lim_{N\to\infty} \frac{1}{2NQ} \sum_{l=2} \sum_{m} \left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2l(l+1)} \right)^{-1},$$

are well-defined and finite, and the saddle point satisfies the condition

$$\eta(\beta') \geq \eta^* = \frac{\beta' Q}{4} > 0$$

for all $\beta' > 0$.

(B) *A necessary and sufficient condition for the ordered phase(consisting of nonzero ground mode) to be thermodynamically stable and preferred over the disordered phase at some finite positive temperature T, is that the planetary spin satisfies $\frac{\Omega^2 C^2}{Q} > 1$ where C is a geometric constant related to the surface area of the unit sphere.*

(C) For all $\Omega > \Omega_c(Q)$ and for all $\beta' > \beta'_{cc}(Q, \Omega) > 0$, the ordered phase is preferred over the disordered phase, and takes the form of the tiltless $(\alpha_{1,\pm 1} = 0)$ ground mode $\alpha_{10}(\beta', \Omega, Q)\psi_{10}$ with amplitude

$$\alpha_{10} = \frac{-\beta' \Omega C}{2} \left(\frac{2\eta(\beta')}{Q} + \frac{\beta'}{2} \right)^{-1} < 0,$$

which implies that it is aligned with the rotation $\Omega > 0$ (sub-rotating) and is linear in Ω.

These sub-rotating solutions may be compared with the results on barotropic equilibria above topography where they could be analogous to the so called "Fofonoff flows" in oceanography, which are low energy equilibrium states for which the streamlines follows topography contours [33].

VI. CONCLUSIONS

The main physical consequences of the results of the spherical energy-enstrophy theory in this paper are: (i) for all values of the planetary spin and the enstrophy of macroscopic flows, barotropic tiltless super-rotation on a planetary scale is the preferred end-state of this theory at high levels of the energy of macroscopic flow, corresponding to negative statistical temperatures that have small enough absolute values below a critical negative temperature that depends linearly on the enstrophy, and (ii) barotropic tiltless sub-rotating flows on a planetary scale is allowed (has lower free energies) ONLY when the planetary spin is higher than a critical spin that depends on the enstrophy, and then, only when the the macroscopic flow has sufficiently low energy levels, corresponding to positive statistical temperatures below a positive critical temperature that depends on both spin and enstrophy. The tiltless nature of the solid-body rotations follows from the solutions (7) for the amplitudes of the ground modes in the equations of state (4,5,6). In other words, both the condensation to super- and sub- rotations and the prevention of tilt thereof are direct consequences of the non-sticking / reflection of the saddle point in the ground mode phase transition. Moreover, there are no self-organized phase in this theory in which some of the energy is in higher harmonics other than the solid-body rotations or ground modes. This follows from the **Lowest Harmonics Result** and its applications.

It is useful to reword this consequences in the following qualitative way: (a) for all planetary spins, when the energy is high relative to enstrophy, only super-rotation can be observed in barotropic planetary flows, (b) for slow spins, when the energy level is low or intermediate compared to enstrophy, a chaotic or turbulent flow without large-scale coherent structure will be observed, and (c) the sub-rotating planetary flow is an end-state of planetary barotropic flows only for rapidly spinning celestial bodies AND very low energy levels relative to enstrophy.

Many careful Monte-Carlo simulations, including those of this author, over a very wide range planetary and atmospheric parameters for the strictly non-divergent barotropic model, confirm the results for the barotropic model in this paper, that the lowest super-and sub-rotating modes are the only thermo-stable self-organized coherent structures. Isolated higher modes or a small finite range of Higher harmonics were never found numerically in Monte Carlo simulations of the non-divergent barotropic model because they were not thermodynamically stable. Only the mixed state which comprises all the higher ergodic modes are preferred at high absolute values of the statistical temperatures. It remains for future works to show whether a quasi-geostrophic horizontally divergent flow model such as the shallow water equations, can support a richer condensed phase where higher order harmonics co-exist with the solid-body rotations ground modes.

Following directly from these consequences we can further state that sub-rotating barotropic planetary atmospheres should be less common than super-rotating ones amongst extra-solar planetary systems, in agreement with the two data points within our solar system, namely, (I) the extremely slowly spinning Venus and its heavily insolated massive atmosphere, is super-rotating with 200 m/s winds, and (II) the faster spinning large moon, Titan, and its less energetic atmosphere whose energy input is largely from the tidal effects of its Giant planet, is reportedly anti-rotating.

Lastly, we close with the observation that the tiltless super- and sub- rotations in the case here of a spinning solid planet differ from the complete symmetry-breaking phase transition to Goldstone modes reported in the first paper [17], that is, solid-body rotations around arbitrary axis or tilts.

Acknowledgement

This work is supported by ARO grant W911NF-05-1-0001, by the Army Research Office Grant No. W911NF-09-1-0254 and DOE grant DE-FG02-04ER25616. The Monte-carlo simulations by J. Nebus and X. Ding have played an important role in motivating the exact solutions of the Spherical models. Bruce West gave early useful suggestions from his unique historical point of view, having crossed the paths of Kac and Kraichnan. Anonymous referees have made several useful suggestions, in particular requesting the inclusion of the proof of the Lowest Harmonics Lemma, to improve the quality of this paper.

VII. APPENDIX I: LOWEST HARMONICS RESULT

The proof of this result is similar to the free energy comparisons performed in two previous subsections for negative and positive temperatures separately, except that the non-ergodic cutoff is now taken to be a meridional wave-number $l^* > 1$. We will show that making these higher order cutoffs in the steepest descent evaluation of the spherical models' partition function does not lead to any new phase transitions because the corresponding organized macrostates or phases are not thermodynamically stable according to their free energy.

To proceed, for each higher-order non-ergodic cutoff $l^* \geq 2$, there belongs new saddle points η^* which correspond to self-organized macrostates that will be treated again separately for first negative and then positive temperatures. For each of these higher-order saddle points, there belongs a system of equations of state similar to equations (4,5,6):

$$0 = \frac{\partial F}{\partial \eta^*} = \left(1 - \frac{1}{Q}\sum_{l=1}^{l^*}\sum_{m=-l}^{l}\alpha_{lm}^2\right) - \frac{1}{2NQ}\sum_{l>l^*}\sum_{m}\left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2}\lambda_{lm}\right)^{-1} \tag{17}$$

$$0 = \frac{\partial F}{\partial \alpha_{10}} = -\left(\frac{2\eta^*(\beta')}{Q} + \frac{\beta'}{2}\right)\alpha_{10} - \frac{\beta'}{2}\Omega C \tag{18}$$

$$0 = \frac{\partial F}{\partial \alpha_{1,\pm1}} = -\left(\frac{2\eta^*(\beta')}{Q} + \frac{\beta'}{2}\right)\alpha_{1,\pm1} \tag{19}$$

$$0 = \frac{\partial F}{\partial \alpha_{lm}} = \frac{\partial F}{\partial \alpha_{l^*m}} = -\left(\frac{2\eta^*(\beta')}{Q} + \frac{\beta'}{1}\lambda_{l^*m}\right)\alpha_{l^*m} \tag{20}$$

for each $l = 1, ..., l^*$. The coefficient in front of the amplitudes α_{lm} are labeled saddle-point multiplication factors and used below. The following analysis shows through free energy comparisons that for negative temperatures, because of the spectral gap between the eigenvalues of the lowest modes $l = 1$ and higher-order $l > 1$ modes, the free energy at these higher order saddle points are not favorable - that is, at these higher order saddle point temperatures

$$T'' < T' < 0 \tag{21}$$

(where $T' < 0$ is the ground mode or negative critical temperature originally calculated), they have much smaller internal energy $U'' < U'$ due (a) to the lower energy of the higher modes (which depends on the reciprocal of their eigenvalues) AND (b) to the above proven fact that at negative temperatures, these higher order saddle points are associated also with a sub-rotating ground mode ($\alpha_{10} < 0, l = 1$) which is an energy minimum [15]. Together with the entropy term comparisons, namely,

$$0 < -T''S'' < -T''S' \tag{22}$$

because the entropy S' is clearly greater than S'' since it contains the higher $l > 1$ modes that are absent from S'', the free energy estimates are

$$F'' = U'' - T''S'' < U' - T''S' = F'(T'') \tag{23}$$

where the last term on the right denotes the free energy of the ground mode $l = 1$ saddle point at the more negative temperature $T'' < 0$. Furthermore at $T'' < 0$,

$$F'(T'') < F(T'') = U(T'') - T''S \tag{24}$$

where the right hand side denotes the larger free energy of the original (preferred) mixed state at $T'' < T' < 0$ in which all of the fixed enstrophy are shared amongst the higher modes in the entropy term, and the ground modes $l = 1$ have zero amplitudes. Since the most probable and thermo-stable macrostate is given by that with the maximum free energy at any given negative temperatures (and minimum free energy at positive temps), there cannot be a second critical temperature $T'' < T' < 0$ more negative than the original $l = 1$ one.

To eliminate the possibility of any transition at positive temperatures $T'' > T' > 0$ to higher order harmonics, we can again use free energy comparisons but a shorter argument eliminating phase transitions at positive temperatures to organized phases involving higher harmonics is based on equations of state (22-25) for the non-ergodic modes. Since the saddle-point multiplication factors in the corresponding equations of state eqns (22-25), are strictly positive for each of the higher harmonics $1 < l \leq l^*$ at positive temperatures, their associated amplitudes must vanish, $\alpha_{lm} = 0$, except for the sub-rotating ground mode $\alpha_{10} < 0$ which has a counter-term in the form of the positive planetary spin eqn (23). This sub-rotating ground mode has been shown in the calculations in the previous section to have higher free energy than the mixed phase at all positive temperatures $T > T' > 0$ where $T' > 0$ is the lowest order positive critical temperature previously calculated in the case of large enough planetary spin. This completes the proof of the non-existence of higher order critical temperatures.

VIII. APPENDIX II: NON-STICKING SADDLE POINTS

From this discussion of (i) $\beta' \leq \beta'_c < 0$, and after substituting the nonzero solution (6) of the equations of state back into the saddle point equation,

$$\left(1 - \frac{1}{Q}\alpha_{10}^2\right) = \lim_{N\to\infty} \frac{1}{2NQ} \sum_{l=2} \sum_{m} \left(\frac{\eta(\beta')}{Q} + \frac{\beta'}{2l(l+1)}\right)^{-1}, \tag{25}$$

we derive a single equation,

$$\left(1 - \frac{(\beta')^2\Omega^2C^2}{16Q}\left(\frac{\eta(\beta',\Omega,Q)}{Q} + \frac{\beta'}{4}\right)^{-2}\right) = \lim_{N\to\infty} \frac{1}{2NQ} \sum_{l=2} \sum_{m} \left(\frac{\eta(\beta',\Omega,Q)}{Q} + \frac{\beta'}{2l(l+1)}\right)^{-1} \tag{26}$$

for the saddle point $\eta(\beta',\Omega,Q) \geq \eta^*$ when $\beta' \leq \beta'_c < 0$.

The RHS of this equation can be made larger (resp. smaller) than 1 by choosing $\eta(\beta') < \eta^*$ ($> \eta^*$ resp.) and since $Q = \sum_{l=1} \sum_m \alpha_{lm}^2$ is the relative enstrophy, we must have

$$0 \leq \left(1 - \frac{1}{Q}\alpha_{10}^2\right) \leq 1$$

which means that its LHS lie between 0 and 1. Thus, by choosing a suitable $\eta(\beta') \geq \eta^*$ we should be able to satisfy (26) for $\beta' \leq \beta'_c < 0$. It remains to check that this is consistent with the property $\alpha_{10}^2 \leq Q$ of the ordered solution (6), that is, for all Ω,

$$\frac{(\beta')^2\Omega^2C^2}{16Q}\left(\frac{\eta(\beta',\Omega,Q)}{Q}+\frac{\beta'}{4}\right)^{-2} \leq 1. \tag{27}$$

Thus, to prove that the saddle point condition (26) has solutions $\eta(\beta',\Omega,Q) \geq \eta^*$ for all $\Omega > 0$, $Q > 0$, and for all $\beta' \leq \beta'_c(Q) < 0$, it is sufficient to note that for fixed $\beta' \leq \beta'_c(Q)$, its RHS($\eta(\beta')$) < 1, decreases as $\eta > \eta^*$ increases, while its LHS($\eta(\beta')$) < 1, increases as $\eta > \eta^*$ increases; and in such a way that RHS($\eta(\beta')$) is surjective on the interval $(0,1)$ with $\lim_{\eta(\beta')\nearrow\infty} RHS(\eta(\beta')) = 0$ for any fixed $\beta' < \beta_c^{'}$, and $RHS(\eta^*(\beta'_c)) = 1$, and LHS($\eta(\beta')$) is surjective on $(0,1)$ with $\lim_{\eta(\beta')\nearrow\infty} LHS(\eta(\beta')) = 1$ for any fixed $\beta' < \beta_c^{'}$,and $LHS(\bar{\eta}(\beta')) = 0$ for the solution

$$\bar{\eta}(\beta') = -\frac{\beta'\Omega C\sqrt{Q}}{4} - \frac{\beta' Q}{4} > \eta^*$$

of

$$\frac{(\beta')^2\Omega^2C^2}{16Q}\left(\frac{\bar{\eta}(\beta',\Omega,Q)}{Q}+\frac{\beta'}{4}\right)^{-2} = 1.$$

Condition (27) then implies that for all $\Omega > 0$, $Q > 0$, and for all $\beta' \leq \beta'_c(Q) < 0$,the saddle point $\eta(\beta',\Omega,Q)$ satisfies

$$\eta(\beta',\Omega,Q) \geq -\frac{\beta'\Omega C\sqrt{Q}}{4} - \frac{\beta' Q}{4} > \eta^*$$

which proves the reflection property in the following remark. We also note that when the planetary spin $\Omega = 0$, this saddle point has the standard property of sticking to η^* for all $\beta' < \beta'_c(Q) < 0$.

Remark 1: *Since the extreme saddle point*

$$\eta^* = -\frac{\beta' Q}{4}$$

satisfies the saddle point conditions (7) and (10) only at the single value of the temperature $T_c^{'} < 0$ that separates the disordered phase not at other $T < 0$, we have shown that the usual phenomenon known as, sticking of the saddle point in the ordered phase, does not hold here. A more appropriate label for this new saddle point behaviour seen in the spherical models for barotropic flows on a rotating sphere, is jumping and reflection of the saddle point at the negative critical point. Indeed the proof above shows that, for all $\Omega > 0$ and $Q > 0$, and for all $\beta' < \beta'_c(Q) < 0$, the saddle point $\eta(\beta') \geq -\frac{\beta'\Omega C\sqrt{Q}}{4} - \frac{\beta' Q}{4} > \eta^*$.

IX. APPENDIX III: SUB-ROTATION IS PREFERRED FOR SMALL ENOUGH POSITIVE T

To compare the free energy per site of these two phases we ignore for the moment the infinite sum of logarithmic terms in F and focus on the part which depends on α_{10} :

$$\begin{aligned}
\chi(\alpha_{10};\beta,Q,\Omega) &= \eta'(\beta')\left(1-\frac{\alpha_{10}^2}{Q}\right) - \frac{\beta'}{4}\left(\alpha_{10}^2 + 2\Omega C\alpha_{10}\right) \\
&= \eta'(\beta')\left(1-\frac{(\beta')^2\Omega^2C^2}{16Q}\left(\frac{\eta'(\beta',\Omega,Q)}{Q}+\frac{\beta'}{4}\right)^{-2}\right) \\
&\quad -\frac{\beta'}{4}\left[\frac{(\beta')^2\Omega^2C^2}{16}\left(\frac{\eta'(\beta',\Omega,Q)}{Q}+\frac{\beta'}{4}\right)^{-2} - \frac{\beta'\Omega^2C^2}{2}\left(\frac{\eta'(\beta',\Omega,Q)}{Q}+\frac{\beta'}{4}\right)^{-1}\right] \\
&= -\frac{(\beta')^2\Omega^2C^2}{16}\left(\frac{\eta'(\beta',\Omega,Q)}{Q}+\frac{\beta'}{4}\right)^{-2}\left(\frac{\eta'}{Q}+\frac{\beta'}{4}\right) + \frac{(\beta')^2\Omega^2C^2}{8}\left(\frac{\eta'(\beta',\Omega,Q)}{Q}+\frac{\beta'}{4}\right)^{-1} \\
&= \frac{(\beta')^2\Omega^2C^2}{16}\left(\frac{\eta'(\beta',\Omega,Q)}{Q}+\frac{\beta'}{4}\right)^{-1}.
\end{aligned}$$

The same quantity for the disordered phase is given by

$$\chi(\alpha_{10} = 0; \beta, Q, \Omega) = \eta(\beta').$$

Comparing them we get the following inequality which implies that the pure ground mode phase is preferred:

$$\frac{(\beta')^2\Omega^2C^2}{16Q} > \frac{\eta(\beta',Q)}{Q}\left(\frac{\eta'(\beta',\Omega,Q)}{Q} + \frac{\beta'}{4}\right). \tag{28}$$

Fixing $\frac{\Omega^2C^2}{Q} > 1$, we deduce from (14) that (28) holds only if $\beta' > \beta'_{cc}(\Omega, Q)$ where

$$\frac{(\beta'_{cc})^2\Omega^2C^2}{16Q} = \frac{\eta(\beta'_{cc},Q)}{Q}\left(\frac{\eta'(\beta'_{cc},\Omega,Q)}{Q} + \frac{\beta'_{cc}}{4}\right)$$

where such a positive value $\beta'_{cc} < \infty$ exists by virtue of the mean value theorem because for β' near zero,

$$\frac{(\beta')^2\Omega^2C^2}{16Q} < \frac{\eta(\beta',Q)}{Q}\left(\frac{\eta'(\beta',\Omega,Q)}{Q} + \frac{\beta'}{4}\right)$$

since $\eta(\beta', Q)$ increases as β' decreases, and for β' very large,

$$\frac{(\beta')^2\Omega^2C^2}{16Q} > \frac{\eta(\beta',Q)}{Q}\left(\frac{\eta'(\beta',\Omega,Q)}{Q} + \frac{\beta'}{4}\right)$$

since $\eta(\beta', Q)$ decreases down to zero as β' increases to ∞. We used the fact that both saddle points $\eta(\beta', Q)$ and $\eta'(\beta', \Omega, Q)$ are smooth functions of β' in the range $(0, \infty)$.

Returning to the infinite sum of logarithmic terms in F,

$$-\lim_{N\to\infty}\frac{1}{2N}\sum_{l=2}^{\sqrt{N}}\sum_{m=-l}^{l}\ln\left(\frac{N\eta}{Q} + \frac{\beta' N}{2}\lambda_{lm}\right),$$

we note that the value of this convergent sum for saddle point $\eta(\beta', Q)$ is bigger than that for $\eta'(\beta', \Omega, Q)$ in view of (14) but this difference is logarithmic in the difference $\eta'(\beta', \Omega, Q) - \eta(\beta', Q) > 0$, and is therefore dominated by the algebraic difference

$$\chi(\alpha_{10}; \beta, Q, \Omega) - \chi(\alpha_{10} = 0; \beta, Q)$$

discussed above.

This completes the proof that the disordered phase is preferred at high positive temperatures but the ordered phase is preferred at low enough temperatures where, unlike the negative critical point, the threshold value $\beta'_{cc}(\Omega, Q)$ depends on both relative enstrophy Q and planetary spin Ω. From (16) we deduce that $\eta'(\beta', \Omega, Q)$ is linear in Ω. Since $\eta(\beta', Q)$ does not depend on Ω, this implies $\beta'_{cc}(\Omega, Q)$ decreases as planetary spin Ω increases. Thus, as Ω decreases to zero, the threshold value $\beta'_{cc}(\Omega, Q)$ tends to ∞, and the disordered phase is preferred at all positive temperatures in the case of a non-rotating massive sphere.

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