

EXTREMAL FREE ENERGY IN A SIMPLE MEAN FIELD THEORY FOR A COUPLED BAROTROPIC FLUID - ROTATING SPHERE SYSTEM

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ABSTRACT. A family of spin-lattice models are derived as convergent finite dimensional approximations to the rest frame kinetic energy of a barotropic fluid coupled to a massive rotating sphere. A simple mean field theory for this statistical equilibrium model is formulated and solved, providing precise conditions on the planetary spin and relative enstrophy in order for phase transitions to occur at positive and negative critical temperatures, T_+ and T_- . When the planetary spin is relatively small, there is a single phase transition at $T_- < 0$, from a preferred mixed vorticity state $v = m$ for all positive temperatures and $T < T_-$ to an ordered pro-rotating (west to east) flow state $v = n_u$ for $T_- < T < 0$. When the planetary spin is relatively large, there is an additional phase transition at $T_+ > 0$ from a preferred mixed state $v = m$ above T_+ to an ordered counter-rotating flow state $v = n_d$ for $T < T_+$.

1. Introduction. Consider the system consisting of a rotating high density rigid sphere of radius R , enveloped by a thin shell of non-divergent barotropic fluid. A comparison with the divergent case - coupled shallow water model - will be given in the appendix. The barotropic flow is assumed to be inviscid, apart from an ability to exchange angular momentum and energy with the infinitely massive solid sphere. In addition we assume that the fluid is in radiation balance and there is no net energy gain or loss from insolation. This provides a crude model of the complex planet - atmosphere interactions, including the enigmatic torque mechanism responsible for the phenomenon of atmospheric super-rotation - one of the main applications motivating this work. We construct a simple mean field theory for the equilibrium statistical mechanics of this problem with the specific purpose of investigating its critical phenomenology - phase transitions that are dependent on a few key parameters in the problem. The adjective simple refers to the fact that in this mean field theory, a comparison of the free energy between two special macrostates in the problem - a disordered vorticity state and a unmixed vorticity state representing rigidly rotating flows - is made to determine the critical temperatures. Comparisons of the results in this paper with those obtained by a zero temperature variational method and also those predicted by a more sophisticated mean field theory based on the Bragg method underscore the fact that this simple mean field theory is nonetheless

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quite powerful in predicting the critical phenomenology of the coupled barotropic flow - rotating solid sphere system.

This paper will therefore contain new results - the simple mean field theory and its implications - together with a fairly extensive survey of the main ideas, approaches and results of applying statistical mechanics to complex geophysical flows in the past two decades. The reader is invited to consult the book [36] for the details of many topics mentioned here.

For a geophysical flow problem concerning super-rotation on a spherical surface, one of the key parameters is angular momentum of the fluid. In principle, the total angular momentum of the fluid and solid sphere is a conserved quantity but by taking the sphere to have infinite mass, the active part of the model is just the fluid which relaxes by exchanging angular momentum with an infinite reservoir located on the sphere.

Inverse and forward cascades in quasi - 2d geophysical relaxation problems such as this one involve energy and enstrophy respectively. The total or rest frame energy of the fluid and sphere is conserved. Since we have assumed the mass of the solid sphere to be infinite, we need only keep track of the kinetic energy of the barotropic fluid - in the non-divergent case, there is no gravitational potential energy in the fluid since it has uniform thickness and density, and its upper surface is a rigid lid. Unlike the situation in decaying 2d turbulence on a periodic domain, the usual Principle of Selective Decay - that enstrophy decays at a faster rate than energy piled up at large scales - does not apply to nearly inviscid 2D flows in complex boundaries. Here, enstrophy has a significantly slower decay rate at small / viscous scales than the rates at which energy and angular momentum are transferred at complex boundaries by a combination of torques, normal forces and shear. At this point there are in principle two distinct infinite reservoirs in the relaxation problem, namely, the energy and angular momentum ones.

The author's approach to this class of geophysical problems differs from previous works in three major aspects, namely (A) angular momentum of the fluid is not conserved but rather the fluid relaxes by exchanging energy and angular momentum with the solid sphere, (B) the energy and angular momentum reservoirs are combined into a single one and (C) enstrophy is constrained microcanonically. (A) is justified by one of the aims of this approach - to model and give a relaxational explanation of the enigmatic super-rotation problem in the Venusian atmosphere - and by the physics of angular momentum of the coupled barotropic fluid - sphere systems on which the approach is based. Indeed, previous works on barotropic flows on a rotating sphere with trivial topography [3], [12] that are based on the classical energy - enstrophy theory for the barotropic vorticity equation - hence, conserving the fluid's angular momentum - have failed to discover any phase transitions within the valid temperature range of the corresponding Gaussian models.

The justification of (B) takes the form of the fact - easily seen by a calculation on the reader's part - that the rest frame kinetic energy of the non-divergent barotropic fluid has two distinct parts, the second of which is proportional to the fluid's changing net angular momentum relative to a frame that is rotating at the fixed angular velocity of the sphere. The separation of two time scales at large and small spatial scales in decaying coupled flows in complex domains, introduced earlier through the inverse and forward cascades, justifies the combination (A), (B) and (C) because in the fast time-scale for energy and angular momentum transfer, enstrophy changes little and can be treated as fixed.

Conservation of relative enstrophy, treated here as a microcanonical constraint, modifies the classical energy-enstrophy theories [9], [3], [12], [33] in substantial ways. Since the classical energy-enstrophy theories are all doubly canonical in the energy and enstrophy, they are essentially equivalent to the Gaussian model which is the simplest of the few exactly-solvable statistical mechanics models. Hence, previous applications of equilibrium statistical mechanics to geophysical flows [3], [12], [33] have largely used it. However, replacing the canonical constraint on enstrophy by a microcanonical one yields a significant benefit, namely, statistical equilibrium models, known collectively as the spherical model, that are well-defined for all positive and negative temperatures. For the aims of this paper, which is to investigate the precise conditions for phase transitions in the equilibrium statistics of the coupled barotropic fluid - rotating solid sphere system, this is important because a phase transition could in principle, occur at any positive or negative temperature. For the reason that angular momentum is fixed in vortex gas models, they are not suitable for coupled flows.

Higher vorticity moments are less significant than enstrophy in statistical equilibrium models of quasi-2d geophysical flows because they are not conserved by finite dimensional truncations such as in the absolute equilibrium approach, even in the case of uncoupled flows on simple domains. Moreover, only the second moment enters into the classical picture of the inverse and forward cascades. We discuss this point in greater detail in [40].

Any statistical mechanics theory on a subject matter other than the traditional domain of atoms, requires some discussion of what we mean by temperature and entropy. Statistical temperature for macroscopic flows is by now a recognized scientific concept. Unlike the standard notion of temperature which measures the average kinetic energy of molecular motions, macroscopic flow temperature is a measure of the average kinetic energy contained in eddies which can vary over a wide range of length scales. Given a macroscopic flow state or macrostate vorticity distribution, there belongs a suitable flow temperature that depends on the average energy of the eddies in the flow. We will refer to corresponding energy reservoirs instead of heat baths. Thus, roughly speaking, transfers of energy and angular momentum at complex boundaries contributes to a change in the kinetic energy and angular momentum of organized/ large eddies and the kinetic energy and entropy of small eddies.

2. Summary of objectives, approach and results. It is difficult to solve the spherical model for the coupled barotropic fluid - rotating solid sphere system analytically in closed form. Progress on this current topic is discussed in [41], and on the author's webpage: www.rpi.edu/~limc. Other more tenable methods - both numerical and analytical ones - will therefore have to be used to address this problem. Three related methods used in the energy-enstrophy theory of coupled barotropic flows system are (i) a simple mean field approach - presented here - which is based on the notion of conserving an averaged relative enstrophy, (ii) Monte - Carlo simulations of the spherical model (with the microcanonical constraint on relative enstrophy), and (iii) the Bragg mean field method. In this paper we present the first. The second is presented in another paper [14]. The third is based on an intermediate mean field method [8] which gives a renormalized expression for the free energy in terms of the coarse-grained barotropic non-divergent vorticity, without using a

relative enstrophy constraint [15]. These approximate expressions for the free energy yield values for positive and negative critical temperatures for the coupled barotropic fluid - rotating solid sphere system which are consistent with the mean field results in this paper, and the Monte-carlo simulations in [14]. Similar to the Curie-Weiss theory for phase transitions in ferromagnets [8], the special form of the hyperbolic tangent plays a key role in [15] and will be taken up again in a paper on the associated nonlinear fixed point result.

The specific aim for formulating a simple mean field theory in this paper is to find precise conditions on planetary spin, enstrophy and energy for phase transitions in the equilibrium statistical mechanics of the coupled barotropic fluid - rotating solid sphere system. The formulation of this simple mean field theory is based on common features of several recent mean field theories for the 2D Euler equations but not equivalent to any single one of them - see Joyce and Montgomery [29], Lundgren and Pointin [30], and Caglioti et al [34].

Moreover, the above mean field theories are all known to be asymptotically exact in a non-extensive continuum limit; proofs of this property were mainly constructed via large deviations methods and result in nonlinear elliptic mean field equations as in [34]. Although a rigorous proof for this property has not yet been constructed in the case of the coupled barotropic fluid - rotating solid sphere system, we are certain that only the details differ from the general line of approach used in [34]. Our point of departure, then, is to assume that the mean field is asymptotically exact for the coupled barotropic fluid - rotating solid sphere system. Asymptotic exactness [34], [29], [30] of the mean field is a useful property for it implies that (a) a non-extensive continuum/ thermodynamic limit is well defined for the family of finite size spin-lattice models H_N , that we will derive, as finite dimensional approximations of the rest frame barotropic kinetic energy of non-divergent flows, and (b) the thermal properties of this family of statistical equilibrium spin-lattice models, in the thermodynamic limit as mesh size N tends to ∞ , is completely determined by the mean field.

The mean field results in this paper are obtained without solving any nonlinear elliptic PDEs, an important point which will be discussed further, in view of our main aim of calculating precise critical temperatures. In particular, using this simple mean field theory, we predict at least one and as many as two critical temperatures for the equilibrium statistical mechanics of barotropic flows on a sphere, depending on the planetary spin Ω and relative enstrophy Q_r . For relatively low values of the planetary spin $\Omega > 0$, there is no positive temperature phase transition, just a negative critical temperature $T_c^-(\Omega, Q_r) < 0$ between a mixed vorticity state $v = m$ (without any long range order) for T , positive or negative, less hot than T_c^- and the prograde solid-body flow state $v = n_u$ for negative T hotter than T_c^- . For large enough planetary spins, there is in addition a phase transition at positive critical temperature $T_c > 0$ between a hot mixed state $v = m$ and the cold retrograde solid-body flow state $v = n_d$.

The history of applying statistical equilibrium methods to 2D turbulence is characterized by several distinctive findings, such as Onsager's discovery of negative temperatures when the vortices are confined, the presence of coherent structures and an inverse cascade of energy to large scales [9], [11]. Two dimensional vortex statistics also have the dubious distinction of not supporting any phase transitions at positive temperatures (cf. [31], [11], [9], [29], [38], [39], [28], [24]). Until recently,

quasi 2D turbulence in the context of barotropic rotating flows over trivial topography appeared to retain many of these properties, including the lack of phase transition. Nontrivial topography appears to have qualitatively different and anisotropic effects. In their classic paper [33], Salmon et al derived the statistical equilibrium states of an inviscid unforced single-layer of fluid in a periodic box with nontrivial bottom topography in the beta plane approximation, and reported correlations between bottom topography and expected stream function when flow energy is low. Frederiksen [12] reported statistical equilibrium results for the coupled barotropic fluid - rotating solid sphere system with nontrivial topography, that are consistent with several numerical and experimental findings, namely, topographic effects are asymmetric in the sense that eastward solid-body flows over nontrivial topography is less stable compared with westward solid-body flows.

However, when the topography is trivial, Frederiksen and Sawford [3] reported that the eastward and westward solid-body flows are both stable but did not find any phase transitions. By going to a simple but powerful mean field approach, not based on solving PDEs numerically, we calculate asymmetric positive and negative critical temperatures in terms of planetary spin and relative enstrophy. We believe that our result on the existence of a positive critical temperature in the rotating barotropic flows system when the planetary spin is sufficiently large, and a negative critical temperature for all values of the planetary spin, is the first in the large literature on the equilibrium statistical mechanics of geophysical flows since the papers by Salmon, Holloway and Hendershott [33], Frederiksen and Sawford [3] and Carnevale and Frederiksen [35].

This paper is organized into the following main sections and appendix: (3) the extended Planck's theorem, (4) derivation of Ising model-like spin-lattice approximation for the kinetic energy of the barotropic flows, (5) the classical energy-enstrophy theory based on this spin-lattice model, and the spherical model for the barotropic flows on a rotating sphere, (6) simple mean field theories of the spin-lattice model and its thermal properties, (7) conclusion and (8) appendix: background and properties of the coupled barotropic fluid - rotating solid sphere system.

3. Equilibrium statistical mechanics. Planck's theorem will be used to find the most probable equilibrium states in an isothermal setting: minima of a Gibbs free energy per lattice site f determines the thermodynamically stable states. It is well known since Onsager's seminal paper [31] that 2D vortex statistics is characterized by negative temperatures at high values of flow kinetic energy. Since we are interested in exploring the entire range of flow kinetic energy in our model, we will have to extend Planck's theorem for negative temperatures T :

Extended Planck's Theorem: *The most probable (thermodynamically stable) state at negative (positive) temperature T corresponds to maxima (minima) of the Gibbs free energy per site*

$$f = U - TS$$

where

$$U = \langle H \rangle$$

is the internal energy per site and

$$S = -k_B \int ds p(s) \ln p(s)$$

is the mixing entropy per site, given in terms of the probability distribution $p(s)$ for site value s . Here k_B is the Boltzmann constant.

Remark 0: It follows that the most probable state, also the thermodynamically stable state, is not necessarily the one, call it m' that maximizes $W = \exp(S/k_B)$, but rather that which maximizes the product $W \exp(-\beta U)$. Nonetheless, at temperatures T where $|T|$ is small, it is likely that the state extremizing F is closer to one, call it m that extremizes U instead of m' maximizing S ; and when $|T|$ is large, it is likely that the state extremizing F is closer to m' that maximizes S than that extremizing U . We note the significant point that for systems that support negative temperatures such as barotropic flows in particular, the specific heat is nonetheless positive even when the temperature is negative.

A precise correspondence exists between the statistical equilibrium properties of barotropic flows on one hand and its dynamical properties on the other. This correspondence is encapsulated in the Minimum Enstrophy or its equivalent, the extremal kinetic energy principle, which form a pair of dual variational principles for the steady-states of the barotropic flows (cf. Leith [4], Young [2], Prieto and Schubert [24] and Lim [13]). The maximal kinetic energy steady-state of the barotropic flows is closely related to the most probable flow state at hot enough negative temperatures in the corresponding statistical mechanics model. Likewise, the minimal kinetic energy steady-state of the barotropic flows is related to the most probable flow state at low positive temperatures. At temperatures sufficiently close to 0, this correspondence follows directly from the form of the Gibbs free energy $f = U - TS \simeq U$, no matter in which of the two general categories below the barotropic flows thermal system happens to be.

In general, the Gibbs canonical ensemble consists of the standard form for the probability measure

$$P_G(w) = \frac{1}{Z[\beta, \mu]} \exp(-\beta H[w] - \mu \Gamma[w])$$

where $H[w]$ is the energy and β an inverse temperature while $\Gamma[w]$ represents a significant conserved quantity and μ a chemical potential or Lagrange Multiplier conjugate to $\Gamma[w]$. The partition function or configurational integral $Z[\beta, \mu]$ provides the normalization required to make $P_G[w]$ a probability measure. It is customary to include only the key conserved quantities as canonical constraints $\Gamma[w]$. Indeed, in the case of quasi-2D turbulence, there are an infinite number of conserved quantities, namely the moments of the relative vorticity or higher order enstrophies, and it is unnecessary to include all of them as canonical constraints in the Gibbs probability P_G .

4. Spin-lattice approximation. Given the well known fact that Gibbs' canonical ensemble and the corresponding partition function for the spherical model - to be discussed in detail below - are closely related to path-integrals and therefore extremely complex mathematical objects, a rational approximation procedure based on finite dimensional spin-lattice models, will have to be devised to simulate their critical phenomenology on the computer as well as to solve them by analytical means. Such a rational approximation scheme must satisfy two basic requirements when the size or order of the approximation is taken to infinity: (A) the resulting family of finite dimensional models converge to the correct energy functional and constraints of the problem and (B) the thermodynamic limit - in this case, the

non-extensive continuum limit - of this family of approximate models exists. (A) is shown to be true for the family of spin-lattice models given next. (B) turns out to be true if exact solutions to the spherical models - to be obtained by the Kac-Berlin method of steepest descent in [41] - yield valid free energy expressions in terms of the associated saddle points in the non-extensive continuum limit. For the purpose of this paper, the assumption of the validity of (B) is subsumed under the earlier assumption that the mean field is asymptotically exact in this class of problems for the coupled barotropic flows on a rotating sphere.

Using a uniform mesh M of N points $\{x_1, \dots, x_N\}$ on S^2 and the Voronoi cells based on this mesh - see Lim and Nebus [18] - we approximate the relative vorticity by

$$w(x) = \sum_{j=1}^N s_j H_j(x)$$

where

$$s_j = w(x_j) \in (-\infty, \infty)$$

and H_j is the indicator function on the Voronoi cell D_j centered at x_j , that is,

$$H_j(x) = 1 \text{ for } x \in D_j, \tag{1}$$

$$H_j(x) = 0 \text{ for } x \notin D_j. \tag{2}$$

The real valued spins s_j should henceforth be viewed as coarse-grained or block averaged vorticity - resulting from a single step renormalization procedure outlined in [15] - that is ideally suited to the mean field approach in this paper. Another point of view describing the real valued spin states $\{s_j\}$ as macrostates can be found in the book [36].

Recall that the Voronoi cell D_j is defined to be all the points in S^2 that are nearer to x_j than to any other points in the mesh M . This confers the essential property that the Voronoi cells is a disjoint cover for S^2 , that is,

$$D_j \cap D_k = \emptyset, \tag{3}$$

$$\bigcup_{j=1}^N D_j = S^2. \tag{4}$$

Uniformity of the mesh M confers the other essential property that the areas A_j of the cells are equal, that is,

$$A_j = |D_j| = \frac{4\pi}{N}.$$

The rest frame kinetic energy given in terms of the parameters of a frame that is rotating at the fixed angular velocity of the solid sphere has been shown to be

$$H = -\frac{1}{2} \int_{S^2} q\psi dx \tag{5}$$

$$= -\frac{1}{2} \int_{S^2} dx (w + 2\Omega \cos \theta) G(w) \tag{6}$$

where the fundamental solution of the Laplace-Beltrami operator on S^2 is

$$\psi(x) = G(w) = \int_{S^2} dx \ln |1 - x \cdot x'| w(x').$$

Under the above approximation for the relative vorticity w , the truncated or lattice approximate energy

$$\begin{aligned} H_N &= -\frac{1}{2} \int_{S^2} dx \left(\sum_{j=1}^N s_j H_j(x) + 2\Omega \cos \theta \right) G \left(\sum_{j=1}^N s_j H_j(x) \right) \\ &= -\frac{1}{2} \sum_{j=1}^N \sum_{k=1}^N \left[\int_{S^2} dx H_j(x) G(H_k) \right] s_j s_k - \Omega \sum_{j=1}^N \left[\int_{S^2} dx \cos \theta G(H_j) \right] s_j \\ &\rightarrow H \text{ as } N \rightarrow \infty \end{aligned}$$

under the calculus rules of (Lebesgue) integration. With the interactions

$$J_{jk} = \int_{S^2} dx H_j(x) G(H_k)$$

and the external fields

$$F_j = \Omega \int_{S^2} dx \cos \theta G(H_j),$$

the truncated energy takes the standard form of a spin lattice model,

$$H_N = -\frac{1}{2} \sum_{j=1}^N \sum_{k=1}^N J_{jk} s_j s_k - \sum_{j=1}^N F_j s_j. \quad (7)$$

The interactions J_{jk} are logarithmic and thus, long range. The external fields F_j are non-uniform and linear in the spin $\Omega \geq 0$, and represent the coupling between the local relative vorticity or spin s_j and the planetary vorticity field $p \cos \theta$. They are turned off for the special case of single layer inviscid vortex dynamics on a non-rotating sphere. The presence of this inhomogeneous term when $\Omega > 0$, is the source of the much richer mathematical and physical properties of the coupled barotropic flows on a rotating sphere.

Evaluating the integral and using the essential properties of H_j we obtain

$$\begin{aligned} J_{jk} &= \int_{S^2} dx H_j(x) \int_{S^2} dx' \ln |1 - x \cdot x'| H_k(x') \\ &\rightarrow \frac{16\pi^2}{N^2} \ln |1 - x_j \cdot x_k| \text{ as } N \rightarrow \infty. \end{aligned} \quad (8)$$

For the integral in the external fields F_j we obtain

$$F_j = \Omega \int_{S^2} dx \cos \theta \int_{S^2} dx' \ln |1 - x \cdot x'| H_j(x') \quad (9)$$

$$= \Omega \int_{S^2} dx' H_j(x') \int_{S^2} dx \cos \theta \ln |1 - x \cdot x'| \quad (10)$$

after using the symmetry of the inverse G to the Laplace-Beltrami operator on S^2 . Then

$$\begin{aligned} F_j &= \Omega \|\cos \theta\|_2 \int_{S^2} dx' H_j(x') \int_{S^2} dx \psi_{10}(x) \ln |1 - x \cdot x'| \\ &= -\frac{1}{2} \Omega \|\cos \theta\|_2 \int_{S^2} dx' H_j(x') \psi_{10}(x') \\ &\rightarrow -\frac{2\pi}{N} \Omega \|\cos \theta\|_2 \psi_{10}(x_j) \text{ as } N \rightarrow \infty, \end{aligned} \quad (11)$$

where $\|\cos \theta\|_2$ is the L_2 norm of the function $\cos \theta$, that is,

$$\|\cos \theta\|_2 = \sqrt{\int_{S^2} dx \cos^2 \theta},$$

and the spherical harmonic ψ_{10} which represents the relative vorticity of solid-body rotation, satisfies the eigenvalue relation,

$$\begin{aligned} \psi_{10}(x) &= \frac{\cos \theta}{\|\cos \theta\|_2} = -2G(\psi_{10}) \\ \lambda_{lm} &= -l(l+1) = -2 \text{ for } l = 1, m = 0. \end{aligned} \tag{12}$$

The truncated relative enstrophy is given by

$$\begin{aligned} \Gamma_N &= \int_{S^2} dx w^2 = \int_{S^2} dx \left(\sum_{j=1}^N s_j H_j(x) \right)^2 \\ &= \frac{4\pi}{N} \sum_{j=1}^N s_j^2 \rightarrow \Gamma_r \text{ as } N \rightarrow \infty, \end{aligned} \tag{13}$$

after using the two essential properties of the indicator functions H_j .

Lastly, the truncated total circulation is given by

$$TC_N = \int_{S^2} dx w = \int_{S^2} dx \sum_{j=1}^N s_j H_j(x) \tag{14}$$

$$= \frac{4\pi}{N} \sum_{j=1}^N s_j \rightarrow TC \text{ as } N \rightarrow \infty. \tag{15}$$

5. Classical and recent energy-enstrophy theories. To put the mean field theory here in perspective, we briefly review the two main theories - both of which are based on Kac's inventions circa 1952 - used to investigate the statistical relationships between energy and enstrophy in macroscopic flows. The first is the Gaussian model which forms the basis of all the classical energy-enstrophy theories. The second is the spherical model for which the canonical enstrophy constraint is replaced by the microcanonical form. Versions of the spherical models for new energy - enstrophy -circulation models of macroscopic flows were first introduced by the author beginning in early 2000 - see the Springer-Verlag book by Lim and Nebus [36].

5.1. Gaussian model. Almost all the early papers on variations of this model used a spectral formulation in terms of a truncated set of orthonormal eigenfunctions for the flow domain. Many authors presented results on the applications of this theory to a large range of topics in geophysical flows including two-layer flows over nontrivial bottom topography, and quasi-geostrophic f-plane and beta plane flows [33]. We will use a spatial lattice formulation instead for reasons already mentioned in the introduction. It is easy to show that this classical energy-enstrophy theory is identical to the well-known Gaussian Model introduced by Kac [32] which is exactly-solvable.

The classical energy-entropy theory as formulated using a spatial discretization is now given in terms of the truncated energy H_N (7) and relative entropy Γ_N (13) by the Gaussian partition function,

$$Z_N = \left(\frac{b}{2\pi}\right)^{N/2} \int \left[\prod ds_j\right] \exp\left(-b \sum_{j=1}^N s_j^2\right) \exp(-\beta H_N[S; \Omega]),$$

in terms of the spin (vorticity) state

$$S = \{s_1, \dots, s_N\}, \quad (16)$$

$$s_j \in (-\infty, \infty). \quad (17)$$

In this model, the standard deviation of the above Gaussian is given in terms of the parameter

$$b = \frac{4\pi\mu}{N}$$

where μ is the chemical potential or Lagrange multiplier associated with the relative entropy constraint. The factor $\left(\frac{b}{N}\right)^{N/2}$ is needed to make

$$W[S] = \left(\frac{b}{2\pi}\right)^{N/2} \exp\left(-b \sum_{j=1}^N s_j^2\right) \text{ a probability distribution.}$$

The exact solution of this Gaussian model where it is well-defined, anchors many of the previous works on statistical equilibrium in geophysical flows (cf. Salmon, Holloway and Hendershott [33], Frederiksen and Sawford [3] and Carnevale and Frederiksen [35], amongst many others).

It turns out, as will be shown next, that the choice of a canonical constraint on total kinetic energy $H[q]$ and a microcanonical constraint on the relative entropy Γ_r gives a spherical model formulation which although difficult to solve analytically, is amenable to numerical simulations and mean field methods. More important is the fact that the spherical model is well-defined for all temperatures positive or negative, while the Gaussian model is not defined for certain temperatures.

5.2. Spherical model for the coupled barotropic flows. We need to fix the low temperature defect of the classical energy-entropy theory of the barotropic flows which was mentioned above. We will do this by replacing the canonical constraint on relative entropy by a microcanonical constraint, which yields a version of Kac's spherical model for the spin-lattice model. The spherical model formulation of an equilibrium statistical energy-entropy theory for the barotropic flows on a rotating sphere is based on the spin lattice partition function,

$$Z_N = \int (\prod ds_j) \delta(NQ_r - \sum s_j^2) \exp[-\beta H_N] \quad (18)$$

where H_N , J_{jk} and the external fields F_j are given in (7), (8) and (11).

The microcanonical relative entropy constraint takes the Laplace integral form and gives

$$Z_N = \frac{1}{4\pi i} \int (\prod ds_j) \int_{a-i\infty}^{a+i\infty} d\eta \exp\left[\frac{1}{2}\eta NQ_r - \frac{1}{2}\langle S|K|S\rangle\right] \exp\left[\beta \sum_{j=1}^N F_j s_j\right]$$

in terms of the matrix

$$K_{jk} = \eta - \beta J_{jk}$$

and the spin vector $S = \{s_1, \dots, s_N\}$.

6. Mean field theory. The spherical model for the energy-entropy theory of the rotating barotropic flows is difficult to solve but it fixes the above foundational difficulty of the classical energy-entropy theory of the coupled barotropic flows. We therefore use mean field methods to derive thermal properties of the spin-lattice models,

$$H_N = -\frac{1}{2} \sum_{j=1}^N \sum_{k=1}^N J_{jk} s_j s_k - \sum_{j=1}^N F_j s_j.$$

This yields physically significant phase transitions for the new energy-relative entropy theory of the coupled barotropic flows on a rotating sphere. The main step of this mean field theory is to calculate the change in free energy per site between two fundamentally different vorticity states, namely, the *mixed macrostate* where the coarse-grained vorticity of opposite sign are juxtaposed in such a way that a site has equal probability of having spin $\pm s_0$ and the *unmixed macrostate* where the coarse-grained vorticity of opposite sign are separated into hemispheres on S^2 in a way that a site has probability 0 (resp. probability 1) of having spin s_0 (resp. $-s_0$) - depending on where the given site is located relative to the hemispherical pattern.

Many detailed aspects of the formulation below can be changed without any loss of accuracy or precision to the stated results. Some of these are the neighborhood size z which determines the range of the interactions in the model, and the simplified mean spin values $\pm s_0$. For the energy-entropy spin-lattice models, the actual neighborhood $N(j)$ equals the set of all other sites in the mesh. However, although this signifies a global interaction in H_N , it is a short range one by the accepted definition - the integral of interactions over all sites is finite - and therefore, its mean field properties are not dependent on z . Choosing the simplified mean spin values to be $\pm s_0$ appears to be a more serious restriction - changing the real valued spins in the original lattice models H_N to binary spins that make the resulting model more of an Ising type. Again, it is not hard to see - by a single step of spatial renormalization or block spin averaging - that the binary spin models we use in the ensuing mean field theory are equivalent to the original.

6.1. Set-up for coupled barotropic flows. Since the coarse-grained spin or vorticity $s_0(x)$ satisfies the following integral relations,

$$\Gamma = \int_{S^2} dx s_0^2(x) = Q \quad (19)$$

$$\int_{S^2} dx s_0(x) = 0 = TC, \quad (20)$$

we assume, for the mean field theory, the existence of a site dependent volume fraction or probability distribution function $\nu_x(s_0)$ such that

$$\int_{-M}^M ds_0 \nu_x(s_0) = 1, \quad (21)$$

where M denotes the limiting values of the coarse-grained spin $s_0(x)$ over S^2 and

$$\int_{S^2} dx \int_{-M}^M ds_0 \nu_x(s_0) s_0^2 = Q \quad (22)$$

$$\int_{S^2} dx \int_{-M}^M ds_0 \nu_x(s_0) s_0 = 0. \quad (23)$$

Next we assume that the mixed state and unmixed mean field state, denoted respectively by

$$v = m, n, \quad (24)$$

are characterized by m where the mean spins $s_0(x)$ are independent at neighboring sites and n where the neighboring spins are correlated, that is, $s_0(x) = s_0(x') \in \{s_0^\pm\}$. Equivalently, m is characterized by

$$\nu_x^m(\pm s_0) = \frac{1}{2} \text{ for all } x \in S^2; \quad (25)$$

and n by

$$\nu_{x^+}^n(s_0) = 1 \text{ for all } x^+ \in S^2, \quad (26)$$

$$\nu_{x^-}^n(-s_0) = 1 \text{ for all } x^- \in S^2 \quad (27)$$

where x^\pm are lattice sites in respectively the positive and negative hemispheres of the unmixed state n .

The notation $N(j)$ denotes the neighborhood of site j , that is, all lattice sites k which are connected to site j . Let $|N(j)| = z$ be the common size of neighborhoods $N(j)$ or coordination number for the lattice. By varying z we can model interactions of different range in a variety of spin-lattice models. Let ε denote the interaction energy scale (obtained by averaging J_{jk} over $N(j)$) for H_N . For the purpose of modeling the energy-entropy theories, $\varepsilon < 0$ which represents an anti-ferromagnetic interaction.

In addition to the spin-lattice models H_N , we will need a crude lattice approximation for the per site mixing entropy. Following the same approach for J_{jk} and F_j in (8) and (11) respectively, we derive the lattice approximation for the total mixing entropy in the ensuing steps,

$$S[\nu] = -k_B \int_{S^2} dx \int_{-M}^M ds \nu_x(s) \ln \nu_x(s) \quad (28)$$

$$\simeq -k_B \sum_{j=1}^N \int_{S^2} dx H_j(x) \int_{-M}^M ds \nu_x(s) \ln \nu_x(s) \quad (29)$$

$$= -\frac{4\pi k_B}{N} \sum_{j=1}^N \int_{-M}^M ds \nu_{x_j}(s) \ln \nu_{x_j}(s). \quad (30)$$

6.2. Proofs and results when the solid sphere does not rotate. First we compute the entropy per site for the mixed state $v = m$,

$$S_m = -\frac{4\pi k_B}{N} \int_{-M}^M ds \nu_x \ln \nu_x = \frac{4\pi k_B}{N} \ln 2. \quad (31)$$

Similarly, the per site entropy in the unmixed state $v = n$ is given by

$$S_n = 0,$$

because by the defining property of the unmixed state $v = n$, neighboring mean values $s_0(x)$ are correlated perfectly.

Next we compute the mixed state internal energy per site,

$$\begin{aligned}
 u_m &= -\frac{\varepsilon}{2} \int_{-M}^M ds \nu_x(s) \int_{-M}^M ds' \nu_{x'}(s') s s' z \\
 &= -\frac{\varepsilon z}{2} \left[\int_{-M}^M ds \nu_x(s) s \right]^2 \\
 &= 0
 \end{aligned} \tag{32}$$

We compute the unmixed state per site internal energy by using the property that in the unmixed state, neighboring mean spins $s(x_k)$ satisfy

$$s(x_k) = s(x_j) = \pm s_0 \text{ for all } k \in N(j) \tag{33}$$

$$\text{with } |N(j)| = z \text{ for all } j = 1, \dots, N. \tag{34}$$

Then in this mean field theory,

$$\begin{aligned}
 u_n &= -\frac{\varepsilon z}{2N} \int_{S^2} dx \int_{-M}^M ds_0 \nu_x s_0^2 = -\frac{\varepsilon z}{2} \int_{-M}^M ds_0 \nu_x s_0^2 \\
 &= -\frac{\varepsilon z}{8\pi} Q
 \end{aligned} \tag{35}$$

after using (22).

Given the enstrophy $Q > 0$ and temperature T , and treating the mean spin distribution ν_x and state $v = m, n$ as free parameters, the isothermal free energy difference per site between the mixed and unmixed states is given by

$$\begin{aligned}
 \Delta f &= f_m - f_n \\
 &= (u_m - u_n) - T(S_m - S_n) \\
 &= \frac{\varepsilon z}{8\pi} Q - \frac{4\pi k_B T}{N} \ln 2.
 \end{aligned} \tag{36}$$

Equivalently,

$$\begin{aligned}
 f_m &= u_m - T S_m \\
 &= -\frac{4\pi k_B T}{N} \ln 2;
 \end{aligned} \tag{37}$$

and

$$\begin{aligned}
 f_n &= u_n - T S_n \\
 &= -\frac{\varepsilon z}{8\pi} Q.
 \end{aligned} \tag{38}$$

We have nearly finished proving the following mean field result for the non-rotating barotropic flows using the extension of Planck's theorem that for negative temperatures $T < 0$, thermodynamically stable statistical equilibria corresponds to maximizers of the free energy:

Theorem 5: *If $\varepsilon < 0$, then (i) for all $T > 0$, the mixed state $v = m$ is preferred. For $\varepsilon < 0$ and $T < 0$, there is a finite N critical temperature*

$$T_c(N) = \frac{\varepsilon z N}{8\pi S_{m,s}} Q < 0 \tag{39}$$

such that if (ii)

$$0 > T > T_c(N)$$

then $v = n$ is preferred, and (iii) if

$$T < T_c(N) < 0,$$

then $v = m$ is preferred with maximum mean spin entropy

$$S_{ms} = 4\pi k_B \ln 2. \quad (40)$$

(iv) In the non-extensive continuum limit - assumed to exist - as $N \rightarrow \infty$, $T_c(N)$ tends to a finite negative critical temperature

$$T_c(Q) = \frac{\frac{Q}{2} \int_{S^2} dx \psi_{10}(x) \int_{S^2} dx' \psi_{10}(x') \ln |1 - x \cdot x'|}{S_{ms}} \quad (41)$$

$$= \frac{\frac{Q}{2} \int_{S^2} dx \psi_{10}(x) G(\psi_{10})(x)}{S_{ms}} \quad (42)$$

$$= \frac{-\frac{Q}{4} \int_{S^2} dx \psi_{10}^2(x)}{S_{ms}} = -\frac{Q}{4S_{ms}} < 0 \quad (43)$$

such that if

$$0 > T > T_c$$

then $v = n$ is preferred and if

$$T < T_c < 0,$$

then $v = m$ is preferred.

Proof. The proof of (i) follows from the above calculations. To prove cases (ii) and (iii), we compare the per site free energy

$$f_m(\max) = -\frac{T}{N} S_{ms} > 0 \quad (44)$$

with the same in the unmixed state $v = n$,

$$f_n(\max) = -\frac{\varepsilon z}{8\pi} Q > 0. \quad (45)$$

When

$$-\frac{T}{N} S_{ms} < -\frac{\varepsilon z}{8\pi} Q$$

or equivalently $0 > T > T_c(N)$, (ii) $v = n$ is preferred, and vice-versa for (iii). Since $\varepsilon < 0$, $Q > 0$ and by definition, $S_{ms} > 0$, the critical temperature $T_c(N) < 0$.

The proof of (iv) follows from consideration of the nonextensive continuum limit under which the free energy

$$\frac{\varepsilon z N}{8\pi} Q \rightarrow \frac{Q}{2} \int_{S^2} dx \psi_{10}(x) \int_{S^2} dx' \psi_{10}(x') \ln |1 - x \cdot x'| \quad (46)$$

$$= -\frac{Q}{4} < 0 \quad (47)$$

as the number N of lattice sites tend to ∞ , together with the fact that the denominator in $T_c(N)$ - total entropy S_{ms} - (40) does not depend on N . \square

Remark 1: When the interaction energy scale $\varepsilon < 0$, there is a negative temperature transition between the mixed state at hot $T < T_c < 0$ and the unmixed state at very hot $T_c < T < 0$.

Remark 2: The mixed state free energy per site is entirely entropic. The unmixed state free energy f_n is purely an internal energy term which is linear in the entrophy Q .

6.3. Mean field theory when the solid sphere rotates. Let us denote the two parts of H_N by

$$H_N = H_N^{(1)} + H_N^{(2)} \quad (48)$$

$$H_N^{(1)} = -\frac{1}{2} \sum_{j=1}^N \sum_{k=1}^N J_{jk} s_j s_k \quad (49)$$

$$H_N^{(2)} = \frac{2\pi\Omega}{N} \sum_{j=1}^N s_j \cos \theta_j. \quad (50)$$

Then we write the corresponding two parts of the internal energies,

$$u_m = u_m^{(1)} + u_m^{(2)} \quad (51)$$

$$u_n = u_n^{(1)} + u_n^{(2)} \quad (52)$$

with

$$u_m^{(1)} = 0 \quad (53)$$

$$u_n^{(1)} = -\frac{\varepsilon z}{8\pi} Q_r \quad (54)$$

by (32) and (35) respectively.

Next we assume that the mixed state $v = m$ is defined in (24) and the unmixed mean field states denoted by $v = n$, n_u and n_d are characterized by correlated neighboring spins, $s(x) = s(x') \in \{s_0^\pm\}$. Moreover, the unmixed states $v = n$, n_u and n_d are both exactly correlated by hemispheres into opposite values s_0^\pm such that

$$s_0^+ + s_0^- = 0 \quad (55)$$

$$2\pi ((s_0^+)^2 + (s_0^-)^2) = Q_r. \quad (56)$$

Solving for s_0^\pm we get

$$s_0^+ = -s_0^- = \sqrt{\frac{Q_r}{4\pi}}. \quad (57)$$

The unmixed states $v = n_u$ and n_d are further characterized by hemispherically correlated spins s_0^\pm which satisfies the expressions (74) and (75) respectively, that is, aligned and anti-aligned with the northern and southern hemispheres corresponding to the planetary spin $\Omega > 0$.

The entropy calculations are exactly the same as when planetary spin $\Omega = 0$ because the entropies S_m and S_n depends only on the statistical distribution $v_x(s_0)$ of the mean field relative vorticity s_0 .

Next we compute the values of the per site internal energies $u_m^{(2)}$ and $u_n^{(2)}$ due to the nonzero planetary spin $\Omega > 0$ of H_N : using definition (24) for the mixed state

$v = m$, we get

$$u_m^{(2)} = \frac{2\pi\Omega}{N^2} \left\langle \sum_{j=1}^N s_j \cos \theta_j \right\rangle \quad (58)$$

$$= \frac{2\pi\Omega}{N^2} \sum_{j=1}^N \cos \theta_j \langle s_j \rangle \quad (59)$$

$$= \frac{2\pi\Omega}{N^2} \left(\int_{-M}^M ds \nu_x(s) s \right) \sum_{j=1}^N \cos \theta_j \quad (60)$$

$$= 0 \quad (61)$$

since (25). Denoting by

$$x_j^\pm, \theta_k^\pm$$

the lattice sites and co-latitudes that fall respectively into the hemispheres determined by the correlated means s_0^\pm , we obtain for the generic unmixed state $v = n$,

$$u_n^{(2)} = \frac{2\pi\Omega}{N^2} \left\langle \sum_{j=1}^N s_j \cos \theta_j \right\rangle = \frac{2\pi\Omega}{N^2} \sum_{j=1}^N \langle s_j \rangle \cos \theta_j \quad (62)$$

$$\equiv \frac{2\pi\Omega}{N^2} \left\{ \left(\sum_{j=1}^{N/2} \langle s(x_j^+) \rangle \cos \theta_j^+ \right) + \left(\sum_{k=1}^{N/2} \langle s(x_k^-) \rangle \cos \theta_k^- \right) \right\} \quad (63)$$

$$= \frac{2\pi\Omega}{N^2} \left(s_0^+ \sum_{j=1}^{N/2} \cos \theta_j^+ + s_0^- \sum_{k=1}^{N/2} \cos \theta_k^- \right) \quad (64)$$

where the hemispheres defined by s_0^\pm need not be aligned with the northern and southern hemispheres corresponding to the spin $\Omega > 0$.

Next we compute the per site free energies

$$\begin{aligned} f_m &= u_m - TS_m \\ &= \frac{4\pi k_B T}{N} \ln 2 \end{aligned} \quad (65)$$

and

$$\begin{aligned} f_n &= u_n - TS_n \\ &= -\frac{\varepsilon z}{8\pi} Q_r + \frac{2\pi\Omega}{N^2} \left(s_0^+ \sum_{j=1}^{N/2} \cos \theta_j^+ + s_0^- \sum_{k=1}^{N/2} \cos \theta_k^- \right). \end{aligned} \quad (66)$$

Therefore, the per site change in free energy is now given by

$$\Delta f = f_m - f_n \quad (67)$$

$$= \frac{4\pi k_B T}{N} \ln 2 \quad (68)$$

$$+ \frac{\varepsilon z}{8\pi} Q_r - \frac{2\pi\Omega}{N^2} \left(s_0^+ \sum_{j=1}^{N/2} \cos \theta_j^+ + s_0^- \sum_{k=1}^{N/2} \cos \theta_k^- \right).$$

Remark 3: Application of this formulation to the non-rotating barotropic flows on the sphere at both positive and negative temperatures requires consideration of the nonextensive continuum limit under which

$$\varepsilon z = \varepsilon(N)z(N) \rightarrow O(N^{-1}) < 0, \quad (69)$$

$$\frac{2\pi}{N^2} \sum_{j=1}^{N/2} s_0^+ \cos \theta_j^+ + \frac{2\Omega}{N^2} \sum_{k=1}^{N/2} s_0^- \cos \theta_k^- \rightarrow O(N^{-1}). \quad (70)$$

6.3.1. *Positive temperature.* For $T > 0$, we will need to compare the minimum free energy per site in the mixed state $v = m$,

$$f_m(\min) = -\frac{T}{N} S_{ms} = \frac{4\pi k_B T}{N} \ln 2 \quad (71)$$

with the same in the unmixed state $v = n$,

$$f_n(\min) = -\frac{\varepsilon z}{8\pi} Q_r + \frac{2\pi\Omega}{N^2} \left(s_0^+ \sum_{j=1}^{N/2} \cos \theta_j^+ + s_0^- \sum_{k=1}^{N/2} s_0^- \cos \theta_k^- \right). \quad (72)$$

The extreme value of $f_n(\min)$ is obtained at the most negative value of

$$\frac{2\pi\Omega}{N^2} \left(s_0^+ \sum_{j=1}^{N/2} \cos \theta_j^+ + s_0^- \sum_{k=1}^{N/2} \cos \theta_k^- \right)$$

which occurs when the hemispheres associated with the correlated means s_0^\pm are anti-correlated with those corresponding to the planetary spin $\Omega > 0$.

For any $T > 0$, if

$$\frac{2\pi\Omega}{N^2} \left(s_0^+ \sum_{j=1}^{N/2} \cos \theta_j^+ + s_0^- \sum_{k=1}^{N/2} \cos \theta_k^- \right) > \frac{\varepsilon z}{8\pi} Q_r - \frac{T}{N} S_{ms} \quad (73)$$

then $v = m$ is preferred; if the inequality is reversed then an unmixed state is preferred.

Thus, if $\varepsilon < 0$, then for all $T > 0$ and $Q_r > 0$, the RHS of (73) is negative. This implies that for all $T > 0$, $Q_r > 0$, and $\Omega > 0$, the mixed state $v = m$ is preferred over any $v = n_u$ state where the hemispheres associated with the means s_0^\pm are correlated with those corresponding to spin $\Omega > 0$, i.e.,

$$\frac{1}{N} \left(s_0^+ \sum_{j=1}^{N/2} \cos \theta_j^+ + s_0^- \sum_{k=1}^{N/2} \cos \theta_k^- \right) > 0. \quad (74)$$

Next, using (73), we compare the mixed state $v = m$ with the unmixed states $v = n_d$ where the hemispheres associated with the means s_0^\pm are anti-correlated with those corresponding to spin $\Omega > 0$, i.e.,

$$\frac{1}{N} \left(s_0^+ \sum_{j=1}^{N/2} \cos \theta_j^+ + s_0^- \sum_{k=1}^{N/2} \cos \theta_k^- \right) < 0. \quad (75)$$

For $\varepsilon < 0$ and any fixed relative enstrophy $Q_r > 0$, there is a positive finite size N critical temperature depending on Ω and Q_r ,

$$T_c(\Omega, Q_r; N) = \frac{2\pi\Omega I_- - \frac{\varepsilon z}{8\pi} N Q_r}{-S_{ms}(Q_r)} > 0 \quad (76)$$

where

$$-\infty < I_- \equiv \min_{s_0^\pm} \frac{1}{N} \left(s_0^+ \sum_{j=1}^{N/2} \cos \theta_j^+ + s_0^- \sum_{k=1}^{N/2} \cos \theta_k^- \right) < 0, \quad (77)$$

provided the spin Ω is large enough relative to Q_r , that is,

$$\Omega > \Omega_+(Q_r) = \frac{\varepsilon z}{16\pi^2 I_-} N Q_r \quad (78)$$

which converges to a finite limit as $N \rightarrow \infty$. The minimum in I_- (77) is taken over all possible orientations of the hemispheres associated with s_0^\pm , and by (57),

$$I_- = O(\sqrt{Q_r}).$$

In summary, (a) for $T > T_c(N) > 0$, the mixed state $v = m$ is preferred over any unmixed state $v = n_u, n_d$ and (b) for $0 < T < T_c(N)$, the unmixed state $v = n_d$ satisfying (75) is preferred over the mixed state $v = m$; (b) in the nonextensive continuum limit as $N \rightarrow \infty$, the positive finite size N critical temperature

$$T_c(\Omega, Q_r; N) \rightarrow T_c(\Omega, Q_r) < \infty$$

and moreover, if

$$\Omega > \Omega_+(Q_r) = \frac{\varepsilon z}{16\pi^2 I_-} N Q_r$$

holds, then

$$T_c(\Omega, Q_r) > 0.$$

Remark 4: Thus, there is a positive temperature phase transition if the planetary spin is large enough.

6.3.2. Negative temperature. An extension of Planck's theorem to negative temperature states that for $T < 0$, the state with maximum free energy is preferred. Thus, we compare the per site free energies of the mixed $v = m$ and unmixed states $v = n_u$ and n_d :

$$f_m(\max) = -\frac{T}{N} S_{ms}, \quad (79)$$

$$f_n(\max) = -\frac{\varepsilon z}{8\pi} Q_r + \frac{2\pi\Omega}{N^2} \left(s_0^+ \sum_{j=1}^{N/2} \cos \theta_j^+ + s_0^- \sum_{k=1}^{N/2} \cos \theta_k^- \right). \quad (80)$$

For $T < 0$,

$$-\frac{\varepsilon z}{8\pi} Q_r + \frac{2\pi\Omega}{N^2} \left(s_0^+ \sum_{j=1}^{N/2} \cos \theta_j^+ + s_0^- \sum_{k=1}^{N/2} \cos \theta_k^- \right) > -\frac{T}{N} S_{ms} > 0 \quad (81)$$

implies that the unmixed states are preferred. Solving this inequality for the finite size N critical temperature we get

$$T_c^-(\Omega, Q_r; N) = \frac{2\pi\Omega I_+ - \frac{\varepsilon z}{8\pi} N Q_r}{-S_{ms}} < 0 \quad (82)$$

where

$$\infty > I_+ \equiv \max_{s_0^\pm} \frac{1}{N} \left(s_0^+ \sum_{j=1}^{N/2} \cos \theta_j^+ + s_0^- \sum_{k=1}^{N/2} \cos \theta_k^- \right) = -I_- > 0 \quad (83)$$

and

$$I_+ = O(\sqrt{Q_r}).$$

In this case when (c) $T < T_c^-(N) < 0$, the mixed state $v = m$ is preferred over any aligned unmixed state $v = n_u$ and thus, over any anti-aligned unmixed state $v = n_d$ as well since the left hand side of (81) evaluated at $v = n_d$ and $v = n_u$ satisfies

$$\left[-\frac{\varepsilon z}{8\pi} Q_r + \frac{2\pi\Omega}{N^2} \left(s_0^+ \sum_{j=1}^{N/2} \cos \theta_j^+ + s_0^- \sum_{k=1}^{N/2} \cos \theta_k^- \right) \right] (n_d) <$$

$$\left[-\frac{\varepsilon z}{8\pi} Q_r + \frac{2\pi\Omega}{N^2} \left(s_0^+ \sum_{j=1}^{N/2} \cos \theta_j^+ + s_0^- \sum_{k=1}^{N/2} \cos \theta_k^- \right) \right] (n_u) < -\frac{T}{N} S_{ms}.$$

When (d) $T_c^-(N) < T < 0$, the aligned unmixed state $v = n_u$ is preferred over $v = m$. It is important to note that a negative finite size N critical temperature $T_c^-(\Omega, Q_r; N) < 0$ exists for any spin $\Omega > 0$, unlike the positive finite size critical temperature $T_c(N)$ which exists only for spins satisfying (78).

Remark 5: Under the nonextensive continuum limit $N \rightarrow \infty$,

$$T_c^-(\Omega, Q_r; N) \rightarrow T_c^-(\Omega, Q_r) < \infty \quad (84)$$

$$T_c^-(\Omega, Q_r) < 0, \quad (85)$$

because in (82), the denominator is negative by definition and does not depend on N , and the numerator tends to a finite positive limit by virtue of Remark 3. This implies that there is a negative temperature phase transition for all values of the planetary spin.

Since the numerator in (82) contains the positive term $-\frac{\varepsilon z}{8\pi} N Q_r$ (for $\varepsilon < 0$), we can define a second negative critical temperature

$$T_c^{--}(\Omega, Q_r; N) = \frac{2\pi\Omega I_- - \frac{\varepsilon z}{8\pi} N Q_r}{(-S_{ms})} < 0 \quad (86)$$

provided the spin $\Omega > 0$ is small enough, that is,

$$\Omega < \Omega_+(Q_r) = \frac{\varepsilon z}{16\pi^2 I_-} N Q_r. \quad (87)$$

We note that $T_c^{--}(\Omega, Q_r; N) < 0$ and $T_c(N) > 0$ in (76) are the same expression corresponding to the two sides of the equality in (87). In this case when (e) $T < T_c^{--}(N) < 0$, the mixed state $v = m$ is preferred over the anti-aligned unmixed state $v = n_d$ and when (f) $T_c^{--}(N) < T < 0$, the anti-aligned unmixed state $v = n_d$ is preferred over $v = m$.

Comparing (82) to (86), we deduce that when the spin Ω satisfies (87),

$$T_c^-(N) < T_c^{--}(N) < 0.$$

By comparing the per site free energies of unmixed states $v = n_u$ and n_d when $T < 0$ (more specifically $T_c^- < T < T_c^{--} < 0$ and $T_c^- < T_c^{--} < T < 0$),

$$f_n^u(\max) = -\frac{\varepsilon z}{8\pi}Q_r + \frac{2\pi\Omega}{N}I_+ \quad (88)$$

$$f_n^d(\max) = -\frac{\varepsilon z}{8\pi}Q_r + \frac{2\pi\Omega}{N}I_-, \quad (89)$$

we deduce at once that the aligned unmixed state $v = n_u$ is always preferred over the anti-aligned unmixed state $v = n_d$ for $T < 0$.

Remark 6: Under the nonextensive continuum limit $N \rightarrow \infty$,

$$T_c^{--}(N) \rightarrow T_c^{--} < \infty$$

and

$$T_c^- < T_c^{--} < 0$$

provided the planetary spin is small enough, that is,

$$\Omega < \Omega_+(Q_r) = \frac{\varepsilon z}{16\pi^2 I_-} N Q_r.$$

Collecting together the above computations, we summarize the phase transitions in this simple mean field theory for the coupled barotropic flows - rotating sphere system:

Theorem 6: (A) for large enough spins $\Omega > 0$, the anti-aligned unmixed state $v = n_d$ changes into the mixed state $v = m$ at $T_c(\Omega, Q_r) > 0$, (B) this mixed state $v = m$ continues to be preferred for all positive $T > T_c$ and negative $T < T_c^-$, changing to the aligned unmixed state $v = n_u$ at $T_c^-(\Omega, Q_r) < 0$, and (C) (i) for large enough spins Ω , the unmixed state $v = n_u$ persists for all T such that $T_c^- < T < 0$, and (ii) for small enough spins $\Omega < \Omega_+(Q_r)$, the state $v = n_u$ persists as the preferred state for all T such that $T_c^- < T < 0$, but for T such that $T_c^{--} < T < 0$, the state $v = n_d$ is preferred over $v = m$ but not $v = n_u$.

In the nonextensive continuum limit, the mean field critical temperatures of the energy-relative enstrophy theory for this problem:

$$\infty > T_c(\Omega, Q_r) = \lim_{N \rightarrow \infty} \frac{2\pi\Omega I_- - \frac{\varepsilon z}{8\pi} N Q_r}{(-S_{ms})} \quad (90)$$

$$= \frac{\min_{w, Q_r} (\int_{S^2} dx \Omega C \psi_{10} G[w(x')]) + \frac{Q_r}{4}}{(-S_{ms})} \quad (91)$$

$$= \frac{\frac{1}{2}\Omega C \sqrt{Q_r} - \frac{Q_r}{4}}{S_{ms}} > 0 \quad (92)$$

precisely when

$$\Omega > \Omega_+^\infty(Q_r) = \frac{\sqrt{Q_r}}{2C} \quad (93)$$

$$> 0; \quad (94)$$

and for all planetary spins $\Omega > 0$,

$$-\infty < T_c^-(\Omega, Q_r) = \lim_{N \rightarrow \infty} \frac{2\pi\Omega I_+ - \frac{\varepsilon z}{8\pi} N Q_r}{(-S_{ms})} \quad (95)$$

$$= \frac{\max_{w, Q_r} \left(\int_{S^2} dx \Omega \cos \theta(x) G[w(x')] \right) + \frac{Q_r}{4}}{(-S_{ms})} \quad (96)$$

$$= \frac{-\frac{1}{2}\Omega C \sqrt{Q_r} - \frac{Q_r}{4}}{S_{ms}} < 0, \quad (97)$$

where I^\pm are given by (83) and (77), $C = \sqrt{\int_{S^2} dx \cos^2 \theta}$, the \min_{w, Q_r} (resp. \max_{w, Q_r}) is taken over all relative vorticity $w(x)$ with fixed relative enstrophy Q_r and the total entropy is

$$S_{ms} \equiv -4\pi k_B \ln 2 > 0.$$

Proof. Using the definitions of the interaction energy scale $\varepsilon(N)$ and the size $z(N)$ of the interaction neighborhood, we deduce that the term $-\frac{\varepsilon z}{8\pi} N Q_r$ in the numerator of $T_c(N)$, being the finite dimensional representation of the per site spin-spin interaction internal energy in the unmixed states $v = n_u$ or n_d (as opposed to the spin- Ω interaction term), tends to

$$-\frac{Q_r}{2} \int_{S^2} dx \psi_{10}(x) \int_{S^2} dx' \psi_{10}(x') \ln |1 - x \cdot x'| = \frac{Q_r}{4} > 0. \quad (98)$$

Thus, from the definition of the minimum I_- , and the derivation of the spin-lattice models H_N from the rest frame pseudo kinetic energy H of the coupled barotropic flows, the finite N critical temperature

$$T_c(\Omega, Q_r; N) = \frac{2\pi\Omega \min_{s_0^\pm} \frac{1}{N} \left(s_0^+ \sum_{j=1}^{N/2} \cos \theta_j^+ + s_0^- \sum_{k=1}^{N/2} \cos \theta_k^- \right) - \frac{\varepsilon z}{8\pi} N Q_r}{(-S_{ms})}$$

$$\rightarrow T_c(\Omega, Q_r) = \frac{\min_{w, Q_r} \left(\int_{S^2} dx \Omega \cos \theta(x) G[w(x')] \right) - \frac{Q_r}{4}}{(-S_{ms})}.$$

where the denominator does not depend on N . The minimum in the numerator is taken over all relative vorticity $w(x)$ with fixed relative enstrophy Q_r , and is a well-defined finite negative quantity that is proportional to Ω and to $\sqrt{Q_r}$ -

$$\min_{w, Q_r} \left(\int_{S^2} dx \Omega C \psi_{10} G[w(x')] \right) = \int_{S^2} dx \Omega C \psi_{10} G[\sqrt{Q_r} \psi_{10}] \quad (99)$$

$$= -\frac{1}{2} \Omega C \sqrt{Q_r} \int_{S^2} dx \psi_{10}^2 \quad (100)$$

$$= -\frac{1}{2} \Omega C \sqrt{Q_r} \quad (101)$$

Since

$$\Omega_+(Q_r) = \frac{\varepsilon z}{16\pi^2 I_-} N Q_r \quad (102)$$

$$\rightarrow \Omega_+^\infty(Q_r) = \frac{-Q_r}{\min_{w, Q_r} \left(\int_{S^2} dx 4C \psi_{10} G[w(x)] \right)} > 0, \quad (103)$$

when $\Omega > \Omega_+^\infty(Q_r)$, the mean field critical temperature

$$T_c(\Omega, Q_r) > 0.$$

A similar argument based on

$$\max_{w, Q_r} \left(\int_{S^2} dx \Omega C \psi_{10} G[w(x')] \right) = \int_{S^2} dx \Omega C \psi_{10} G[-\sqrt{Q_r} \psi_{10}] \quad (104)$$

$$= \frac{1}{2} \Omega C \sqrt{Q_r} \quad (105)$$

proves the existence of the continuum limit of T_c^- to conclude the proof. \square

It is useful to restate and separate the above phase transitions into two categories depending on planetary spin.

Theorem 7: *When the planetary spin $\Omega < \Omega_+(Q_r)$, (i) there is no positive temperature phase transition and the mixed state $v = m$ is preferred for all $T > 0$, (ii) there is a negative temperature phase transition at $T_c^- < 0$ (which exists irrespective of the value of planetary spin), where the preferred mixed state $v = m$ for all $T < T_c^- < 0$, changes into the aligned state $v = n_u$ which is preferred over both $v = m$ and $v = n_d$ for all $T_c^- < T < 0$, and (iii) there is a secondary transition at the hotter temperature $T_c^{--} < 0$ (that is, $T_c^- < T_c^{--}$), where the intermediate state $v = n_d$ changes place with $v = m$ in order of thermal preference. Letting \prec denote ‘has smaller free energy than’, we summarize the state preference for the case $\Omega < \Omega_+(Q_r)$:*

$$n_d \prec n_u \prec m \text{ for } T < T_c^- < 0 \quad (106)$$

$$n_d \prec m \prec n_u \text{ for } T_c^- < T < T_c^{--} \quad (107)$$

$$m \prec n_d \prec n_u \text{ for } T_c^{--} < T < 0. \quad (108)$$

When the planetary spin $\Omega > \Omega_+(Q_r)$, (iv) there is a positive critical temperature $T_c(\Omega, Q_r)$ given by (76) at which the preferred state changes from $v = n_d$ for $0 < T < T_c$ to $v = m$ for all $T > T_c$ and all negative $T < T_c^- < 0$, and (v) there is a negative critical temperature $T_c^- < 0$ that exists irrespective of the value of the planetary spin, at which the preferred state changes from $v = m$ to $v = n_u$ for all negative $T > T_c^-$. We summarize the state preference for the case $\Omega > \Omega_+(Q_r)$:

$$n_d \prec n_u \prec m \text{ for } T < T_c^- < 0 \quad (109)$$

$$n_d \prec m \prec n_u \text{ for } T_c^- < T < 0. \quad (110)$$

7. Conclusion. A related approach used to derive nonlinear stability properties of steady-states in the coupled barotropic flow - rotating sphere system is a variational formulation that is based on extremizing the rest frame kinetic energy of flow for fixed relative enstrophy [10], [13]. This approach is the dual of the Minimum Enstrophy method where enstrophy is extremized under fixed kinetic energy without fixing angular momentum. The Minimum Enstrophy Method is in turn related to the dynamical asymptotic principle of Selective Decay for damped 2D turbulence which states that the quotient of enstrophy to energy tends to a minimum in time. Amongst many others, the Minimum Enstrophy methods with fixed angular momentum were first used by Leith [4], Young [2] and Prieto and Schubert [24] in geophysical flows. The main result in [13] gives precise necessary and sufficient conditions for the existence and nonlinear stability of prograde and retrograde solid-body flows in the coupled barotropic fluid - rotating solid sphere system in terms of the planetary spin, relative enstrophy and rest frame kinetic energy. These conditions have been useful also as starting points in Monte-Carlo simulations of

the spherical model for barotropic flows in [14] and will be compared to the results of the mean field theory here as well as the results in [15].

Results from all three methods are closely correlated with recent rigorous results on the nonlinear stability of steady-states of barotropic flows [10], [13] and results from direct numerical simulations of a pde closely related to the coupled barotropic fluid - rotating solid sphere system [5]. The relationship between nonlinear stability and statistical equilibrium properties of geophysical flows is however, not new. Carnevale and Frederiksen discussed this relationship in their paper [35], and Shepherd et al [19], [20], [21] are good references on applications of the energy-Casimir method in geophysical flows. In particular, Shepherd et al [1] have found convincing arguments for zonal anisotropy in rapidly rotating barotropic flows (and the lack of such in slowly or non-rotating flows), in the sense that, the Arnold stability theorems used to prove the nonlinear stability of zonal basic flows, fail when the planetary spin is small.

In conclusion and comparison with the extensive Monte-Carlo simulation results on the same problems (cf. Ding and Lim [14]), the numerical asymptotic results for the dissipative coupled barotropic fluid - rotating solid sphere system in Cho and Polvani [5] and the variational results in Lim [13], we observe that (1) *the mixed state $v = m$ is preferred for all positive temperature except when the planetary spin is large compared to relative enstrophy, whence the counter-rotating unmixed state $v = n_d$ arises at low enough positive temperatures*, (2) *the mixed state $v = m$ is preferred for negative temperatures that are more negative than $T_c^- < 0$ (when the system has intermediate kinetic energies) while the pro-rotating state $v = n_u$ is preferred for negative temperatures hotter or less negative than T_c^- (when the system has extremely high kinetic energies)*, (3a) *the positive critical temperature T_c increases with planetary spin Ω and (3b) the negative critical temperature T_c^- decreases linearly in the planetary spin Ω . Its dependence on relative enstrophy is more complex, being a sum of two terms, one of which is linear in Q_r and other is proportional to $\sqrt{Q_r}$; an increase in Q_r results in a decrease in T_c^- , that is, T_c^- becomes more negative and less hot.*

Given the accepted temperature convention that the more negative $T < 0$ is less hot (or less energetic), and negative temperatures are hotter (or more energetic) than positive ones, we note that the Monte-Carlo results of Ding and Lim [14], the dynamic simulation results of Cho and Polvani [5] and the variational steady state results of Lim [13] are all consistent with the mean field results in this paper and [15]. Moreover, they complement each other.

In particular, when T is negative and hotter than $T_c^-(\Omega, Q_r)$ (or have relatively high kinetic energy), the prograde solid-body state $v = n_u$ is preferred over both the mixed state $v = m$ and the retrograde solid-body state $v = n_d$, according to result (2) in our mean field theory; according to simulations [14] of the statistical equilibrium at very hot negative temperatures, the prograde solid-body flow is the most probable vortex state; according to the numerical results on the dissipative barotropic flows in Cho (cf. fig. 8 in [5]), there is a robust relaxation to the prograde solid-body state when the planetary spin Ω is not too large for a given range of flow kinetic energy, which is consistent with result (3b) that T_c^- decreases linearly in the planetary spin Ω ; and finally according to the steady-state / nonlinear stability results in Lim [13], the prograde solid-body state is allowed only when the planetary spin is small compared to the kinetic energy, and in this case, it is nonlinearly stable (and hence observable).

According to result (1) in our mean field theory, the mixed vortex state $v = m$ is preferred for all positive temperatures, unless planetary spin is large compared to relative enstrophy, in which case, the retrograde solid-body state $v = n_d$ is preferred over both the mixed state and the prograde solid-body state $v = n_u$ when positive T is lower than $T_c(\Omega, Q_r)$. According to the MC-simulator results in Ding and Lim [14], the statistical equilibria of the barotropic flows relaxes to the retrograde solid-body state for relatively low positive temperatures and large planetary spins; this is consistent with our result (3a) that T_c increases with the planetary spin Ω . According to the numerical results on the dissipative coupled barotropic fluid - rotating solid sphere system in Cho (cf. fig. 9 in [5]), the asymptotic state reached when planetary spin is relatively large, is dominated by large anti-cyclonic polar vortices; allowing for the fact that the large polar vortex can be expressed in the form of a superposition of mainly the retrograde solid-body state and some zonally symmetric spherical harmonics of low order, their numerical result is partially consistent with our results (1) and (3a). Finally, according to the steady-state / nonlinear stability results in Lim [13], the retrograde solid-body state arises for all values of the planetary spin and relative enstrophy, but it is nonlinearly stable (and hence observable) only when the planetary spin is large compared to relative enstrophy; this is consistent with our mean field results.

8. Appendix: non-divergent vs divergent flows. The non-divergent case of the coupled barotropic fluid - rotating solid sphere system in this paper is much simpler to treat than the divergent case which is of course more realistic. This more realistic model is essentially based on coupling the shallow-water equations (coupled SWE) model to a rotating solid sphere through a complex torque mechanism that allows the divergent fluid in this case to exchange energy as well as angular momentum with the sphere. We will summarize aspects of the coupled SWE in order to state explicitly the approximations in the coupled barotropic fluid - rotating solid sphere system. For this purpose let us denote by U , L and H , the velocity, length and depth scales respectively. Then two important dimensionless numbers are the Rossby and Froude numbers respectively,

$$R = \frac{U}{2\Omega L}, \quad F = \frac{U}{\sqrt{gH}},$$

where g is the gravitational constant. Within the coupled SWE model, the relative importance of convective / local flow to rotational effects is measured by the Rossby number R . The Froude number F measures the relative importance within the coupled SWE model of the convective / local flow effects to gravity-depth effects. In a definite sense, a small Rossby number $R \ll 1$ signals the importance of rotational effects: Ω has to be relatively large or the scale L of the flow has to be relatively large in order for rotation of the planet to be important. On the other hand, a large Froude number $F \gg 1$ implies the importance of gravity effects, since in this case, the gravity waves are relatively slow, and cannot be time-averaged out of the problem.

The Rossby Radius of deformation,

$$L_R = \frac{\sqrt{gH}}{2\Omega} = \frac{RL}{F},$$

measures the relative importance of gravity-depth effects to rotational effects. When it is of $O(1)$, both gravity and rotational effects are relevant to the problem, and

only when $L_R \gg 1$ that rotational and convective or inertial effects dominate. It is convenient to label the square of L_R/L by the Burgers number

$$B = \frac{R^2}{F^2} = \frac{L_R^2}{L^2}.$$

Small values of B signals the importance of gravity-depth effects over rotational effects; it includes the case when the Rossby number R itself is relatively small, that is, when rotational effects dominate convective or inertial effects, as well as the case when R is $O(1)$, that is, when rotational effects are comparable to convective or inertial effects.

The coupled barotropic fluid - rotating solid sphere system can be characterized as the limit of the coupled SWE when L_R tends to ∞ or equivalently when the depth scale H tends to ∞ with Ω and g fixed. The flow (u, v) in the coupled barotropic fluid - rotating solid sphere system model is strictly 2D, that is, ω is a scalar field and the top and bottom boundary conditions are idealized away by taking, in effect, the depth scale H to ∞ . Thus, in a definite sense, the coupled barotropic fluid - rotating solid sphere system models a rotating fluid of infinite depth. This fact partly accounts for its tractability relative to the more complex coupled SWE model where boundary conditions at the top and bottom of the fluid are retained. In this definite sense, the coupled barotropic fluid - rotating solid sphere system model is non-divergent, i.e., $\text{div}(u, v) = 0$, while the coupled SWE is a divergent model.

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