Hwks
LA 20.
Homework assignments are due in grader's box (email: sunj8@rpi.edu) in AE 301 by Friday 6pm one week after. Grader will grade those circled in red but you are responsible for both red and blue homework problems in tests.

All one hr tests are closed book and there will be no make-ups without written permission. Test 1 will be at end of week 4 (this week counts as wk 1).

Homeworks are posted on my webpage:

www.rpi.edu/~limc before Friday classes.

Office hrs: Weds 11-1pm or by appointment.
1.31 The number $-1$ times a vector

$(-1)v = -v$ for every $v \in V$.

**Proof**  For $v \in V$, we have

$$v + (-1)v = 1v + (-1)v = (1 + (-1))v = 0v = 0.$$  

This equation says that $(-1)v$, when added to $v$, gives 0. Thus $(-1)v$ is the additive inverse of $v$, as desired.

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**EXERCISES 1.B**

1. Prove that $-(-v) = v$ for every $v \in V$.

2. Suppose $a \in F$, $v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

3. Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

4. The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in 1.19. Which one?

5. Show that in the definition of a vector space (1.19), the additive inverse condition can be replaced with the condition that

$$0v = 0$$

for all $v \in V$.

Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of $V$. (The phrase “a condition can be replaced” in a definition means that the collection of objects satisfying the definition is unchanged if the original condition is replaced with the new condition.)

6. Let $\infty$ and $-\infty$ denote two distinct objects, neither of which is in $\mathbb{R}$. Define an addition and scalar multiplication on $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbb{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases}$$

$$t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

$$t + \infty = \infty + t = \infty,$$  

$$t + (-\infty) = (-\infty) + t = -\infty,$$  

$$\infty + \infty = \infty,$$  

$$(-\infty) + (-\infty) = -\infty,$$  

$$\infty + (-\infty) = 0.$$  

Is $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ a vector space over $\mathbb{R}$? Explain.
Sums of subspaces are analogous to unions of subsets. Similarly, direct sums of subspaces are analogous to disjoint unions of subsets. No two subspaces of a vector space can be disjoint, because both contain \(0\). So disjointness is replaced, at least in the case of two subspaces, with the requirement that the intersection equals \(\{0\}\).

The result above deals only with the case of two subspaces. When asking about a possible direct sum with more than two subspaces, it is not enough to test that each pair of the subspaces intersect only at \(0\). To see this, consider Example 1.43. In that nonexample of a direct sum, we have \(U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_3 = \{0\}\).

**EXERCISES 1.C**

1. For each of the following subsets of \(F^3\), determine whether it is a subspace of \(F^3\):

   (a) \(\{(x_1, x_2, x_3) \in F^3 : x_1 + 2x_2 + 3x_3 = 0\}\);
   (b) \(\{(x_1, x_2, x_3) \in F^3 : x_1 + 2x_2 + 3x_3 = 4\}\);
   (c) \(\{(x_1, x_2, x_3) \in F^3 : x_1x_2x_3 = 0\}\);
   (d) \(\{(x_1, x_2, x_3) \in F^3 : x_1 = 5x_3\}\).

2. Verify all the assertions in Example 1.35.

3. Show that the set of differentiable real-valued functions \(f\) on the interval \((-4, 4)\) such that \(f'(-1) = 3f(2)\) is a subspace of \(R^{(-4,4)}\).

4. Suppose \(b \in R\). Show that the set of continuous real-valued functions \(f\) on the interval \([0, 1]\) such that \(f_0^1 f = b\) is a subspace of \(R^{[0,1]}\) if and only if \(b = 0\).

5. Is \(R^2\) a subspace of the complex vector space \(C^2\)?

6. (a) Is \(\{(a, b, c) \in R^3 : a^3 = b^3\}\) a subspace of \(R^3\)?
(b) Is \(\{(a, b, c) \in C^3 : a^3 = b^3\}\) a subspace of \(C^3\)?

7. Give an example of a nonempty subset \(U\) of \(R^2\) such that \(U\) is closed under addition and under taking additive inverses (meaning \(-u \in U\) whenever \(u \in U\)), but \(U\) is not a subspace of \(R^2\).

8. Give an example of a nonempty subset \(U\) of \(R^2\) such that \(U\) is closed under scalar multiplication, but \(U\) is not a subspace of \(R^2\).
9 A function $f : \mathbb{R} \to \mathbb{R}$ is called periodic if there exists a positive number $p$ such that $f(x) = f(x + p)$ for all $x \in \mathbb{R}$. Is the set of periodic functions from $\mathbb{R}$ to $\mathbb{R}$ a subspace of $\mathbb{R}^\mathbb{R}$? Explain.

10 Suppose $U_1$ and $U_2$ are subspaces of $V$. Prove that the intersection $U_1 \cap U_2$ is a subspace of $V$.

11 Prove that the intersection of every collection of subspaces of $V$ is a subspace of $V$.

12 Prove that the union of two subspaces of $V$ is a subspace of $V$ if and only if one of the subspaces is contained in the other.

13 Prove that the union of three subspaces of $V$ is a subspace of $V$ if and only if one of the subspaces contains the other two.

14 Verify the assertion in Example 1.38.

15 Suppose $U$ is a subspace of $V$. What is $U + U$?

16 Is the operation of addition on the subspaces of $V$ commutative? In other words, if $U$ and $W$ are subspaces of $V$, is $U + W = W + U$?

17 Is the operation of addition on the subspaces of $V$ associative? In other words, if $U_1, U_2, U_3$ are subspaces of $V$, is

$$(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)?$$

18 Does the operation of addition on the subspaces of $V$ have an additive identity? Which subspaces have additive inverses?

19 Prove or give a counterexample: if $U_1, U_2, W$ are subspaces of $V$ such that

$$U_1 + W = U_2 + W,$$

then $U_1 = U_2$.

20 Suppose

$$U = \{(x, x, y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}.$$

Find a subspace $W$ of $\mathbb{F}^4$ such that $\mathbb{F}^4 = U \oplus W$. 

This exercise is surprisingly harder than the previous exercise, possibly because this exercise is not true if we replace $\mathbb{F}$ with a field containing only two elements.]