A Classical version of the Einstein-de Haas Effect

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The Einstein-de Haas effect is a striking example of macroscopic manifestation of microscopic quantum phenomena, in this case magnetization of electron orbital and intrinsic spins in a ferromagnetic rod by an external magnetic field that leads to a net global angular momentum having to be compensated by an opposing twist in the rod. This letter gives a system consisting of a rotating fluid layer coupled to a massive sphere which displays a classical version of the Einstein-de Haas effect, is simple enough to be solved exactly for phase transitions, and is relevant to some large-scale phenomena in planetary atmospheres.

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When a ferromagnetic rod suspended by a thread is magnetized it twists because of the ordering of electron spins and / or orbital momenta into a macroscopic angular momentum. The compensation of this macroscopic angular momentum of the electrons by the solid substrate (rod) is the well-known Einstein-de Haas effect [2]. A change of magnetization states is therefore prohibited by conservation law unless a substantial solid / elastic substrate can absorb the resulting change in angular momentum [2]. The main aim here is to demonstrate an entirely classical version of this phenomena. This note describes the exact solution of a family of classical spin-lattice models, not related to electron spins or explicitly to any quantum systems, which exhibits a classical version of this phenomena. This note describes intuitively - and also rigorously through the explicit transformations given below - this coupled system can be viewed as classical 2D fluid motion restricted to the surface of a solid sphere with which it exchanges energy and angular momentum. The CEDH effect then arises in the form of spontaneous organization of random local vorticity of the fluid (with initial total angular momentum set to zero) into a macroscopic rotational flow with nonzero angular momentum, compensated according to conservation of angular momentum, by an opposing rotation of the solid sphere. This critical phenomenon was first discovered in Monte-Carlo simulations of the lattice model given below in equations (1), (3) and (4) [6], [5]. Robust condensation of random vorticity in this coupled system into a global spin with net angular momentum was later confirmed by mean field methods in [7]. A picture of the self-organized vorticity state that arises consistently at the highest energies is given in figure 1.

Denoting by $\gamma_{jk}$ the angle subtended at the center of the unit sphere $S^2$ by the lattice sites $x_j, x_k \in S^2$, we introduce the classical lattice model [11]

$$H_N[S] = -\frac{1}{2} \sum_{j \neq k}^N J_{jk} \vec{s}_j \cdot \vec{s}_k - \vec{h} \cdot \sum_{j=1}^N \vec{s}_j \quad (1)$$

$$J_{jk} = \frac{16\pi^2}{N^2} \ln(1 - \cos \gamma_{jk})$$

$$-\vec{h} = \frac{2\pi}{N} \Omega \vec{n}$$

for the kinetic energy of 2D ideal fluid measured in the fixed frame, where local spins $\vec{s}_j = s_j \vec{n}_j$ represent a discrete approximation of the relative vorticity field with $s_j \in (-\infty, \infty)$ and $\vec{n}_j$ the outward unit normal to the sphere $S^2$ at $x_j$. External field $\vec{h}$ is given by the angular velocity $\Omega \vec{n}$ of a rotating frame in which an infinitely massive sphere is at rest. Here $\vec{n}$ is taken to be the outward unit normal at the north pole of $S^2$, and $S = (s_1,...,s_N)$ is the local spin configuration or microstate.

Without further conditions, this classical lattice model cannot support nonzero magnetization

$$L_N[S] = \frac{4\pi}{N} \sum_{j=1}^N s_j > 0 \quad (2)$$

because - by mapping rotation to an external magnetic field - it is prohibited by the Bohr-van Leeuwen theorem when $s_j \in (-\infty, \infty)$. Quantity $L$ is usually called the magnetization in spin-lattice models but in the context

FIG. 1: Super-rotating ordered vorticity pattern (red = pos) shows hemispheric separation into positive (neg) regions.
of 2D fluid flows as well as in other rotation problems, $\vec{L}$ is the angular momentum of the fluid. It turns out that a natural constraint for the physics of 2D fluid flows is the conservation of enstrophy or square-norm of the vorticity [8, 9, 13]

$$\Gamma_N[S] = \frac{4\pi}{N} \sum_{j=1}^{N} \vec{s}_j \cdot \vec{n}_j = Q < \infty. \tag{3}$$

As this is equivalent to a finite bound on the local spin magnitudes $s_j$, the Bohr-Van Leeuwen theorem is no longer valid. In addition, Stokes theorem implies that it is natural to treat only the case of zero circulation (or neutral charge),

$$\frac{4\pi}{N} \sum_{j=1}^{N} \vec{s}_j \cdot \vec{n}_j = 0. \tag{4}$$

Together with these two constraints we have in $H_N$ a family of classical lattice models constituting a version of Kac’s spherical model with global long-range interactions— the enstrophy is a spherical constraint on the local spins which restricts the domain of the configurational integral in the Gibbs partition function $Z_N(\beta) = \int D[S] e^{-\beta H_N[S]} \delta(\Gamma[S] - Q)$. As will be shown below via the exact solution of the spherical model, this interaction gives rise to a well-defined nonextensive thermodynamic limit.

Next, we derive the lattice energy functional $H_N$ from the rest-frame kinetic energy of 2D ideal fluid flow on $S^2$ in rotating frame variables [3], thus relating it directly to a problem of rotations in classical physics: Expressed in a frame that is rotating at the fixed angular velocity $\Omega \vec{n}$ of the infinitely massive solid sphere, the total kinetic energy of macroscopic flow of a thin layer of incompressible fluid on $S^2$ (nondivergent fluid here means that the fluid velocity relative to the rotating frame has zero horizontal divergence) is

$$H_T[q] = \frac{1}{2} \int_{S^2} dx \left[ (v_x + u_p)^2 + v^2 \right] = \frac{1}{2} \int_{S^2} dx \left[ (u_x^2 + v^2) + v^2 + 2u_p u_p \right] + \frac{1}{2} \int_{S^2} dx \frac{d}{dx} \cos \theta$$

where $u_x$, $v_x$ are the zonal (east-west) and meridional (north-south) components of the relative velocity, $u_p$ is the zonal component of the velocity of the rotating frame (the meridional component being zero), and $\theta, \phi$ are respectively the co-latitude and longitude, being the spherical coordinates in which the area variable of integration $dx = \sin \theta \, d\theta \, d\phi$. Dropping the constant second term $\frac{1}{2} \int_{S^2} dx \frac{d}{dx} (u_p^2)$ (constant because the layer thickness is fixed) and using properties of the Green’s function $G[f](x) = -\frac{1}{2} \int_{S^2} dx' f(x') \ln \frac{1}{|x - x'|}$ of the 2D Poisson-Beltrami equation $\Delta \psi = w$ on $S^2$, the energy functional

$$H[w] = -\frac{1}{2} \int dx \psi(x)w(x) - \Omega \int dx \psi(x) \cos \theta = -\frac{1}{2} \int_{S^2} dx \, wG[w] + \frac{\Omega}{2} \int_{S^2} dx \, w \cos \theta(x);$$

note that $\psi_0(x) = C \cos \theta$ is the spherical harmonic (eigenfunction of $G$) with wavenumber $l = 1$, $m = 0$ and eigenvalue $-1/2$. From standard results in the physics of rotating systems, we see that the second term is the product of the rotation field $\Omega \vec{n}$ and the angular momentum of the fluid in the rotating frame $\int_{S^2} dx \, w(x) \cos \theta(x)$. This is directly mapped to the usual product $\vec{h}$ \cdot $\vec{L}$ of an external magnetic field and the magnetization. Given $N$ fixed mesh points $x_k$ on $S^2$ and the Voronoi cells based on this mesh, we approximate the relative vorticity $w(x)$ by discretizing the vorticity field as a piecewise constant function, $w(x) \approx \sum_{j=1}^{N} s_j H_j(x)$, where $s_j = w(x_j)$ and $H_j(x)$ is the characteristic function for the Voronoi domain $D_j$ containing the site $x_j$. In terms of these piecewise constant approximations of the relative vorticity $w$, the lab frame kinetic energy $H[w]$ (modulo a constant term proportional to the moment of inertia of the fluid layer) takes the form of the classical lattice models $H_N$ in (1) for any finite $N$. Existence of the limits as $N$ tends to $\infty$,

$$\Gamma_N[S] \rightarrow \int_{S^2} dx w^2(x) < \infty \tag{5}$$

and

$$\frac{4\pi}{N} \sum_{j=1}^{N} \vec{s}_j \cdot \vec{n}_j \rightarrow \int_{S^2} dx w(x) = 0, \tag{6}$$

follows.

Using the method of steepest descent which is exact in the nonextensive thermodynamic limit given below, we solve for the spherical model partition function and free energy corresponding to $H_N$ (1) in the case of a stationary and infinitely massive rigid substrate, that is $\Omega = 0$ in $\vec{h}$ and $H_N$. The remaining case of a fluid coupled to an infinitely massive sphere rotating at fixed angular velocity $\Omega > 0$ is technically harder and will be presented in a longer paper. The partition function for the spherical Ising model has the form

$$Z_N(\beta) \propto \int D[S] \exp \left( -\beta H_N[S] \right) \delta \left( \frac{Q}{4\pi} \sum_{j=1}^{N} \vec{s}_j \cdot \vec{s}_j \right)$$

where the integral is a path-integral taken over all the microstates $S$ with zero circulation. Considering the integral as $N \rightarrow \infty$, this partition function can be calculated using Laplace’s integral form,

$$\int D[S] \exp \left( -\beta H_N[S] \right)$$
\[ \left( \frac{1}{2\pi} \int_{a=-\infty}^{a=\infty} d\eta \exp \left( \eta \left( \frac{Q N}{\pi} - \sum_{j=1}^{N} \bar{s}_j \cdot \bar{s}_j \right) \right) \right) \times \left( \frac{1}{2\pi} \int_{a=-\infty}^{a=\infty} d\eta \exp \left( \eta \left( \frac{Q N}{\pi} - \sum_{j=1}^{N} \bar{s}_j \cdot \bar{s}_j \right) \right) \right) \times \left( \frac{1}{2\pi} \int_{a=-\infty}^{a=\infty} d\eta \exp \left( \eta \left( \frac{Q N}{\pi} - \sum_{j=1}^{N} \bar{s}_j \cdot \bar{s}_j \right) \right) \right) \]

In terms of the eigenvalues \( \lambda_{l,m} = \frac{1}{l(l+1)}, \quad l = 1, 2, \cdots, \sqrt{N}, m = -l, -l+1, \cdots, l \) of the Green’s function for the Laplace-Beltrami operator on \( S^2 \), and Fourier amplitudes \( \alpha_{l,m} \) such that \( \bar{s}_j = w(x_j) \approx \sum_{l=1}^{\sqrt{N}} \sum_{m=-l}^{l} \alpha_{l,m} \psi_{l,m}(x_j) \) for each of the lattice sites \( x_j \),

\[
\int D[\alpha] \exp \left( -\beta \sum_{l=1}^{N} \alpha_{l,m}^2 \right)
\times \left( \frac{1}{2\pi} \int_{a=-\infty}^{a=\infty} d\eta \exp \left( \eta N \left( 1 - \frac{4\pi}{Q} \sum_{l=1}^{N} \sum_{m=-l}^{l} \alpha_{l,m}^2 \right) \right) \right)
\times \int_{\Omega \geq 2} D[\alpha] \exp \left( -\sum_{l=2}^{N} \sum_{m=-l}^{l} \left( \frac{\beta N \lambda_{l,m}}{2} + \eta N \frac{4\pi}{Q} \alpha_{l,m}^2 \right) \right).
\]

Solving explicitly the inner integral - it is the product of a collection of Gaussian integrals - gives

\[
\Pi_{l=2}^{\sqrt{N}} \Pi_{m=-l}^{l} \left( \frac{\pi}{N \eta \frac{4\pi}{Q} + \frac{\beta N}{2} \lambda_{l,m}} \right)^{1/2}
\]

provided

\[
\beta \lambda_{l,m} + \eta \frac{4\pi}{Q} > 0, \quad l = 2, 3, \cdots, \sqrt{N},
\]

\[
m = -l, -l+1, \cdots, 0, \cdots, l
\]

Thus, the partition function becomes

\[
\int_{a=-\infty}^{a=\infty} d\eta \exp \left( N \left[ \eta \left( 1 - \frac{4\pi}{Q} \sum_{m=-1}^{1} \alpha_{l,m}^2 \right) \right] \right)
\times \exp \left( -\frac{1}{2} \sum_{l=2}^{N} \sum_{m=-l}^{l} \log \left( N \eta \frac{4\pi}{Q} + \frac{\beta N}{2} \lambda_{l,m} \right) \right)
\]

which we can cast in a form suitable for the saddle point method or method of steepest descent, \( Z \propto \lim_{N \to \infty} \frac{1}{\sqrt{N}} \int_{a=-\infty}^{a=\infty} d\eta \exp N F(\eta, Q, \beta) \). In the thermodynamic limit as \( \sqrt{N} \to \infty \), the free energy per site - after separating out the 3-fold degenerate ground states \( \psi_{1,0}, \psi_{1,1}, \psi_{1,-1} \) - is, modulo a factor of \(-\beta'\), given by

\[
F(\eta, Q, \beta') = \eta \left( 1 - \frac{4\pi}{Q} \sum_{m=-1}^{1} \alpha_{l,m}^2 \right) - \frac{\beta'}{2} \sum_{m=-1}^{1} \lambda_{l,m} \alpha_{l,m}^2
\]

\[
- \frac{1}{2N} \sum_{l=2}^{N} \sum_{m=-l}^{l} \log \left( N \eta \frac{4\pi}{Q} + \frac{\beta N}{2} \lambda_{l,m} \right).
\]

The saddle point condition is

\[
0 = \frac{\partial F}{\partial \eta} = \left( 1 - \frac{4\pi}{Q} \sum_{m=-1}^{1} \alpha_{l,m}^2 \right)
\]

\[
- \frac{2\pi \sqrt{\frac{Q}{\beta N}} \sum_{m=-l}^{l} \left( N \eta \frac{4\pi}{Q} + \frac{\beta N}{2} \lambda_{l,m} \right)^{-1}}{Q}.
\]

To close the system we need a set of three additional constraints given by the equations of state for \( m = -1, 0, 1 \)

\[
0 = \frac{\partial F}{\partial \alpha_{l,m}} = \left( \frac{8\pi \eta}{Q} + \beta \lambda_{1,m} \right) \alpha_{l,m}
\]

which have solutions

\[
\alpha_{1,m} = 0 \quad \text{or} \quad \frac{8\pi \eta}{Q} + \beta \lambda_{1,m} = 0, \quad \text{for each} \; m.
\]

This means that in order to have nonzero amplitudes in at least one of the ground or condensed modes, which are the only ones to have angular momentum,

\[
\frac{4\pi \eta}{\Omega} = -\frac{\beta'}{4}
\]

which implies that the inverse temperature must be negative, \( \beta' < 0 \). Moreover, the Gaussian condition on the modes with \( l = 2 \) namely \( \frac{\beta'}{12} - \frac{\beta'}{2} > 0 \) can only be satisfied by a negative temperature, \( \beta' < 0 \), when there is any energy in the angular momentum containing ground modes.

Substituting this nonzero solution \( \alpha_{1,m} \neq 0 \) into the saddle point equation yields an explicit equation for the negative critical temperature

\[
0 = \left( 1 - \frac{4\pi}{Q} \sum_{m=-1}^{1} \alpha_{l,m}^2 \right) - \frac{4\pi}{Q N} \sum_{l=2}^{N} \sum_{m=-l}^{l} \left( \lambda_{l,m} - \frac{1}{2} \right)^{-1}
\]

\[
= \left( 1 - \frac{4\pi}{Q} \sum_{m=-1}^{1} \alpha_{l,m}^2 \right) - \frac{T}{T_{c}^{2}}
\]

which proves that \( k_B T_c = (\beta')^{-1} \) has a finite limit as \( N \) tends to \( \infty \), and is inversely proportional to the enstrophy \( Q \), equivalently for any finite \( N \),

\[
-\infty < \beta' = \frac{4\pi}{\Omega N} \sum_{l=2}^{N} \sum_{m=-l}^{l} \left( \lambda_{l,m} - \frac{1}{2} \right)^{-1} < 0.
\]
The saddle point equation also allows us to compute the equilibrium amplitudes of the ground modes for temperatures $T$ that are hotter than the negative critical temperature $T_c$, that is for temperatures $T$ so that $T_c < T < 0$,

$$
\sum_{m=-1}^{1} \alpha_{1,m}^2(T) = \frac{\Omega}{4\pi} \left( 1 - \frac{T}{T_c} \right) \tag{14}
$$

This argument shows that at positive temperatures, there cannot be any energy in the solid-body rotating modes and there is no phase transition at positive temperatures. In summary we have found a problem from classical rotational physics whose lattice representation exhibit self-organization of initially random vortical energy into a macroscopic coherent flow at very high energies in the form of symmetry-breaking Goldstone modes. These extremely high energy condensed modes carry a nonzero angular momentum or magnetization $\vec{L}[\text{group}] = \lim_{N \to \infty} \frac{\vec{L}}{N}$ that can be directed along an arbitrary axis while initial vorticity states have zero global angular momentum, as is confirmed in numerous Monte-Carlo simulations of this problem in Lim and Nebus [6] and Ding and Lim [5]. Moreover, these condensed modes correspond to nearly rigid rotation of the fluid layer about an axis in the direction of $\vec{L}$. The main mathematical results of this note are contained in the expressions (12) and (14) for respectively the negative critical temperature $T_c$ and the amount of energy in the condensed phase. By the law of conservation of angular momentum in an isolated system such as this nondivergent ideal fluid layer coupled to a massive rigid sphere, the extremely energetic end-states’ net angular momentum must be accompanied by a spontaneous and opposite rotation of the massive sphere. In the case of a very massive sphere, the resulting angular velocity of the sphere is very small. If the sphere is anchored but have a high elastic rigidity, then the net angular momentum of the fluid layer will be absorbed by a twisting of the elastic medium [2]. Without the rigid or elastic substrate - that is deprived of a reservoir of angular momentum - the fluid layer like its quantum counterparts, cannot transition to states with net angular momentum or magnetization. Venus’ super-rotating middle atmosphere of heavy CO$_2$ provides a plausible real system in which the massive solid sphere rotates very slowly, once every 240 earth days, and where seasonal variations in angular momentum of the atmosphere, rotating on average 60 times faster, is compensated by opposite, very small changes in the length of the Venusian day. Gas giants Jupiter, Saturn and Uranus also exhibit persistent differential rotations at the equator but this classical model for a thin layer of fluid may not be applicable to deep rotating fluids.

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