Triangularization of finite-banded non-self-adjoint Matrices by Generating Functions Based on $SL(2,\mathbb{R})$

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Abstract

Starting with the spaces of all degree N homogeneous polynomials with real coefficients, in the variables x, y, and the corresponding space of linear differential operators of fixed order m that leave them invariant, we provide a new method based on transformations of the Euclidean plane generated by $SL(2,\mathbb{R})$, to simultaneously triangularize a subset of the associated family of $(N+1)$ by $(N+1)$ non-self-adjoint $(2m+1)-$ banded matrices by explicit solutions of all its eigenvalues and eigenvectors. We prove necessary and sufficient conditions that characterize the triangularizable vector subspaces for each g in $SL(2,\mathbb{R})$ and show in a special case, that constant column sum (a property equivalent to singly stochastic matrices) and linearity in the drift velocity (of the associated stochastic differential equations) are the necessary and sufficient conditions for triangularization by this procedure. For any element in $SL(2,\mathbb{R})$ we show that the corresponding subspace of triangularizable matrices has no less than 6 dimensions in a 9 dimensional vector space. This class of matrices include well known Markov transition operators from physical, biological and sociological applications, many of which are challenging non-self adjoint problems involving non time-reversible stochastic dynamics with absorbing states.

1 Introduction

Triangularization of finitely-banded real non-symmetric matrices with the aim of ultimately solving the spectral problem for these non-self adjoint linear operators continues to be an active field of research in applied mathematics. The closed-form or exact solutions for the eigenvalues and right and left eigenvectors of these non-symmetric matrices of arbitrary dimensions is a nontrivial project. We present in this paper a new method and rigorous results that shows this project can be solved for a large and more to the point, useful class of linear operators. It is well known that the spectral theory of non-self adjoint linear operators presents many significant challenges that are not present for symmetric matrices.

The specific method and similarity transforms introduced here is based on well-defined vector spaces of homogeneous polynomials in two variables and the corresponding class of linear partial differential operators that leave invariant these polynomials. To fix notation we will specify these vector spaces and linear operators in this introductory part of the paper. We show below that the linear operators are related to the class of banded non-symmetric matrices of arbitrary dimensions that we want to solve for eigenvalues and left- and right-eigenvectors. It is natural and we will prove that the method herein is related to a set of transformations of the Euclidean plane generated by the group $SL(2,\mathbb{R})$. We provide details on those banded matrices that can be solved by a typical (and especially familiar) element, namely, the one that generates the transformation $u = x - y$, $v = y$.

In an earlier paper, we applied a variant of this method to a class of second order linear partial differential operators with monomial coefficients in x and y whose degree equals the order of the partial derivative ($m + n = m' + n' = k \leq 2$) which correspond to stochastic matrices associated with urn models [6]. Surprisingly, it turns out that the restriction to stochastic matrices is a necessary condition for triangularizability in the more general context of the current paper, a result that will be proved below. Extensions and several applications of the methods and results in this paper have been motivated by classical problems in random walk and Markov models, including the basic mathematical models of network science such as the voter model, the naming game model, the Bernoulli-Laplace model for diffusion and models of genetics and ecology [4, 5, 7, 6, 3, 1]. In the recent paper [6], we applied a nonlinear extension of the general method for triangularization to explicit solutions of the eigenvalues and both left- and right-eigenvectors. These spectral properties of the Bernoulli-Laplace problem were used to construct new elementary proofs of the classical results of Diaconis et al. [1] on tight bounds for mixing times, that were previously obtained by applications of deep theorems in group representations and Gelfand pairs.

To the $(N+1)$-dimensional vector space $P^{(N)}$ of homogeneous polynomials of degree N, in two variables,

$$G(x,y)=\sum_{i=0}^{N}c_i x^i y^{N-i} \tag{1}$$

there belongs naturally a vector space W of all linear partial differential operators (in x and y), spanned by elements of the form

$$x^m y^n \partial_{x^{m'}}\partial_{y^{n'}}, \tag{2}$$

$$\tag{3}$$

for $m+n=m'+n'=k=0,1,2,\ldots$, where

$$\partial_{x^{m'}}=(\partial_x)^{m'} \tag{4}$$

$$\partial_{y^{n'}}=(\partial_y)^{n'}, \tag{5}$$

that fixes $P^{(N)}$. The discrepancy

$$\Delta m = m - m', \tag{6}$$

will be shown in the next section to correspond to the position of a diagonal (resp. off-diagonal) band in a $(N+1)$ dimensional matrix M_{N+1}. The homogeneous polynomials of degree N form a vector space $P^{(N)}$ with coordinates $\{c_0,...,c_N\}$ which is invariant under the linear operators In this paper we focus on non-symmetric pentadiagonal matrices and leave the general theory for another paper. When $|\Delta m|\leq 2$, we will show that a 14-dimensional matrix space $W_{14}=\{M_{N+1}(\alpha)\mid \alpha\in\mathbb{R}^{14}\}$ of pentadiagonal matrices is equivalent to the above linear partial differential operators, similarly restricted. W_{14} can be viewed as the algebra of all second order linear partial differential operators, $W^{(2)}$ acting on the $(N+1)-$dimensional vector space $P^{(N)}$ by

$$L_\alpha G(x,y)\in P^{(N)},\ \text{for each}\ L_\alpha\in W^{(2)}, \tag{7}$$

where

$$W^{(2)}=span\{x^m y^n\partial_{x^{m'}}\partial_{y^{n'}}|0\leq m+n=m'+n'\leq 2\}, \tag{8}$$

is a vector space parametrized by the 14 real valued components of the vector α that specifies the corresponding operator L_α.

We then find three necessary and sufficient conditions on the vector α characterizing the subclass of W_{14} that is triangularizable by *any* element $g\in SL(2,\mathbb{R})$ into a lower triangular matrix M'_{N+1} which is similar to the original pentadiagonal matrix M_{N+1}. Interestingly, for the special $g\in SL(2,\mathbb{R})$ corresponding to $u=x-y$, these conditions are then shown to be equivalent to a pair of physically meaningful conditions, namely, (a) the column sums of each such triangularizable matrix are a fixed constant dependent on the matrix, and (b) the deterministic drift of the stochastic dynamics governed by these singly stochastic or Markov transition matrices, is linear. By triangularizing the matrix, we calculate all eigenvalues and eigenvectors of the original matrix explicitly for any N, even for potentially large matrices. This therefore provides a method for explicitly solving a family of pentadiagonal non-symmetric matrices. In terms of the original matrix, the partial differential equation method for solving for the eigenvalues and eigenvectors and triangularization is equivalent to a similarity transformation by the upper Pascal matrix as defined in Ref. [2].

The paper is organized as follows: (I) In Sec. 2 we define the class of pentadiagonal matrices associated with the $14-$dimensional vector space $W^{(2)}$ of second order linear partial differential operators, and give necessary and sufficient conditions for the transformation $u=x-y$, $v=y$ to triangularize $W^{(2)}$ which yields an $11-$dimensional subspace $W_{11}<W^{(2)}$ of diagonalizable pentadiagonal matrices or second order linear partial differential operators; (II) In section 2.1, we give the method for finding exact formulae for all eigenvalues and eigenvectors of the 11-dimensional W_{11}; (III) In Sec. 3, we introduce a reduction

of $W^{(2)}$ to purely second order operators which is characterized as a 9−dimensional vector space $W'^{(2)}$, which would allow us to reduce the dimension of W_{14} without loosing generality. Transformations of the Euclidean plane with coordinates x, y given by $g \in SL(2,\mathbb{R})$ are now considered as candidates for the triangularization of $W'^{(2)}$, and the resulting necessary and sufficient conditions for triangularization by any $g \in SL(2,\mathbb{R})$ are given in terms of the nullspaces $N(A,B,C)$ of a three parameters $(A,B,C)(g)$ family of 3 by 9 real matrices.

2 Pentadiagonal Matrices: triangularization, eigenvalues and eigenvectors

Each term (such as $x^2\partial_{xy}$) in the partial differential operator L_α here corresponds to a diagonal or off-diagonal band in a matrix M_{N+1}, according to the discrepancy Δm between the degrees of x in the monomial coefficient and the orders m' in the associated partial derivative $\partial_{x^{m'}y^{n'}}$ where non-negative integers m, n satisfy $m+n=k\leq 2$. For example, $-y^2\partial_{yy}$ corresponds to a diagonal term because the degree of y (resp. x) in the coefficient $-y^2$ equals the order of partial differentiation in y (resp. x) in ∂_{yy}. The term $x^2\partial_{yy}$ corresponds to an off-diagonal band in M_{N+1} since the monomial coefficient x^2 has degree 2 (resp. 0) in x (resp. y),which differs from the order 0 (resp. 2) of differentiation with respect to x (resp. y). This is demonstrated in more detail in Lemma 1 in Sec. 2. For a vector whose $(N+1)$ components are real numbers c_i, we define the homogeneous polynomial G of degree N,

$$G(x,y)=\sum_{i=0}^{N} c_i x^i y^{N-i}, \tag{9}$$

We consider all linear partial differential operators acting on G of the form

$$L_\alpha(G)\equiv\{xx\}G_{xx}+\{xy\}G_{xy}+\{yy\}G_{yy}+\{x\}G_x+\{y\}G_y+\{1\}G \tag{10}$$

where

$$\{xx\}=\eta x^2+\zeta y^2+\rho xy \tag{11}$$
$$\{yy\}=\eta' x^2+\zeta' y^2+\rho' xy \tag{12}$$
$$\{xy\}=\eta'' x^2+\zeta'' y^2+\rho'' xy \tag{13}$$
$$\{x\}=ax+by \tag{14}$$
$$\{y\}=cx+dy. \tag{15}$$
$$\{1\}=\theta. \tag{16}$$

This is the largest class of linear partial differential operators in x and y for $0\leq k\leq 2$. The differential operator $L_\alpha(.)$ is specified by fourteen parameters, given by

$$\boldsymbol{\alpha}=(\eta,\eta',\eta'',\zeta,\zeta',\zeta'',\rho,\rho',\rho'',a,b,c,d,\theta)^T. \tag{17}$$

We show that we may interpret L_α acting on a given $G(x,y)$ to be equivalent to an $(N+1)\times(N+1)$ matrix M_{N+1} acting on a vector whose components are c_i. This relationship is shown in the following lemma:

Lemma 1 *The differential operator $L_\alpha(G)$ is equivalent to a pentadiagonal matrix*

$$M_{N+1}(\alpha)=\begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & p^{(2)}_{i-2} & p^{(1)}_{i-1} & r_i & q^{(1)}_{i+1} & q^{(2)}_{i+2} & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}_{(N+1)\times(N+1)} \tag{18}$$

multiplied onto a column vector with components c_i, where

$$p_i^{(2)} = \eta'(N-i)(N-i-1) \tag{19}$$

$$p_i^{(1)} = \eta'' i(N-i) + \rho'(N-i)(N-i-1) + c(N-i) \tag{20}$$

$$r_i = \theta + \eta i(i-1) + \rho'' i(N-i) + \zeta'(N-i)(N-i-1) + ai + d(N-i) \tag{21}$$

$$q_i^{(1)} = \rho i(i-1) + \zeta'' i(N-i) + bi \tag{22}$$

$$q_i^{(2)} = \zeta i(i-1). \tag{23}$$

Proof. Let us consider the $\{xx\}G_{xx}$ term of $P(G)$. Without loss in generality, similar analysis holds for the the other terms. We first extend c_i to include all $i \in \mathbb{Z}$, with $c_i = 0$ if $i \notin \{0 \dots N\}$. So, for $G = \sum_i c_i x^i y^{N-i}$, we have

$$\eta x^2 G_{xx} = \sum_i c_i \eta i(i-1) x^i y^{N-i} \tag{24}$$

$$\zeta y^2 G_{xx} = \sum_i c_i \zeta i(i-1) x^{i-2} y^{N-(i-2)} \tag{25}$$

$$= \sum_i c_{i+2} \zeta (i+1)(i+2) x^i y^{N-i} \tag{26}$$

$$\rho xy G_{xx} = \sum_i c_i \rho i(i-1) x^{i-1} y^{N-(i-1)} \tag{27}$$

$$= \sum_i c_{i+1} \rho i(i+1) x^i y^{N-i}. \tag{28}$$

The $\eta x^2 G_{xx}$ term contributes to the diagonal elements of the matrix. The $\zeta y^2 G_{xx}$ and $\rho xy G_{xx}$ terms contribute to the second and first superdiagonals respectively. Similar calculations, based on the discrepancy $\Delta m = m - m'$, for each term (with monomial expressions in x and y) in $L_\alpha(G)$ yield $p_i^{(2)}, p_i^{(1)}, r_i, q_i^{(1)}$, and $q_i^{(2)}$ given above. That is, $\Delta m < 0$,(resp. $= 0$, > 0) corresponds to $\Delta m - th$ superdiagonal (resp. diagonal, $\Delta m - th$ subdiagonal) band in $M_{N+1}(\alpha)$. ■

Using $L_\alpha(G)$, we seek to transform the matrix $M_{N+1}(\alpha)$ given in Lemma 1 into lower triangular form. We do this by taking $u = x - y$ and $G(x,y) = H(u,y) = \sum_i b_i u^i y^{N-i}$. We note that L is a linear operator and $u = x - y$ is a linear change of variables. This implies that the polynomial coefficients do not change degree and partial derivatives do not change their order (and are expressed linearly). Therefore, we have $L_\alpha(G) \to L'_\alpha(H)$, where $L'_\alpha(H)$ takes the same form as $L_\alpha(G)$ in the sense that x is replaced by u and there are new parameters for the transformed operator. Looking for lower triangular form of the associated transformed matrix, $M'_{N+1}(\alpha)$, we show in the following theorem that there are three orthogonality conditions that must be satisfied.

Theorem 1 *$L'_\alpha(H)$ corresponds to a lower triangular matrix if and only if $\boldsymbol{\alpha}$ is in the nullspace of*

$$A = \begin{bmatrix} 2 & 4 & -3 & 0 & 2 & -1 & 1 & 3 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2N+2 & N-1 & 0 & -2N+2 & N-1 & 0 & -2N+2 & N-1 & 1 & 1 & -1 & -1 & 0 \\ 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{29}$$

Proof. Given that $u = x - y$, we have $G_x = H_u$ and $G_y = H_y - H_u$. Using these to transform the derivatives and the polynomial coefficients, we deduce that

$$\{1\}G = \theta H \tag{30}$$

$$\{x\}G_x = \{x\}H_u = [a(u+y) + by]H_u \tag{31}$$

$$\{y\}G_y = \{y\}(H_y - H_u) = [c(u+y) + dy](H_y - H_u) \tag{32}$$

$$\{xx\}G_{xx} = (\eta x^2 + \zeta y^2 + \rho xy)H_{uu} = [\eta(u+y)^2 + \zeta y^2 + \rho(u+y)y]H_{uu} \tag{33}$$

$$\{yy\}G_{xx} = [\eta'(u+y)^2 + \zeta' y^2 + \rho'(u+y)y](H_{yy} - 2H_{uy} + H_{uu}) \tag{34}$$

$$\{xy\}G_{xx} = [\eta''(u+y)^2 + \zeta'' y^2 + \rho''(u+y)y](H_{uy} - H_{uu}). \tag{35}$$

$$\tag{36}$$

By Lemma 1, the transformed operator $L'_\alpha(H)$ corresponds to a pentadiagonal matrix of similar form. However, to require lower triangular form, we require the first and second superdiagonal terms to vanish. In the context of Lemma 1, the term that contributes to the second superdiagonal is

$$(\eta + \eta' - \eta'' + \zeta + \zeta' - \zeta'' + \rho + \rho' - \rho'')y^2 H_{uu}. \tag{37}$$

For the second superdiagonal to vanish, the first orthogonality condition we obtain is

$$(1, 1, -1, 1, 1, -1, 1, 1, -1, 0, 0, 0, 0, 0)\boldsymbol{\alpha} = 0 \tag{38}$$

which is the third row of A. Now, the three terms that that contribute to the first superdiagonal are yH_u, y^2H_{uy}, and uyH_{uu}. The uyH_{uu} term is given by

$$[2\eta + \rho + 2\eta' + \rho' + 2\eta'' + \rho'']uyH_{uu}. \tag{39}$$

The y^2H_{uy} terms are

$$[\eta'' + \zeta'' + \rho'' + -2\eta' - 2\zeta' - 2\rho']y^2H_{uy}. \tag{40}$$

The yH_u terms are given by

$$[a + b - c - d]yH_u. \tag{41}$$

Applying Lemma 1 to $L'(H)$ gives the first superdiagonal. We set this to zero to obtain

$$0 =[\rho + \rho' + \rho'' + 2(\eta + \eta' - \eta'')]i(i + 1) \tag{42}$$
$$+ [\eta'' + \zeta'' + \rho'' - 2(\eta' + \zeta' + \rho')](N - i - 1)(i + 1) \tag{43}$$
$$+ [a + b - c - d](i + 1). \tag{44}$$

Dividing by $i + 1$ gives

$$0 =[\rho + \rho' + \rho'' + 2(\eta + \eta' - \eta'')]i \tag{45}$$
$$+ [\eta'' + \zeta'' + \rho'' - 2(\eta' + \zeta' + \rho')](N - i - 1) \tag{46}$$
$$+ [a + b - c - d]. \tag{47}$$

Let

$$\boldsymbol{v}_1 = (2, 4, -3, 0, 2, -1, 1, 3, -2, 0, 0, 0, 0, 0) \tag{48}$$
$$\boldsymbol{v}_2 = (0, -2N + 2, N - 1, 0, -2N + 2, N - 1, 0, -2N + 2, N - 1, 1, 1, -1, -1, 0) \tag{49}$$

This condition for the first superdiagonal to vanish can be written as

$$(\boldsymbol{v}_1 + i\boldsymbol{v}_2)\boldsymbol{\alpha} = 0 \tag{50}$$

for all $i \in \{0, \dots, N\}$. Therefore, it is also necessary for $\boldsymbol{\alpha}$ to be orthogonal to $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ for triangularization. The row vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are the first and second rows of the matrix A. Therefore, the three orthogonality conditions given above is equivalent to requiring $\boldsymbol{\alpha}$ to be in the nullspace of A. ■

The orthogonality conditions in the theorem appear to be arbitrary. However they correspond to elegant properties of the matrix M_{N+1} that can be easily checked and are physically meaningful. For the orthogonality conditions in the theorem to hold, the sum of the columns must be a constant value and the drift velocity of the corresponding stochastic dynamics must satisfy a linearity condition. These are simple conditions to verify given a matrix M. These claims are shown in Theorem 2.

Theorem 2 *$\boldsymbol{\alpha}$ is in the nullspace of A (given in the theorem) if and only if the sum of the columns of $M_{N+1}(\alpha)$ is constant and the drift velocity $v_i = 2p_i^{(2)} + p_i^{(1)} - q_i^{(1)} - 2q_i^{(2)}$ is linear in i.*

Proof. If $\boldsymbol{\alpha}$ is in the nullspace of A, then $A\boldsymbol{\alpha} = 0$. We first show that if the sum of the columns of M is constant and v_i is linear in i, then there are three orthogonality conditions on $\boldsymbol{\alpha}$ that would need to be satisfied, which can be expressed as $A'\boldsymbol{\alpha} = 0$. Then, we construct an invertible matrix matrix Z such that $A' = ZA$. This proves that these conditions are equivalent because $A\boldsymbol{\alpha} = 0 \to ZA\boldsymbol{\alpha} = A'\boldsymbol{\alpha} = 0$ and $A'\boldsymbol{\alpha} = 0 \to Z^{-1}A'\boldsymbol{\alpha} = A\boldsymbol{\alpha} = 0$.

Here we find A'. We obtain two conditions by requiring a constant column sum and one condition for requiring v_i to be linear in i. For v_i to be linear in i, we require the i^2 terms to vanish. Therefore, we require

$$2\eta' - \eta'' - 2\zeta + \zeta'' - \rho + \rho' = 0. \tag{51}$$

So, we take the first row of A' to be

$$(0, 2, -1, -2, 0, 1, -1, 1, 0, 0, 0, 0, 0, 0). \tag{52}$$

The column sum is given by

$$p_i^{(2)} + p_i^{(1)} + r_i + q_i^{(1)} + q_i^{(1)} = c_0 + c_1 i + c_2 i^2 \tag{53}$$

where

$$c_0 = -N(N-1)(\eta' + \zeta' + \rho') + N(c+d) + \theta \tag{54}$$

$$c_1 = -\eta + (-2N+1)\eta' + N\eta'' - \zeta + (-2N+1)\zeta' + N\zeta'' - \rho + (-2N+1)\rho' + N\rho' + a + b - c - d \tag{55}$$

$$c_2 = \eta + \eta' - \eta'' + \zeta + \zeta' - \zeta'' + \rho + \rho' - \rho''. \tag{56}$$

For the sum of the columns to be independent of the i, we require $c_2 = c_1 = 0$, which give two orthogonality conditions. Therefore, the second and third rows of A' are given to be

$$(-1, -2N+1, N, -1, -2N+1, N, -1, -2N+1, N, 1, 1, -1, -1, 0) \tag{57}$$

and

$$(1, 1, -1, 1, 1, -1, 1, 1, -1, 0, 0, 0, 0, 0). \tag{58}$$

Therefore, we write

$$A' = \begin{bmatrix} 0 & 2 & -1 & -2 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2N+1 & N & -1 & -2N+1 & N & -1 & -2N+1 & N & 1 & 1 & -1 & -1 & 0 \\ 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{59}$$

A simple matrix multiplication shows that

$$Z = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \tag{60}$$

implies that $A' = ZA$. Also, Z is clearly invertible, since $det(Z) = 1$. ■

2.1 Eigenvalues and eigenvectors formulae

The above theorem provides a powerful method for triangularizing a matrix. This implies that the eigenvalues and eigenvectors can be calculated efficiently. We will work out the details only for the special transformation $u = x - y$,and leave the reader to follow the general method in this subsection to calculate the explicit solutions for the eigenvalues and eigenvectors when the transformation of x, y is based on another $g \in SL(2, \mathbb{R})$. We wish to solve $M\boldsymbol{v} = \lambda \boldsymbol{v}$ where $\boldsymbol{v}$ takes components c_i. Applying Lemma 1 to this problem, we express the eigenvalue problem as $L(G) = \lambda G$. Assuming that the premise of the theorem holds, we transform $L(G) \to L'(H)$ and obtain a lower triangular system. Given that $H = \sum_i b_i u^i y^{N-i}$, we apply Lemma 1 to $L'_\alpha(H)$ to obtain

$$L_{i-2}^{(2)} b_{i-2} + L_{i-1}^{(1)} b_{i-1} + L_i^{(0)} b_i = \lambda b_i. \tag{61}$$

where

$$L_i^{(2)} = \eta'(N-i)(N-i-1) \tag{62}$$

$$L_i^{(1)} = (-2\eta' + \eta'')i(N-i) + (2\eta' + \rho')(N-i)(N-i-1) + c(N-i) \tag{63}$$

$$L_i^{(0)} = \theta + (a-c)i + (c+d)(N-i) + (\eta + \eta' - \eta'')i(i-1) + (-4\eta' - 2\rho' + 2\eta'' + \rho'')i(N-1) \tag{64}$$

$$+ (\eta' + \zeta' + \rho')(N-i)(N-i-1). \tag{65}$$

Since the matrix is lower triangular, we have that the set of eigenvalues is

$$\lambda_k = L_k^{(0)} \tag{66}$$

for $k \in \{0, \dots, N\}$. For any k, Eq. (61) can be applied iteratively to obtain all b_i corresponding to eigenvalue λ_k. Since $b_i = 0$ for $i < 0$, we have that $b_i = 0$ until $i = k$. When $i = k$, Eq. (61) is singular, which implies b_k can take any value. This reflects the fact that any constant multiplied by an eigenvector remains an eigenvector. For $k < i \leq N$, b_i will take non-zero values.

Given that $u = x - y$ transforms $L_\alpha(G)$ into a lower triangular form, the original eigenvector components can be found by substitution. Given that $G(x, y) = H(x - y, y)$, we have that

$$G(x, y) = \sum_{i=0}^{N} b_i (x - y)^i y^{N-i} \tag{67}$$

$$= \sum_{i=0}^{N} \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} b_i u^j y^{N-j} \tag{68}$$

$$= \sum_{i=0}^{n} \left[\sum_{j=i}^{N} (-1)^{j-i} \binom{j}{i} b_j \right] u^i y^{N-i} \tag{69}$$

Since the components of G are c_i, we have

$$c_i = \sum_{j=i}^{N} (-1)^{j-i} \binom{j}{i} b_j. \tag{70}$$

This gives the components of the eigenvector in terms of b_i, that are calculated by Eq. (61). Expressing $H(u, y) = G(u + y, y)$ yields

$$b_i = \sum_{j=i}^{N} \binom{j}{i} c_j. \tag{71}$$

This implies that the vector whose components are b_i is given by the vector with c_i multiplied by the upper Pascal matrix, which has elements $\binom{j}{i}$ in column j and row i. If the components of $\boldsymbol{v}$ are c_i and we let U be the upper Pascal matrix, then we have $\boldsymbol{w} = U\boldsymbol{v}$ taking components b_i. Furthermore, given that the conditions in the theorem hold, after transforming to b_i, we have that the matrix is lower triangular. That is, $UMU^{-1}\boldsymbol{w} = \lambda \boldsymbol{w}$ is a lower triangular system, which imply that the eigenvalues are the diagonal elements of UMU^{-1}.

3 Reduction to Purely Second Order Operators

We show here that we can reduce the second order operator in Eq. (10) to a purely second order operator without G_x, G_y, or G terms. This allows us to reduce to a linear operator system of 9 parameters instead of 14. This reduction will be used to simplify the necessary and sufficient conditions for triangularizability by this method. We note that by the definition of G, we can write

$$xyG_{xy} + y^2 G_{yy} = \sum_i c_i [i(N - i) + (N - i)(N - i - 1)] x^i y^{N-i} \tag{72}$$

$$= \sum_i c_i (N - i)(N - 1) x^i y^{N-i} \tag{73}$$

$$= (N - 1) y G_y. \tag{74}$$

similar calculations yield

$$xyG_{xy} + x^2 G_{xx} = (N - 1) x G_x \tag{75}$$

$$x^2 G_{xx} + 2xyG_{xy} + y^2 G_{yy} = N(N - 1) G \tag{76}$$

We therefore can substitute these expressions for G_x, G_y, and G to rewrite the operator as a linear combination of only second order terms. We then relabel our parameters to be the coefficients of this linear combination. After this reduction, we now rewrite the linear operator as a bilinear form given by

$$L_\alpha(G) = (x^2 \;\; xy \;\; y^2)\Theta \begin{pmatrix} \partial_{xx} \\ \partial_{xy} \\ \partial_{yy} \end{pmatrix} G \tag{77}$$

where

$$\Theta = \begin{pmatrix} \eta & \eta'' & \eta' \\ \rho & \rho'' & \rho' \\ \zeta & \zeta'' & \zeta' \end{pmatrix}. \tag{78}$$

Therefore, without loss in generality, we could set $a = b = c = d = \theta = 0$ in the above analysis. We now investigate the effect of a linear transformation specified by $g \in SL(2, \mathbb{R})$, which is characterized in the following theorem:

Lemma 2 *For $g \in SL(2, \mathbb{R})$ given by*

$$g = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{79}$$

and

$$\begin{bmatrix} u \\ v \end{bmatrix} = g \begin{bmatrix} x \\ y \end{bmatrix}, \tag{80}$$

the resulting $N+1 \times N+1$ square matrix of the transformed operator $L'_\alpha(H)$ is lower triangular if and only if α is in the nullspace of

$$\begin{bmatrix} -2A^2BD & 2ACB^2 & A^2B^2 \\ -2B^3D & 2B^3D & B^4 \\ -2AB^2D & (AD+BC)B^2 & AB^3 \\ -2A^3C & 2A^3C & A^4 \\ -2AB^2C & 2A^2BD & A^2B^2 \\ -2A^2BC & A^2(AD+BC) & A^3B \\ A^2(AD+BC) & -2A^2BC & -A^3 \\ B^2(AD+BC) & -2AB^2D & -AB^3 \\ AB(AD+BC) & -AB(AD+BC) & -A^2B^2 \end{bmatrix}^T, \tag{81}$$

where $\alpha = (\eta, \eta', \eta'', \zeta, \zeta', \zeta'', \rho, \rho', \rho'')^T$

Proof. Since $g \in SL(2, \mathbb{R})$, we have $AD - BC = 1$, and

$$x = Du - Bv \tag{82}$$
$$y = -Cu + Av \tag{83}$$
$$\partial_x = A\partial_u + C\partial_v \tag{84}$$
$$\partial_y = B\partial_u + D\partial_v. \tag{85}$$

By substitution, the transformation $L_\alpha(G(x,y)) \to L'_\alpha(H(u,v))$ takes the form

$$L'_\alpha(H) = (u^2 \;\; uv \;\; v^2)L\Theta R \begin{pmatrix} \partial_{uu} \\ \partial_{uv} \\ \partial_{vv} \end{pmatrix} \tag{86}$$

where,

$$L = \begin{pmatrix} D^2 & -DC & C^2 \\ -2BD & AD+BC & -2AC \\ B^2 & -AB & A^2 \end{pmatrix} \tag{87}$$

$$R = \begin{pmatrix} A^2 & 2AC & C^2 \\ AB & AD+BC & BC \\ B^2 & 2BD & D^2 \end{pmatrix}. \tag{88}$$

In order for the resulting matrix to be lower triangular, we require that uvH_{uu}, v^2H_{uv}, and v^2H_{uu} have zero coefficients since these contribute to the super-diagonal bands. These terms correspond to the sub-diagonal terms of $L\Theta R$. Calculating the matrix multiplication and setting the sub-diagonal elements to zero demonstrates that α is in the nullspace of the following matrix:

$$\begin{bmatrix} -2A^2BD & 2ACB^2 & A^2B^2 \\ -2B^3D & 2B^3D & B^4 \\ -2AB^2D & (AD+BC)B^2 & AB^3 \\ -2A^3C & 2A^3C & A^4 \\ -2AB^2C & 2A^2BD & A^2B^2 \\ -2A^2BC & A^2(AD+BC) & A^3B \\ A^2(AD+BC) & -2A^2BC & -A^3 \\ B^2(AD+BC) & -2AB^2D & -AB^3 \\ AB(AD+BC) & -AB(AD+BC) & -A^2B^2 \end{bmatrix}^T \tag{89}$$

■

A corollary of this lemma is the following result:

Theorem 3 *For any $g \in SL(2,\mathbb{R})$, $\dim N(A,B,C)(g) \geq 6$.*

The significance of this is that for any $g \in SL(2,\mathbb{R})$, the corresponding transformation on x, y generates through the procedure in this paper the simultaneous triangularization of a nonempty subset of $W'^{(2)}$ which is in fact, a vector subspace having dimension no less than 6 in the reduced purely second order system in this section. In the original formulation in section 2, the simultaneous triangularization works on a vector subspace of W_{14} having dimension no less than 11.

We do not fully understand yet how these vector subspaces $N(g)$ fit together inside the 9-dimensional $W'^{(2)}$ as g ranges over the Lie group $SL(2,\mathbb{R})$, and therefore pose it as an open problem.

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