NEW SYMPLECTIC VARIABLES FOR THE FOUR-BODY PROBLEM AND HERMAN RESONANCE

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ABSTRACT

This note presents a new symplectic transformation for the gravitational four body problem, which is based on the combinatorics of the wheel graph $W_4$ with a central vertex (star) and three rim vertices (planets). The rim vertices are connected to the central vertex by inward directed edges, and adjacent rim vertices are connected by anti-clockwise directed edges. The new coordinates $Q_j$ each represent mass-weighted sums of the original coordinates (absolute positions) of three bodies associated with four fundamental cycles in $W_4$. If the system consists of a star whose mass is much greater than three nearly equally massive planets, this transformation puts the original Hamiltonian for four bodies into perturbation form. The leading order term $H_U$ consists of three uncoupled pairs $(Q_k, P_k)$ with unequal gravitational constants and central masses, and have the invariant $P_1$ conjugate to the planetary center of mass. This unperturbed Hamiltonian $H_U$ leads to a first secular system and the derivation of Herman’s resonance.

Keywords: symplectic coordinates; celestial 4-body problems; Herman resonances; perturbation theory

1 Introduction

In this note, we introduce a new symplectic transformation for the four-body problem in celestial mechanics. If the four bodies form a solar system of a massive star $m_1$ and three planets $m_{k>1}$ where $m_{k>1}/m_1 = O(\varepsilon)$, then these new variables provide a perturbation form for the 4-body problem. The combinatorial algorithm on which this symplectic transformation is based cannot be extended to $N > 4$ bodies. Nonetheless, for $N > 3$, the heliocentric N-body problems in celestial mechanics are technically difficult and they have only recently been fully solved for nonlinear stability in terms of KAM tori [13], [8], [9]. We recall that Arnold obtained a full proof of KAM-tori for the three-body problem but his proof for the fully N-body heliocentric problem was not completed until 2004 [13]. In all these projects, symplectic transformations play a critical role but useful ones are few and far in between in the long history of celestial mechanics.

We show below that there is a new symplectic transformation based on the combinatorial topology of a wheel graph $W_4$ with three rim nodes and a center node, that will put the heliocentric 4-body problem of a star and three planets in perturbation form. We also show that under $m_{k>1}/m_1 = O(\varepsilon)$, the final unperturbed term $H'_U$ has a new invariant, which concerns the constancy of the center of mass position of the planets (with masses $m_k = 2, 3, 4$). Compare this to the standard one in celestial mechanics called Jacobi reduction by the center of mass of the whole system [5].

∗Use footnote for providing further information about author (webpage, alternative address)—not for acknowledging funding agencies.
In [1] we introduced an orthogonal 4 by 4 matrix based on $W_4$. The directed edges or arcs in this wheel graph $W_4$ are the spokes connecting each rim node to the center, and the counter-clockwise oriented arcs between adjacent rim nodes. Here we will go a small step further by generating a symplectic transformation from $W_4$. This transformation will take the original variables $(q_j, p_j)$ where for $j = 1, ..., 4$, the $q_j$ denote the absolute position of body $j$ in 3-space, to new coordinates and generalized momenta $(Q_j, P_j)$.

2 Symplectic variables based on $W_4$.

Consider as in [1] the wheel $W_4$ and an incidence matrix for the basic cycles in the directed graph of $W_4$:

\[ M = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & 1 & -1 \end{bmatrix} \]

The first row is clearly the incidence vector of the cycle $c_1$ which consists of the three rim arcs in counterclockwise orientation. The second to fourth rows correspond to incidence vectors of the cycles which consist of one rim arc between two adjacent rim nodes and two arcs (or spokes) from these rim nodes to the center. Following standard practice, each cycle is oriented counterclockwise; in the incidence vector associated with nodes in these cycles, the sign of the entry corresponding to a node is determined in the usual manner: traversing the cycle in the counterclockwise direction, a node is assigned +1 if it is first reached after traversing an arc (directed edge) in the direction of its orientation. These rules generate a sign-pattern $M(W_n)$ for any such wheel graphs $W_n$ with $n$ nodes.

The matrix $M$ is maximally nonsingular [1] where a sign-pattern $H$ of 0, ±1’s is nonsingular when any real matrix $M$ that has this sign-pattern is nonsingular. Using the following assignments of real values, we obtain from $M$ an orthogonal matrix $A$ [1]: for any positive 4-vector $(m_1, m_2, m_3, m_4)$, let

\[ A_{jk} = \text{sgn}(M_{jk}) \frac{1}{\sqrt{3}} i f M_{jk} \neq 0, \]

\[ A_{jk} = 0 i f M_{jk} = 0. \]

We also showed that for $n \geq 5$, there are no assignments of real values to the sign-pattern $M(W_n)$ which will return an orthogonal matrix [1]. This is the reason for restricting to 4-bodies in this note.

In order to generate a symplectic transformation from $M(W_4)$ for the four-body heliocentric problem in 3-space, we can use a relaxed form of the above property that there is an assignment of real values to the sign-pattern $M(W_4)$ which return an orthogonal matrix $A$. We only require that there is a real matrix $A$ with the sign-pattern $M(W_4)$ such that $D = (A^{-1})^t$ also has the same sign-pattern.

Similar to [5], we consider symplectic transformations of block $(A, D)$ form:

\[
\begin{pmatrix} \vec{Q} \\ \vec{P} \end{pmatrix} = T \begin{pmatrix} \vec{Q} \\ \vec{P} \end{pmatrix}
\]

\[ T = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \]

where the symplectic matrix $T$ has diagonal blocks given by the 12 by 12 matrices. The proof of the following proposition consists merely of checking that $AD^t = I$:

**Proposition 1** Let $(m_1, ..., m_4)$ be any positive real numbers, then

\[
A = \begin{bmatrix}
0 & m_1 & m_3 & m_4 \\
m_1 & M(1) & M(3) & M(4) \\
m_3 & M(3) & 0 & m_4 \\
m_4 & M(4) & m_4 & 0
\end{bmatrix}
\]  

\[
D = \begin{bmatrix}
0 & M(1) & M(4) & M(4) \\
M(1) & 0 & M(3) & M(4) \\
M(4) & M(3) & 0 & M(4) \\
M(4) & M(4) & M(4) & 0
\end{bmatrix}
\]
where $M(j) = \sum_{k|M_j \neq 0} m_{j,k}$ yields a symplectic transformation of $(A, D)$ block form.

Each bold entry in $A$ and $D$ represents a real $3 \times 3$ diagonal matrix; for example,

$$-1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\frac{1}{m_2} = \begin{bmatrix} m_2^{-1} & 0 & 0 \\ 0 & m_2^{-1} & 0 \\ 0 & 0 & m_2^{-1} \end{bmatrix}.$$  

In other words, $A$ and $D$ are real $12 \times 12$ matrices consisting of $4 \times 4$ entries, each of which is a $3 \times 3$ diagonal block.

We note the following for later use: (a) the rows 2 - 4 each involves the center node with solar mass $m_1$, and a pair of adjacent rim nodes with smaller planetary masses $m_{j>2}$; for $j = 2, 3, 4$, these rows represent new coordinates $\overrightarrow{Q}_j$, which are mass ratio weighted averages of the star $m_1$ and two planets $m_{j>1}$;

(b) row 1 involves only the rim nodes corresponding to the three planets and moreover row 1 has only positive entries; it represents the new coordinate $\overrightarrow{Q}_1$ which is the center of mass position of the planets.

### 3 Perturbation theory

The full Hamiltonian function of the 4 bodies Newtonian problem is given by

$$H_4 = \frac{1}{2} \sum_{j=1}^{4} m_j v_j^2 - \sum_{j<k}^{4} \frac{m_j m_k}{r_{jk}}$$

where $\overrightarrow{q}_j$ are real $3$-vectors for the absolute positions of the masses, and $\overrightarrow{p}_j$ are real $3$-vectors for their momenta.

The inverse transformation for the relative coordinates $\overrightarrow{q}_j - \overrightarrow{q}_k$, $j > k = 1, ..., 4$ in terms of coordinates $\overrightarrow{Q}_j$, $j = 1, ..., 4$ will be needed below to expand the terms in $H_4$, and are given by:

$$\left( \overrightarrow{q} \right) = A \left( \overrightarrow{Q} \right); \left( \overrightarrow{q} \right) = A^{-1} \left( \overrightarrow{Q} \right)$$

$$\left( \overrightarrow{q} \right)^t = \left( \overrightarrow{Q} \right)^t (A^{-1})^t$$

$$\left( A^{-1} \right)^t = \begin{bmatrix} 0 & M(1) & M(1) & M(1) \\ M(2) & M(2) & M(3) & M(3) \\ M(2) & M(3) & M(3) & 0 \\ M(2) & M(3) & M(3) & 0 \\
M(3) & M(3) & M(3) & 0 \\ M(3) & M(3) & M(3) & 0 \\
M(3) & M(3) & M(3) & 0 \\ M(3) & M(3) & M(3) & 0 \\
M(3) & M(3) & M(3) & 0 \\ M(3) & M(3) & M(3) & 0 \\
M(3) & M(3) & M(3) & 0 \\ M(3) & M(3) & M(3) & 0 \\
M(3) & M(3) & M(3) & 0 \\ M(3) & M(3) & M(3) & 0 \end{bmatrix}.$$  

Thus,

$$\overrightarrow{q}_1 = \frac{1}{3m_1} \sum_{j=2,3,4} M(j) \overrightarrow{Q}_j$$

$$\overrightarrow{q}_2 = \frac{1}{3m_2} \left( M(1) \overrightarrow{Q}_1 + M(2) \overrightarrow{Q}_2 - M(3) \overrightarrow{Q}_3 \right)$$

$$\overrightarrow{q}_3 = \frac{1}{3m_3} \left( M(1) \overrightarrow{Q}_1 - M(2) \overrightarrow{Q}_2 + M(4) \overrightarrow{Q}_4 \right)$$

$$\overrightarrow{q}_4 = \frac{1}{3m_4} \left( M(1) \overrightarrow{Q}_1 - M(4) \overrightarrow{Q}_4 + M(3) \overrightarrow{Q}_3 \right).$$
and

\[
\vec{q}_3 - \vec{q}_2 = \frac{1}{3m_3} \left( M(1) \vec{Q}_1 - M(2) \vec{Q}_2 + M(4) \vec{Q}_4 \right) - \frac{1}{3m_2} \left( M(1) \vec{Q}_1 + M(2) \vec{Q}_2 - M(3) \vec{Q}_3 \right) \quad (19)
\]

\[
\vec{q}_4 - \vec{q}_2 = \frac{1}{3m_4} \left( M(1) \vec{Q}_1 - M(4) \vec{Q}_4 + M(3) \vec{Q}_3 \right) - \frac{1}{3m_2} \left( M(1) \vec{Q}_1 + M(2) \vec{Q}_2 - M(3) \vec{Q}_3 \right) \quad (20)
\]

If we assume in addition to \(m_k > 1/m_1 = O(\varepsilon)\), that the planetary masses are nearly equal as well, i.e., \(m_{k > 1} = m(1 + O(\varepsilon))\) for positive constant \(m\) such that \(m/m_1 = O(\varepsilon)\), then we have, for \(k = 2, 3, 4\),

\[
M(k) = \left( m_1 + 2m(1 + O(\varepsilon)) \right), \quad m_1 + 2m + O(\varepsilon) = M + O(\varepsilon). \quad (22, 23)
\]

After collecting leading order terms (dropping terms of \(O(1)\) and higher) we get

\[
\vec{q}_2 - \vec{q}_1 = \vec{Q}_1 - \frac{2M}{3m} \vec{Q}_4 - \frac{M}{3m_1} \vec{Q}_4 = -\frac{2M}{3m_1} \vec{Q}_3 \quad (24)
\]

\[
\vec{q}_3 - \vec{q}_1 = \vec{Q}_1 - \frac{2M}{3m} \vec{Q}_2 = \frac{M}{3m} \vec{Q}_3 \quad (25)
\]

\[
\vec{q}_4 - \vec{q}_1 = \vec{Q}_1 - \frac{M}{3m} \vec{Q}_2 - \frac{2M}{3m_1} \vec{Q}_4 = -\frac{M}{3m_1} \vec{Q}_2 \quad (26)
\]

and

\[
\vec{q}_3 - \vec{q}_2 = \frac{M}{3m} \left( \vec{Q}_3 + \vec{Q}_4 - 2\vec{Q}_2 \right) \quad (27)
\]

\[
\vec{q}_4 - \vec{q}_2 = \frac{M}{3m} \left( -\vec{Q}_2 - \vec{Q}_4 + 2\vec{Q}_3 \right) \quad (28)
\]

\[
\vec{q}_4 - \vec{q}_3 = \frac{M}{3m} \left( -2\vec{Q}_4 + \vec{Q}_3 + \vec{Q}_2 \right) \quad (29)
\]

Next, we note that each potential energy term in the Hamiltonian

\[
H_4 = \frac{1}{2} \sum_{j=1}^{4} \frac{m_j v_j^2}{r_j} - \sum_{j<k}^{4} \frac{m_j m_k}{r_{jk}} \quad (30)
\]

\[
= \frac{1}{2} \sum_{j=1}^{4} \frac{\vec{q}_j^2}{m_j} - \sum_{j<k}^{4} \frac{m_j m_k}{\|\vec{q}_j - \vec{q}_k\|} \quad (31)
\]

has the form, modulo \(O(m^4/M^2)\) terms,

\[
U_{21} = \frac{m_2 m_1}{r_{21}} = \frac{3m^3/2M}{\|\vec{Q}_3\|} \quad (32)
\]

\[
U_{31} = \frac{m_3 m_1}{r_{31}} = \frac{3m^3/2M}{\|\vec{Q}_2\|} \quad (33)
\]
We have completed the proof of

**Proposition 2** If $\frac{m_{k>1}}{m_1} = O(\varepsilon)$, and the planetary masses are nearly equal as well, i.e., $\frac{m_{k>1}}{m_1} = m(1 + O(\varepsilon))$ for positive constant $m$ such that $\frac{m}{m_1} = O(\varepsilon)$, then the full Hamiltonian in the new symplectic variables is given by

$$H = H_U + O\left(\frac{m^4}{2m + m_1}\right)^2$$

where

$$H_U = \frac{\mathbf{P}_1^2}{6m} + \frac{1}{2(2m + m_1)} \sum_{i=2}^{4} \mathbf{P}_i^2$$

$$-\left[ \frac{3m^3}{2M} \left\| \overrightarrow{Q}_3 \right\| + \frac{9m^3}{2M} \left\| \overrightarrow{Q}_2 \right\| \right. + \left. \frac{3m^3}{M} \left\| \overrightarrow{Q}_3 + \overrightarrow{Q}_4 - 2\overrightarrow{Q}_2 \right\| + \frac{3m^3}{2M} \left\| \overrightarrow{Q}_2 - \overrightarrow{Q}_4 + 2\overrightarrow{Q}_3 \right\| + \frac{3m^3}{M} \left\| -2\overrightarrow{Q}_4 + \overrightarrow{Q}_3 + \overrightarrow{Q}_2 \right\| \right].$$

We recall next, that under $H_4$ (and under $H$), we can wlog set

$$\sum_{j=1}^{4} m_j \overrightarrow{q}_j = m_1 \overrightarrow{q}_1 + \sum_{j=2}^{4} m_j \overrightarrow{q}_j = \overrightarrow{q}.$$  

(41)

From $m_{k>1}/m_1 = O(\varepsilon)$, and the definition of $\overrightarrow{Q}_1$ as the center of mass of the planets, we get

$$-m_1 \overrightarrow{q}_1 = \sum_{j=2}^{4} m_j \overrightarrow{q}_j = M(1) \overrightarrow{Q}_1.$$  

(42)

Thus, while the full dynamics satisfy

$$\overrightarrow{q} = \frac{d}{dt} \sum_{j=1}^{4} \mathbf{P}_j = m_1 \frac{d}{dt} \overrightarrow{q}_1 + \frac{d}{dt} \sum_{j=2}^{4} m_j \overrightarrow{q}_j = m_1 \frac{d}{dt} \overrightarrow{q}_1 + M(1) \frac{d}{dt} \overrightarrow{Q}_1,$$

(43)

the unperturbed dynamics under $H_U$ has the property that

$$3m \frac{d}{dt} \overrightarrow{Q}_1 = \overrightarrow{A},$$

$$\frac{d}{dt} \overrightarrow{P}_1 = \overrightarrow{0},$$

where $\overrightarrow{A}$ is a constant vector.
Since the unperturbed term $H_U$ is derived under the assumption that the planetary masses are equal to $m > 0$, and $m/m_1 = O(\varepsilon)$, it follows that $M(k) = M = 2m + m_1$ for $k = 2, 3, 4$, and thus

$$
\begin{pmatrix}
\overrightarrow{Q}_1 \\
\overrightarrow{Q}_2 \\
\overrightarrow{Q}_3 \\
\overrightarrow{Q}_4
\end{pmatrix} = \begin{bmatrix}
\frac{0}{m} & \frac{1}{m} & \frac{1}{m} & \frac{1}{m} \\
\frac{2m + m_1}{m} & \frac{2m + m_1}{m} & \frac{0}{m} & \frac{2m + m_1}{m} \\
\frac{2m + m_1}{m} & \frac{2m + m_1}{m} & \frac{0}{m} & \frac{2m + m_1}{m} \\
\frac{2m + m_1}{m} & \frac{2m + m_1}{m} & \frac{0}{m} & \frac{2m + m_1}{m}
\end{bmatrix} \begin{pmatrix}
\overrightarrow{Q}_1 \\
\overrightarrow{Q}_2 \\
\overrightarrow{Q}_3 \\
\overrightarrow{Q}_4
\end{pmatrix};
\tag{46}
$$

in other words,

$$
\overrightarrow{Q}_2 + \overrightarrow{Q}_3 + \overrightarrow{Q}_4 = \frac{3m}{M} \overrightarrow{Q}_1 = O(\varepsilon).
\tag{47}
$$

Under the assumption that the planetary masses are equal to $m > 0$, and $m/m_1 = O(\varepsilon)$, we have

$$
\begin{pmatrix}
\overrightarrow{P}_2 \\
\overrightarrow{P}_3 \\
\overrightarrow{P}_4
\end{pmatrix} = \begin{bmatrix}
\frac{M}{m} & \frac{-M}{m} & \frac{-M}{m} & \frac{0}{m} \\
\frac{M}{m} & \frac{-M}{m} & \frac{0}{m} & \frac{-M}{m} \\
\frac{M}{m} & \frac{0}{m} & \frac{-3m}{m} & \frac{0}{m} \\
\frac{-M}{m} & \frac{0}{m} & \frac{-M}{m} & \frac{-3m}{m}
\end{bmatrix} \begin{pmatrix}
\overrightarrow{P}_1 \\
\overrightarrow{P}_2 \\
\overrightarrow{P}_3 \\
\overrightarrow{P}_4
\end{pmatrix},
\tag{48}
$$

from which it follows that

$$
\sum_{j=2}^{4} \overrightarrow{P}_j = \frac{M}{m} \overrightarrow{P}_1(t)
\tag{49}
$$

and

$$
\frac{d}{dt} \sum_{j=2}^{4} \overrightarrow{Q}_j = \sum_{j=2}^{4} \frac{d}{dt} \overrightarrow{Q}_j = \sum_{j=2}^{4} \frac{\partial H_U}{\partial \overrightarrow{P}_j} = M^{-1} \sum_{j=2}^{4} \overrightarrow{P}_j = \frac{\overrightarrow{P}_1(t)}{m}.
\tag{50}
$$

We have completed the proof of

**Proposition 3**  Under the same assumptions on the mass ratios, the dynamics of $H_U$, has the properties

$$
\overrightarrow{P}_1 = A, \text{a constant vector},
\tag{51}
$$

$$
\sum_{j=2}^{4} \overrightarrow{Q}_j = \frac{3m}{M} \overrightarrow{Q}_1(t) = O(\varepsilon).
\tag{52}
$$

Because $\overrightarrow{P}_1 = A$, we can rewrite the unperturbed Hamiltonian $H_U$ above as

$$
H_U = \frac{1}{2(2m + m_1)} \sum_{i=2}^{4} \overrightarrow{P}_i^2 - \overrightarrow{Q}_2 + \overrightarrow{Q}_4 - 2\overrightarrow{Q}_2 = -3\overrightarrow{Q}_2 + \frac{3m}{M} \overrightarrow{Q}_1 = -3\overrightarrow{Q}_2(1 + O(\varepsilon))
\tag{54}
$$

$$
-\overrightarrow{Q}_2 + \overrightarrow{Q}_4 = 3\overrightarrow{Q}_3 - \frac{3m}{M} \overrightarrow{Q}_1 = 3\overrightarrow{Q}_3(1 + O(\varepsilon))
\tag{55}
$$

$$
-2\overrightarrow{Q}_4 + \overrightarrow{Q}_3 + \overrightarrow{Q}_2 = -3\overrightarrow{Q}_4 + \frac{3m}{M} \overrightarrow{Q}_1 = -3\overrightarrow{Q}_4(1 + O(\varepsilon))
\tag{56}
$$

and moving any $O(\varepsilon)$ terms to $H_{\rho}$, we get the final perturbation form in the
Theorem 4 Under the same assumptions on the mass ratios,

\[
H' = H'_U + H'_P, \text{ where } H'_P = O(m^4/M^2), \text{ and }
\]

\[
H'_U = \frac{1}{2(2m + m_1)} \sum_{i=2}^{4} \vec{P}_i^2 - \left[ \frac{5m^3/2M}{\|\vec{Q}_3\|^3} + \frac{11m^3/2M}{\|\vec{Q}_2\|^3} + \frac{3m^3/2M}{\|\vec{Q}_4\|^3} \right].
\]

The Hamilton’s equations are for \(k = 2, 3, 4,\)

\[
\frac{d}{dt} \vec{Q}_k = \frac{\partial H'_U}{\partial \vec{P}_k} = \frac{\vec{P}_k}{M}
\]

and

\[
\frac{d}{dt} \vec{P}_2 = -\frac{\partial H'_U}{\partial \vec{Q}_2} = \frac{11m^3/2M}{\|\vec{Q}_2\|^3} \vec{Q}_2
\]

\[
\frac{d}{dt} \vec{P}_3 = -\frac{\partial H'_U}{\partial \vec{Q}_3} = \frac{5m^3/2M}{\|\vec{Q}_3\|^3} \vec{Q}_3
\]

\[
\frac{d}{dt} \vec{P}_4 = -\frac{\partial H'_U}{\partial \vec{Q}_4} = \frac{3m^3/2M}{\|\vec{Q}_4\|^3} \vec{Q}_4.
\]

Thus,

\[
M \frac{d^2}{dt^2} \vec{Q}_2 = \frac{11m^3/2M}{\|\vec{Q}_2\|^3} \vec{Q}_2
\]

\[
M \frac{d^2}{dt^2} \vec{Q}_3 = \frac{5m^3/2M}{\|\vec{Q}_3\|^3} \vec{Q}_3
\]

\[
M \frac{d^2}{dt^2} \vec{Q}_4 = \frac{3m^3/2M}{\|\vec{Q}_4\|^3} \vec{Q}_4.
\]

which represents Newton’s law \(\vec{F} = m \vec{a}\) for each coordinate \(\vec{Q}_k, k = 2, 3, 4\). In other words, \(H'_U\) represents three uncoupled central force problems, with unequal “gravitational constant times central mass”.

\[
G_k M_k = (11, 5, 3)m^3/2M^2.
\]

We now recall the significant fact that unlike other coordinates used in celestial mechanics including the generalized Jacobi coordinates [lim19a], each of the \(\vec{Q}_k, k = 2, 3, 4\) corresponds to a linear combination or weighted sum of the star’s absolute position \(\vec{q}_1\) and those of two planets \(\vec{q}_{j>1}, \vec{q}_{k>1}\). For instance,

\[
\vec{Q}_1 = \frac{m_1 \vec{q}_1 + m_2 \vec{q}_2 - m_3 \vec{q}_3}{(m_1 + m_2 + m_3)}.
\]

Furthermore, the rotational symmetry implies that the total angular momentum is also conserved.

The remaining questions are typically of KAM type which we hope will be answered by others using these new symplectic variables.
4 Averaging and linear secular system

In this section, we derive the Herman resonances. To derive the first secular system in the 4 – body heliocentric problem, we follow the process which involves first an averaging step and then linearization at the origin. Like the heliocentric planetary case [7, 4], the origin corresponds to 3 circular coplanar orbits.

Recall from the Appendix that each of the uncoupled \((N - 1)\) Keplerian elliptical motions is given by a vector \((a_j, w_1^j, w_2^j, \hat{\ell}_j)\), \(j = 1, ..., N - 1\). The complex parameters \((w_1^j, w_2^j)\) give the inclination, and perihelion with respect to some reference plane. Similar to the general heliocentric planetary case, where all the reference planes of the sun-planet Keplerian ellipses are the same one [7], in the four-body case here, these 3 reference planes are equal. In both the full \(H'\) and the uncoupled term \(H'_U((a_j, w_1^j, w_2^j, \hat{\ell}_j), j = 2, ..., 4)\), in Theorem 4, the new variables \((\tilde{Q}_j, \tilde{P}_j)\) are each associated with a \((a_j, w_1^j, w_2^j, \hat{\ell}_j)\), \(j = 2, ..., 4\). In terms of the latter, the full Newtonian equations of motion after reduction by the center of mass \(\sum_{i=1}^{4} m_i \tilde{r}_i^j\) (where \(\tilde{r}_i^j\) are the original absolute positions of the 4 masses) are: for \(j = 2, ..., 4\),

\[
\frac{d}{dt} a_j = A_j((a_j, w_1^j, w_2^j, \hat{\ell}_j), j = 1, ..., 4) \tag{68}
\]

\[
\frac{d}{dt} w_1^j, 2 = W_1^j, 2((a_j, w_1^j, w_2^j, \hat{\ell}_j), j = 1, ..., 4) \tag{69}
\]

\[
\frac{d}{dt} \hat{\ell}_j = L_j((a_j, w_1^j, w_2^j, \hat{\ell}_j), j = 1, ..., 4). \tag{70}
\]

They can be put in canonical Hamiltonian form with Hamiltonian function \(H'\). This is given in perturbation form in terms of the new symplectic variables by

\[
H'((a_j, w_1^j, w_2^j, \hat{\ell}_j), j = 2, ..., 4) = H'_U(W_4) + H'_P \tag{71}
\]

where \(H'_U\) and \(H'_P\) will be given below. In terms of the virtual masses, \(M_j\) the associated symplectic form is

\[
\omega = M_2 dP_2 \land dQ_2 + M_3 dP_3 \land dQ_3 + M_4 dP_4 \land dQ_4 \tag{72}
\]

by the results in [7].

With abuse of notation, we define the averaged system \([3]\) for \(j = 2, ..., 4\),

\[
\frac{d}{dt} a_j = \frac{1}{(2\pi)^3} \int_0^{2\pi} d \hat{\omega}_2 \ldots \int_0^{2\pi} d \hat{\omega}_4 A_j \tag{73}
\]

\[
\frac{d}{dt} w_1^j, 2 = \frac{1}{(2\pi)^3} \int_0^{2\pi} d \hat{\omega}_2 \ldots \int_0^{2\pi} d \hat{\omega}_4 W_1^j, 2 \tag{74}
\]

\[
\frac{d}{dt} \hat{\ell}_j = \frac{1}{(2\pi)^3} \int_0^{2\pi} d \hat{\omega}_2 \ldots \int_0^{2\pi} d \hat{\omega}_4 L_j. \tag{75}
\]

where the \((a_j, w_1^j, w_2^j, \hat{\ell}_j)\), \(j = 2, ..., 4\) are each associated with the respective new symplectic variables \((\tilde{Q}_j, \tilde{P}_j)\). We will use both systems of variables interchangeably. Applying Proposition 2 in \([4]\) to this system, we find it has a Hamiltonian function given by (3) iterated integrals,

\[
\Pi = \frac{1}{(2\pi)^3} \int_0^{2\pi} d \hat{\omega}_2 \ldots \int_0^{2\pi} d \hat{\omega}_4 H'((a_j, w_1^j, w_2^j, \hat{\ell}_j), j = 2\ to 4) \tag{76}
\]

\[
\Pi_U = \frac{1}{(2\pi)^3} \int_0^{2\pi} d \hat{\omega}_2 \ldots \int_0^{2\pi} d \hat{\omega}_4 H'_U \tag{77}
\]

\[
\Pi_P = \frac{1}{(2\pi)^3} \int_0^{2\pi} d \hat{\omega}_2 \ldots \int_0^{2\pi} d \hat{\omega}_4 H'_P \tag{79}
\]
The coefficients in the symplectic form \( \omega \) in other words, in terms of the elliptical elements \((a_j, w_1^j, w_2^j, \hat{t}_j), j = 2, \ldots, 4\) the Hamiltonian dynamics of \( H^j_{Uj} \) conserve \((a_j, w_1^j, w_2^j, \hat{t}_j)\), for each \( j = 2, \ldots, 4 \). This is the same as the conservation of angular momentum in each of the 3 uncoupled Keplerian motions. Moreover, since each \( \overline{P}_j \) is independent of \( \hat{t}_k \) for \( k \neq j \), we get the averaged uncoupled kinetic energy

\[
\frac{1}{2} \mathcal{K} = \sum_{j=2}^{4} \frac{1}{(2\pi)^2} \int_{0}^{2\pi} dl_{2j} \ldots \int_{0}^{2\pi} dl_{4j} \frac{\overline{P}_j^2}{2M_j} = \sum_{j=2}^{4} \frac{1}{(2\pi)} \int_{0}^{2\pi} dl_{2j} \frac{\overline{P}_j^2}{2M_j}
\]

which is a constant.

After averaging the above \( H \) with respect to the fast angles on the 3 Keplerian ellipses, the resulting averaged Hamiltonian function is

\[
\overline{H} = \overline{H}_U + \overline{H}_P
\]

where after dropping constants from averaging the kinetic energy in \( \overline{H}_U \), each term in \( \overline{H}_U \) has the form,

\[
V_j = \frac{1}{(2\pi)} \int_{0}^{2\pi} dl_j \frac{1}{||\overline{Q}_j||},
\]

This will be calculated by an application of the key Lemma for the case \( \lambda = -1/2 \), for Newtonian potentials. Since we have shown that \( \overline{H}_P \) is smaller than \( \overline{H}_U \), we need only consider the quadratic terms in the expansion of \( \overline{H}_U \) around the origin, to derive the linear secular system.

We use eqn (134) in the appendix for the symplectic form \( \sigma \) of Keplerian motion on a single ellipse, in order to derive the following symplectic form \( \omega \) for motions on 3 ellipses: to first order,

\[
\omega = -2i \sum_{j=2}^{4} \mathcal{M}_j (dg_j \wedge dg_j^* + ds_j \wedge ds_j^*) + ...
\]

The coefficients in the symplectic form \( \omega(T_c(5)) \) for the binary tree \( T_c(5) \) are:

\[
\mathcal{M}_2 = M_2 \sqrt{a_2},
\]

\[
\mathcal{M}_3 = M_3 \sqrt{a_3}, \mathcal{M}_4 = M_4 \sqrt{a_4},
\]

(89)

where \( a_j \) is the semi-major axis for the Keplerian ellipse associated with \( \overline{Q}_j \).

Reverting to the variables \( \overline{w} = (w_1^j, 2, j = 1, \ldots, 4) = ((g_j, s_j), j = 2, \ldots, 4) \in C^8 \), we get the following expansions at the origin which is associated with circular coplanar orbits,

\[
V_j(\overline{w}) = V_j(\overline{Q}) + \sum_{\mu, \nu} \frac{\partial^2 V_j}{\partial w_{\mu} \partial w^*_{\nu}}(\overline{Q}) w_{\mu} w^*_{\nu} + ...
\]

(90)
Using the symplectic form $\omega$, we get the following Hamilton's equations, with $\mu = (2j - 1), 2j, j = 2, \ldots, 4$

$$\frac{dw_\mu}{dt} = -\frac{i}{2M_j} \frac{\partial H_U}{\partial w^*_\mu}$$  \hspace{1cm} (91)

$$\frac{dw^*_\mu}{dt} = \frac{i}{2M_j} \frac{\partial H_U}{\partial w_\mu}$$  \hspace{1cm} (92)

for the function

$$H_U = -\sum_{j=1}^{4} V_j.$$  \hspace{1cm} (93)

In terms of the diagonal matrix $Q$ with entries $d_\mu = M_j$ where $\mu = (2j - 1), 2j$ and $j = 2, \ldots, 4$, the linear secular system is given by

$$\frac{d}{dt} \vec{w} = \frac{1}{2i} Q^{-1} \left[ \partial^* \partial H_U \right] \vec{w}$$  \hspace{1cm} (94)

$$\left[ \partial^* \partial H_U \right]_{\mu\nu} = \frac{\partial^2 H_U}{\partial w^*_\mu \partial w_\nu} \left( \vec{0} \right).$$  \hspace{1cm} (95)

### 4.1 Verification of Herman resonances


**Proposition 5** The trace of $Q^{-1} \left[ \partial^* \partial H_U \right]$ is zero.

Proof: We note that

$$tr Q^{-1} \left[ \partial^* \partial H_U \right] = -\sum_{j=2}^{4} tr Q^{-1} \left[ \partial^* \partial V_j \right]$$  \hspace{1cm} (96)

and for each $j = 2, \ldots, 4$, with $\mu(l) = 2l - 1, \nu(l) = 2l, l = 2, \ldots, 4$,

$$tr Q^{-1} \left[ \partial^* \partial V_j \right] = \sum_{l=1}^{4} \frac{1}{M_l} \left( \frac{\partial^2 V_j}{\partial w^*_{\mu(l)} \partial w_{\mu(l)}} + \frac{\partial^2 V_j}{\partial w_{\nu(l)} \partial w^*_{\nu(l)}} \right) \left( \vec{0} \right)$$  \hspace{1cm} (97)

Next, note that for each $l = 2, \ldots, 4$, and $j$,

$$\Delta w_{\mu(l)} w_{\nu(l)} V_j (\vec{0}) = \left( \frac{\partial^2 V_j}{\partial w^*_{\mu(l)} \partial w_{\mu(l)}} + \frac{\partial^2 V_j}{\partial w_{\nu(l)} \partial w^*_{\nu(l)}} \right) \left( \vec{0} \right) = 0$$  \hspace{1cm} (98)

because $V_j$ is the average over angle $\hat{l}$ of the function $||\overrightarrow{Q_j}||^{-1}$. The key Lemma for the case $\lambda = -1/2$, implies that

$$\Delta w_{\mu(l)} w_{\nu(l)} V_j (\vec{0}) = 0$$  \hspace{1cm} (99)

assuming enough smoothness to pass the Laplacian through the integral over angle $\hat{l}$. This completes the proof.

We are now ready to verify Herman resonances in the $N = 4$ problem. The proof of the following can be found in the $N = 3$ case in [4], and follows from the Proposition above

**Theorem 6** (Herman) The linear secular system can be diagonalized with pure imaginary eigenvalues, $i\lambda_k, k = 1, \ldots, 6$, where $\sum_{k=1}^{6} \lambda_k = 0$. One of the $\lambda_k = 0$. 

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5 Appendix: Transformations for a single ellipse

We summarize here for completeness, the transformations that put the motions along a single ellipse in $3 - space$ into a suitable form [4], [8]. We note here the fact that this part of the work is independent of the number of bodies $N$, and is in fact independent of the subsequent transformations to $N - 1$ uncoupled Keplerian motions. Whether the general Jacobi variables from a binary tree $T(N)$ or some other approach where the $(N - 1)$ two-body motions consist of the Sun-planet pairs, is used to later formulate $H_2(T(N))$, it is immaterial to the work in this subsection. Complex variables will be used for the elements along an ellipse and to locate the ellipse in $3 - space$. The notation used here is that in [4].

Let $a > 0$ be the semi-major axis of the said ellipse. Let Cartesian 3-space and also velocities $(v_x, v_y, v_z)$ be represented in complex form by $(v_x, v_z) \in C \times R$ where $v_x = v_x + iv_y \in C$. Cartesian coordinates for the two-body problem on a fixed ellipse located in the $x - y$ plane are

$$x = a(\cos u - \varepsilon), y = a\sqrt{1 - \varepsilon^2}\sin u, l = u - \varepsilon \sin u$$

where $\varepsilon$ is the eccentricity, $u$ is the eccentric anomaly angle, and $l$ is the mean anomaly angle such that $dl/dt$ is a constant.

To fix the perihelion use the angle $\omega$: thus the complex position of the body is

$$R = (x + iy)e^{i\omega} = ae^{i\omega}(\cos u - \varepsilon + i(1 - \frac{1}{2}\varepsilon^2 + ... \sin u).$$

Regularizing as $\varepsilon \rightarrow 0$, by keeping $a$ and $\bar{u} = \omega + u$ constant, the limiting equation is [4]:

$$\frac{R}{a} = e^{i\bar{u}} - L - \frac{1}{4}e^{2i\bar{u}}LL^* + \frac{1}{4}e^{-i\bar{u}}L^2 + ...$$

where $L = \varepsilon e^{i\omega}$ and the h.o.t. are monomials in $e^{i\bar{u}}, L$ and their complex conjugates. Next, define $\varepsilon = \sin \phi$, $\tau = \tan \frac{\phi}{2}$, and replacing $L$ by $k = \tau e^{i\omega}$, derive the rational expression for the position

$$R = \frac{ae^{i\bar{u}}(1 - ke^{-i\bar{u}})^2}{1 + kk^*}$$

Now put the ellipse into $3 - space$ by using the inclination $0 \leq \delta \leq \pi$, the longitude of the ascending node, $\Omega$ which is the angle that the line of nodes make with the first reference vector (called x-axis) of the reference plane, and the affine transform

$$R \rightarrow \left(\cos^2 \frac{\delta}{2}\right)R + \left(\sin^2 \frac{\delta}{2}\right)R^*$$

followed by multiplying by $e^{i\Omega}$. Introducing

$$s = -i\tan \frac{\delta}{2} e^{i\Omega}, \quad g = ke^{i\Omega} = \tan \frac{\phi}{2} e^{i\bar{\omega}}$$

the above affine transform becomes $R \rightarrow (1 + ss^*)^{-1}(R + ss^*R^*)$ and after defining $\bar{\omega} = \omega + \Omega$, called the longitude of perihelion, and $\bar{u} = \bar{u} + \Omega = u + \bar{\omega}$, called the eccentric longitude, they obtained the complex form of the position $(r_x, r_z) \in C \times R$ of the body in $3 - space$:

$$\begin{align*}
(1 + ss^*)(1 + gg^*)r_e &= e^{i\bar{u}}(1 + g^*s - (g + s)e^{-i\bar{u}}) \times \\
&= (1 - g^*s - (g - s)e^{-i\bar{u}}) \times \\
&= -s^*e^{i\bar{u}}(1 - ge^{-i\bar{u}})^2s^* \\
&= -e^{-i\bar{u}}(1 - g^*e^{i\bar{u}})^2s.
\end{align*}$$

Together with the mean anomaly derivative,

$$\frac{dl}{du} = (1 - ge^{-i\bar{u}})(1 - g^*e^{i\bar{u}}),$$
we now have a complete description of the motion along an ellipse in 3-space.

Looking forward to the averaging and linearization steps in the next subsection, we will give the symplectic form \( \sigma = d\mathbf{T} \wedge d\mathbf{T}' \) for motion on a single Keplerian ellipse in 3-space in terms of the above elements. As before let \( \mathbf{T} = (r_c, r_z) \), \( \mathbf{T}' = (v_c, v_z) \) where \( r_c = r_x + i r_y \) and \( v_c = v_x + i v_y \), and the complex symplectic form is

\[
2\sigma = dv_c \wedge dr_z^* + dv_z^* \wedge dr_c + 2dv_z \wedge dr_z.
\]

To obtain \( \mathbf{T}' \) we follow [4] in using the formula \( v^2 a^3 = \mu \) where \( v = dl/dt \) is the frequency of the Keplerian motion. Thus, \( \mathbf{T}' = \frac{d\mathbf{T}}{d\mu} \), and we have the equations of motion,

\[
(1 + ss^*)(1 + gg^*) \frac{v_c}{i\omega} \frac{dl}{du} = e^{i\bar{u}}(1 - g^2 e^{-2i\bar{u}}) + e^{-i\bar{u}}(1 - (g^*)^2 e^{2i\bar{u}})s^2 \quad (112)
\]

\[
(1 + ss^*)(1 + gg^*) \frac{v_z}{i\omega} \frac{dl}{du} = -e^{i\bar{u}}(1 - g^2 e^{-2i\bar{u}})s^* + e^{-i\bar{u}}(1 - (g^*)^2 e^{2i\bar{u}})s. \quad (113)
\]

We will only need the symplectic form at the origin \( s = g = 0 \), which gives after fixing \( \bar{u} \),

\[
\begin{align*}
\frac{dr_c}{ds} &= -2adg + ... , \frac{dr_z}{ds} = a(-e^{i\bar{u}} ds^* - e^{-i\bar{u}} ds) + ... \quad (115) \\
\frac{dv_c}{ds} &= i\omega a (dg + e^{i\bar{u}} dg^*) + ..., \frac{dv_z}{ds} = i\omega a (-e^{i\bar{u}} ds^* + e^{-i\bar{u}} ds) + ... \quad (116) \\
\sigma &= i\sqrt{\mu a}(-2dg + dg^* - 2ds + ds^*) + ... \quad (117)
\end{align*}
\]

We are now ready to state their key Lemma 4 for a single ellipse in general position, which \( \lambda = \frac{1}{2} \) case is relevant later:

**Lemma 7** Let \( a \) be a nonnegative real number. Let \( E(w_1, w_2) \) be the Keplerian ellipse in 3-space with focus at the origin \( O \), semimajor axis \( a \), and complex coordinates \( (w_1, w_2) \). Let \( B \) be a point in the complement of the reference circle \( E(0,0) \). Let the average be

\[
D_\lambda(w_1, w_2) = \frac{1}{2\pi} \int_0^{2\pi} ||\mathbf{AB}||^{2\lambda} d\bar{l}
\]

where \( A \) is a point on \( E(w_1, w_2) \) with mean longitude \( \bar{l} = \bar{l} + \bar{\omega} \), or sum of the mean anomaly and longitude of perihelion. Then,

\[
\Delta D_\lambda|_{w_1=w_2=0} = \frac{1}{2}(2\lambda + 1)a^2 D_{\lambda-1}|_{w_1=w_2=0}. \quad (119)
\]

In the complex Laplacian \( \Delta = \left( \frac{\partial^2}{\partial w_1 \partial w_2} + \frac{\partial^2}{\partial w_2 \partial w_1} \right) \), the variables \( (w_1, w_2) \) may represent any of the following pairs of elliptic elements, including \( (2g, 2s), \left( \frac{L \bar{c}}{\sqrt{2}}, \frac{\bar{c}}{\sqrt{2}} \right) \),

\[
(L_c, S_c) = \left( \frac{2(g - g^* s^2)}{(1 + ss^*)(1 + gg^*)}, \frac{2s(1 - gg^*)}{(1 + ss^*)(1 + gg^*)} \right), \quad \text{and} \quad (120)
\]

\[
(X, Y) = \left( \frac{2g}{\sqrt{1 + gg^*}}, \frac{2s}{\sqrt{1 + gg^*}} \right). \quad (121)
\]

The usage of the subscript \( c \) is the same as above to denote the horizontal or complex part of a 3-vector. The vectors \( \mathbf{S}, \mathbf{L} \) are respectively, the normalized angular momentum and eccentricity vector, with \( ||\mathbf{S}|| = \sqrt{1 - \varepsilon^2}, ||\mathbf{L}|| = \varepsilon \), and \( \mathbf{S} \cdot \mathbf{L} = 0 \). They satisfy the property that \( \sqrt{\mu a} \mathbf{S} \) is the angular momentum where \( \mu \) is the gravitational constant, and \( \mathbf{L} \) points towards the perihelion of the ellipse. Thus, \( \mathbf{S} \) and \( \mathbf{L} \) are the Souriau vectors defined by \( \mathbf{S} = \mathbf{S} + \mathbf{L} \) and \( \mathbf{S} = \mathbf{S} - \mathbf{L} \), with \( ||\mathbf{S}|| = ||\mathbf{L}|| = 1 \).

Lastly, the real variables \( X \) and \( Y \) have the following property: let \( I_i = \sqrt{\mu a} \) be the conjugate to the mean anomaly angle \( \bar{l} \); then the real and imaginary parts of \( \sqrt{I_i X} = x_1 + i x_2 \) and \( \sqrt{I_i Y} = y_1 + iy_2 \) are symplectic variables, known as the Poincare variables, \( (I_i, l, x_2, x_1, y_1, -y_2) \). The canonical symplectic form is \( dI_i \wedge dl + dx_2 \wedge dx_1 + dy_2 \wedge dy_1 \).
References

[1] CC Lim, Nonsingular sign patterns and the orthogonal group *Linear algebra and its applications* V. 184 (1993), 1-12


