Abstract. Maximum principle or positivity-preserving property holds for many mathematical models. When the models are approximated numerically, it is preferred that these important properties can be preserved by numerical discretizations for the robustness and the physical relevance of the approximate solutions. In this paper, we investigate such discretizations of high order accuracy within the central discontinuous Galerkin framework. More specifically, we design and analyze high order maximum-principle-satisfying central discontinuous Galerkin methods for scalar conservation laws, and high order positivity-preserving central discontinuous Galerkin for compressible Euler systems. The performance of the proposed methods will be demonstrated through a set of numerical experiments.

Key words. Hyperbolic conservation laws; Central discontinuous Galerkin method; Maximum principle; Positivity preserving; High order accuracy

1. Introduction. Maximum principle or positivity-preserving property holds for many mathematical models, which are often in the form of PDEs. Such property plays an important role for the theoretical understanding of the equations. When these models are approximated numerically, it is preferred that such properties can be preserved by numerical discretizations for the robustness and the physical relevance of the approximate solutions. In this paper, we are concerned with the preservation of these properties within high order central discontinuous Galerkin framework.

One mathematical model we will consider is the scalar conservation law, given as

\begin{align}
  u_t + F(u)_x &= 0, \quad u(x,0) = u_0(x) 
\end{align}

in one dimension, and

\begin{align}
  u_t + F(u)_x + G(u)_y &= 0, \quad u(x,y,0) = u_0(x,y) 
\end{align}

in two dimensions. Here the initial data \( u_0 \) is a function with bounded variation.

The unique entropy solution to (1.1), similarly to (1.2), satisfies a strict maximum principle [3]. That is, if

\[
m = \min_x (u_0(x)), \quad M = \max_x (u_0(x)),
\]
then \( u(x, t) \in [m, M] \) for any \( x \) and \( t \). This property is also desired for numerical schemes solving (1.1) and (1.2), for instance in some applications where \( u \) represents a volume ratio, and it should always be in the range of \([0, 1]\).

Many numerical schemes have been developed for solving the conservation laws (1.1) and (1.2), with examples including Runge-Kutta discontinuous Galerkin (DG) methods [2], weighted essentially non-oscillatory (WENO) finite difference or finite volume schemes [11, 5], and central schemes such as central DG methods [12]. However, these methods, especially when they are of high order accuracy, do not in general satisfy the strict maximum principle. One important breakthrough was made in [15], where the authors proposed a very general framework to achieve maximum principle especially for high order methods. There a sufficient condition was first given to ensure that the cell averages of the numerical solution, which is from a high order finite volume WENO method or a DG method with the Euler forward method in time, satisfy the maximum principle. A linear scaling limiter was then designed and applied to enforce this condition without destroying the local conservation and accuracy. For higher order temporal accuracy, strong stability preserving (SSP) time discretizations were used, and they can be expressed as a convex combination of the first order Euler forward method and therefore keep the maximum principle property.

With the success in [15], the authors further generalized the techniques to compressible Euler equations in [16]. Here high order positivity-preserving DG methods were developed, which preserve the positivity of density and pressure. Such property is important for the well-posedness of the equations, and the violation of numerical schemes can lead to instability and the breakdown of the simulation. A simpler and more robust strategy was later proposed in [14]. In [1], two authors of the present paper extended the positivity-preserving techniques in [16, 14] to ideal MHD equations, which, in the absence of the magnetic field, will become compressible Euler equations. In [1], high order DG methods and central DG methods were considered, and theoretical analysis was established in one-dimensional case; in two dimensions, the positivity-preserving property of the methods was mainly verified numerically. Our development with the central DG methods in [1], together with the missing analysis in two dimensional case for ideal MHD equations, motivates us to examine both theoretically and numerically the central DG methods in this paper for solving scalar conservation laws and compressible Euler equations while preserving maximum principle or positivity of density and pressure.

Central DG methods are a family of high order numerical methods defined on overlapping meshes, which combine features of central schemes and DG methods, and were introduced originally for hyperbolic conservation laws [12] and then for diffusion equations [13]. By evolving two sets of numerical solutions without using any numerical flux at element interfaces as in Godunov schemes, central DG methods provide new opportunities to designing accurate and stable schemes such as for Hamilton-Jacobi equations [8], ideal MHD equations [6, 7], Camassa-Holm equation [9], and Green-Naghdi equations [10].

In this paper, we first develop a maximum-principle-satisfying high order central DG method for the one- and two-dimensional scalar conservation laws in Section 2. A sufficient condition is proved for the cell averages of the numerical solution to be bounded in \([m, M]\) when the Euler forward method is used in time. This condition can be ensured through a linear scaling limiter as in [15] without destroying accuracy and conservation under a suitable CFL condition. In Section 3, we construct and analyze positivity-preserving central DG methods for compressible Euler equations in one- and two-dimensional spaces, employing the positivity-preserving limiting tech-

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niques similar to those in [1, 14, 15]. The SSP high order time discretizations will keep the maximum principle and positivity-preserving property. Compared with DG methods, central DG methods do not depend on any numerical Riemann solvers to devise maximum-principle-satisfying or positivity-preserving methods. In Section 4, numerical experiments are presented, which are followed by some concluding remarks in Section 5.


2.1. One-dimensional case. In this section, we consider a one-dimensional scalar conservation law in (1.1) and will develop a maximum-principle-preserving central DG method. Periodic boundary condition is assumed for the equation.

The proposed method is based on the standard central DG method [12], which is formulated on two overlapping meshes and evolves two copies of numerical solutions. Let \( \{x_j\}_j \) be a uniform partition of the computational domain \( \Omega = [x_{\text{min}}, x_{\text{max}}] \), and \( \Delta x \) be the mesh size. With \( x_{j+\frac{1}{2}} = \frac{1}{2}(x_j + x_{j+1}) \), \( I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}) \) and \( I_j + \frac{1}{2} = (x_j, x_{j+1}) \), we define two discrete function spaces \( V_h^C \) and \( V_h^D \), associated with the primal mesh \( \{I_j\}_j \) and the dual mesh \( \{I_j + \frac{1}{2}\}_j \), respectively,

\[
V_h^C = V_h^{C,k} = \{v : v|_{I_j} \in P^k(I_j), \forall j\},
\]

\[
V_h^D = V_h^{D,k} = \{v : v|_{I_j + \frac{1}{2}} \in P^k(I_j + \frac{1}{2}), \forall j\}.
\]

Here \( P^k(I) \) denotes the space of polynomials in \( I \) with degree at most \( k \).

To describe the central DG method, assume the numerical solutions are available at \( t = t_n \), denoted by \( u_h^{n,*} \in V_h^* \), we want to update the numerical solutions \( u_h^{n+1,*} \in V_h^* \) at \( t = t_{n+1} = t_n + \Delta t_n \). Here and below, \( * \) stands for \( C \) and \( D \). Our focus for now will be on the first order forward Euler time discretization, and time discretizations with higher order accuracy will be discussed in Section 2.3.

To obtain \( u_h^{n+1,*} \), we apply to (1.1) the central DG method of [12] in space and the forward Euler method in time. That is, to look for \( u_h^{n+1,*} \in V_h^* \), such that \( \forall v^* \in V_h^* \) with any \( j \),

\[
\int_{I_j} u_h^{n+1,C} \cdot v^* dx = \int_{I_j} \left( \theta_n u_h^{n,D} + (1 - \theta_n) u_h^{n,C} \right) \cdot v^* dx + \Delta t_n \int_{I_j} F(u_h^{n,D}) \cdot v^* dx
\]

\[
= \int_{I_j} \left[ F(u_h^{n,D}(x_{j+\frac{1}{2}})) \cdot v^*(x_{j+\frac{1}{2}}) - F(u_h^{n,D}(x_{j-\frac{1}{2}})) \cdot v^*(x_{j-\frac{1}{2}}) \right] dx.
\]

(2.1)

\[
\int_{I_{j-\frac{1}{2}}} u_h^{n+1,D} \cdot v^* dx = \int_{I_{j-\frac{1}{2}}} \left( \theta_n u_h^{n,C} + (1 - \theta_n) u_h^{n,D} \right) \cdot v^* dx + \Delta t_n \int_{I_{j-\frac{1}{2}}} F(u_h^{n,C}) \cdot v^* dx
\]

\[
= \int_{I_{j-\frac{1}{2}}} \left[ F(u_h^{n,C}(x_j)) \cdot v^*(x_j) - F(u_h^{n,C}(x_{j-1})) \cdot v^*(x_{j-1}) \right] dx.
\]

(2.2)
Here \( \theta_n = \Delta t_n / \tau_n \in [0, 1] \), with \( \tau_n \) being the maximal time step allowed by the CFL restriction at \( t_n \). And \( w(x^\pm) = \lim_{h \to 0^\pm} w(x + h) \). In general, the numerical scheme (2.1)-(2.2) does not satisfy the maximum principle.

### 2.1.1. Maximum-principle-satisfying central DG method

To construct a maximum-principle-satisfying central DG method, the most important step is to achieve such property for the cell averages of the computed solutions. To this end, let \( v^* = 1 \) in (2.1) and (2.2), respectively, we get the scheme satisfied by the cell averages,

\[
\begin{align*}
\bar{u}_{h,j}^{n+1, C} &= (1 - \theta_n) \bar{u}_{h,j}^n + \frac{\theta_n}{\Delta x} \int_{I_j} u_h^{n,D} dx - \lambda_x \left( F(u_{h,j+\frac{1}{2}}^n) - F(u_{h,j-\frac{1}{2}}^n) \right), \\
\bar{u}_{h,j-\frac{1}{2}}^{n+1, D} &= (1 - \theta_n) \bar{u}_{h,j-\frac{1}{2}}^n + \frac{\theta_n}{\Delta x} \int_{I_{j-\frac{1}{2}}} u_h^{n,C} dx - \lambda_x \left( F(u_{h,j}^n) - F(u_{h,j-1}^n) \right),
\end{align*}
\]

where \( \lambda_x = \frac{\Delta t_n}{\Delta x} \), \( \bar{u}_{h,i}^n \) denotes the cell average of \( u_h^n \) on \( I_i \) at time \( t_n \), and \( u_{h,i}^n = u_h^n(x_i, t_n) \).

Assume \( \bar{u}_{h,j}^n, \bar{u}_{h,j-\frac{1}{2}}^n \in [m, M], \forall j \), we want to come up with a sufficient condition to ensure \( \bar{u}_{h,j}^{n+1, C}, \bar{u}_{h,j-\frac{1}{2}}^{n+1, D} \in [m, M], \forall j \). Let \( \hat{L}_1^{1,x} = \{ \hat{x}_j^{1,\beta}, \beta = 1, 2, ..., \hat{N} \} \) and \( \hat{L}_2^{2,x} = \{ \hat{x}_j^{2,\beta}, \beta = 1, 2, ..., \hat{N} \} \) be the Legendre Gauss-Lobatto quadrature points on \([x_{j-\frac{1}{2}}, x_j]\) and \([x_j, x_{j+\frac{1}{2}}]\), respectively, and \( \hat{x}_j^{1,\beta}, \hat{x}_j^{2,\beta} = 1, 2, ..., \hat{N} \) be the corresponding quadrature weights on the interval \([-\frac{1}{2}, \frac{1}{2}]\). This quadrature formula is exact for polynomials of degree up to \( 2\hat{N} - 3 \). We choose \( \hat{N} \) such that \( 2\hat{N} - 3 \geq k \). Note that \( \sum_{\beta=1}^{\hat{N}} \hat{w}_\beta = 1 \), \( \hat{w}_1 = \hat{w}_\hat{N} \), and \( x_{j-\frac{1}{2}} = \hat{x}_j^{1,\hat{N}}, x_j = \hat{x}_j^{2,\hat{N}}, \forall j \).

**Theorem 2.1.** For the scheme (2.3)-(2.4), assume \( \bar{u}_{h,j}^{n, C}, \bar{u}_{h,j-\frac{1}{2}}^{n, D} \in [m, M], \forall j \). If \( u_C^h(x_o, t_n), u_h^D(x_o, t_n) \in [m, M] \) for all \( x_o \in I_j^{1,x}, \forall j \) and \( l = 1, 2 \), then \( \bar{u}_{h,j}^{n+1, C} \) and \( \bar{u}_{h,j-\frac{1}{2}}^{n+1, D} \) will belong to \([m, M], \forall j, \) under the CFL condition

\[
\lambda_x a_x \leq \frac{\theta_n}{2} \hat{w}_1
\]

with \( a_x = \max(||F'(u_h^C(\cdot, t_n))||_\infty, ||F'(u_h^D(\cdot, t_n))||_\infty) \).

**Proof.** Using the Legendre Gauss-Lobatto quadrature rule, one has

\[
\frac{1}{\Delta x} \sum_{j} u_h^{n,D} dx = \frac{1}{2} \left( \sum_{\beta=1}^{\hat{N}} \hat{w}_\beta u_1^{n,D} + \sum_{\beta=1}^{\hat{N}} \hat{w}_\beta u_2^{n,D} \right)
\]
where \( u_{l,j}^{n,D} = u_h^D(\tilde{r}_{l,j}^D, t_n) \). Now, substituting (2.6) into (2.3) yields

\[
\bar{u}_{h,j}^{n+1,C} = (1 - \theta_n)\bar{u}_{h,j}^{n,C} + \frac{\theta_n}{2} \left( \sum_{\beta=1}^{\bar{N}} \tilde{\omega}_\beta u_{l,\bar{N}}^{n,D} + \sum_{\beta=1}^{\bar{N}} \tilde{\omega}_\beta u_{l,\bar{N}}^{n,D} \right) - \lambda_x \left( F(u_{l+1,j}^{n,D}) - F(u_{l,j}^{n,D}) \right)
\]

\[
= (1 - \theta_n)\bar{u}_{h,j}^{n,C} + \frac{\theta_n}{2} \left( \sum_{\beta=1}^{\bar{N}} \tilde{\omega}_\beta u_{l,\bar{N}}^{n,D} + \sum_{\beta=1}^{\bar{N}} \tilde{\omega}_\beta u_{l,\bar{N}}^{n,D} \right) + \left( \frac{\theta_n}{2} \tilde{\omega}_1 u_{l,1}^{n,D} + \lambda_x F(u_{l,1}^{n,D}) \right) + \left( \frac{\theta_n}{2} \tilde{\omega}_1 u_{l,2}^{n,D} - \lambda_x F(u_{l,2}^{n,D}) \right)
\]

\[
= \mathcal{H}(\bar{u}_{h,j}^{n,C}, u_{l,1}^{n,D}, \ldots, u_{l,\bar{N}}^{n,D}, u_{2,1}^{n,D}, \ldots, u_{2,\bar{N}}^{n,D}). \tag{2.7}
\]

where \( \mathcal{H} \) is given as follows. On each mesh element \( K \), we modify the solution \( u_K \) into \( \tilde{u}_K = \alpha_K(u_K - \bar{u}_K) + \bar{u}_K \) such that the modified solutions will satisfy the sufficient condition in Theorem 1, while maintaining accuracy and local conservation (see [15] for the analysis on consistency of the scheme, and can be verified directly. Similarly, one can show

\[
\frac{\partial u_{h,j}^{n+1,C}}{\partial u_{h,j}^{n,C}} = \frac{\partial u_{h,j}^{n+1,C}}{\partial u_{h,j}^{n,D}} \leq \mathcal{H}(u_{h,j}^{n,C}, u_{l,1}^{n,D}, \ldots, u_{l,\bar{N}}^{n,D}, u_{2,1}^{n,D}, \ldots, u_{2,\bar{N}}^{n,D}) = u_{h,j}^{n+1,C}
\]

\[
= \mathcal{H}(M, \ldots, M) = M, \tag{2.8}
\]

hence \( \bar{u}_{h,j}^{n+1,C} \in [m, M], \forall j \). The first and last equalities in (2.8) are related to the non-decreasing with respect to each input, more specifically,

\[
\frac{\partial u_{h,j}^{n+1,C}}{\partial u_{h,j}^{n,C}} = \frac{\partial u_{h,j}^{n+1,C}}{\partial u_{h,j}^{n,D}} \leq \mathcal{H}(u_{h,j}^{n,C}, u_{l,1}^{n,D}, \ldots, u_{l,\bar{N}}^{n,D}, u_{2,1}^{n,D}, \ldots, u_{2,\bar{N}}^{n,D}) = u_{h,j}^{n+1,C}
\]

\[
\leq \mathcal{H}(M, \ldots, M) = M; \tag{2.9}
\]

hence \( \bar{u}_{h,j}^{n+1,C} \in [m, M], \forall j \). The first and last equalities in (2.8) are related to the non-decreasing with respect to each input, more specifically,

\[
\frac{\partial u_{h,j}^{n+1,C}}{\partial u_{h,j}^{n,C}} = \frac{\partial u_{h,j}^{n+1,C}}{\partial u_{h,j}^{n,D}} \leq \mathcal{H}(u_{h,j}^{n,C}, u_{l,1}^{n,D}, \ldots, u_{l,\bar{N}}^{n,D}, u_{2,1}^{n,D}, \ldots, u_{2,\bar{N}}^{n,D}) = u_{h,j}^{n+1,C}
\]

\[
\leq \mathcal{H}(M, \ldots, M) = M; \tag{2.9}
\]

hence \( \bar{u}_{h,j}^{n+1,C} \in [m, M], \forall j \). The first and last equalities in (2.8) are related to the non-decreasing with respect to each input, more specifically,

\[
\frac{\partial u_{h,j}^{n+1,C}}{\partial u_{h,j}^{n,C}} = \frac{\partial u_{h,j}^{n+1,C}}{\partial u_{h,j}^{n,D}} \leq \mathcal{H}(u_{h,j}^{n,C}, u_{l,1}^{n,D}, \ldots, u_{l,\bar{N}}^{n,D}, u_{2,1}^{n,D}, \ldots, u_{2,\bar{N}}^{n,D}) = u_{h,j}^{n+1,C}
\]

\[
\leq \mathcal{H}(M, \ldots, M) = M; \tag{2.9}
\]

hence \( \bar{u}_{h,j}^{n+1,C} \in [m, M], \forall j \). The first and last equalities in (2.8) are related to the non-decreasing with respect to each input, more specifically,

\[
\frac{\partial u_{h,j}^{n+1,C}}{\partial u_{h,j}^{n,C}} = \frac{\partial u_{h,j}^{n+1,C}}{\partial u_{h,j}^{n,D}} \leq \mathcal{H}(u_{h,j}^{n,C}, u_{l,1}^{n,D}, \ldots, u_{l,\bar{N}}^{n,D}, u_{2,1}^{n,D}, \ldots, u_{2,\bar{N}}^{n,D}) = u_{h,j}^{n+1,C}
\]

\[
\leq \mathcal{H}(M, \ldots, M) = M; \tag{2.9}
\]

hence \( \bar{u}_{h,j}^{n+1,C} \in [m, M], \forall j \). The first and last equalities in (2.8) are related to the non-decreasing with respect to each input, more specifically,
with $M_k = \max_{x \in L_k} (u_K)$ and $m_k = \min_{x \in L_k} (u_K)$. $\bar{u}_K$ is a convex combination of the original central DG solution $u_K$ and its average, hence does not change the average of $u_K$. This implies the local conservation property. It can be seen that the number of total quadrature points involved in the maximum-principle-satisfying limiter for the central DG method is twice (resp. four times) of that of the standard DG method in one dimension (resp. two dimensions).

### 2.2. Two-dimensional case

In this section, we will develop a maximum-principle-preserving central DG method to solve the two-dimensional scalar conservation law (1.2). Periodic boundary condition is assumed.

Let $\mathcal{T}_h^C = \{C_{ij}, \forall i,j\}$ and $\mathcal{T}_h^D = \{D_{ij}, \forall i,j\}$ define two overlapping meshes for the computational domain $\Omega = [x_{\min}, x_{\max}] \times [y_{\min}, y_{\max}]$, with $C_{ij} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$ and $D_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, where $\{x_i\}$ and $\{y_j\}$ are uniform partitions of $[x_{\min}, x_{\max}]$ and $[y_{\min}, y_{\max}]$, respectively, and $x_{i+\frac{1}{2}} = \frac{1}{2}(x_i + x_{i+1})$, $y_{j+\frac{1}{2}} = \frac{1}{2}(y_j + y_{j+1})$. The mesh size is $\Delta x$ in $x$ direction and $\Delta y$ in $y$ direction.

Associated with each mesh, we define the following discrete spaces,

$$
\mathcal{W}_h^C = \mathcal{W}_h^{C,i,j} = \{v : v|_{C_{ij}} \in P^k(C_{ij}), \forall i,j\},
\mathcal{W}_h^D = \mathcal{W}_h^{D,i,j} = \{v : v|_{D_{ij}} \in P^k(D_{ij}), \forall i,j\}.
$$

To describe the central DG method, assume the numerical solutions are available at $t = t_n$, denoted by $u_h^{n,*} \in \mathcal{W}_h^*$, and we want to update the numerical solutions $u_h^{n+1,*} \in \mathcal{W}_h^*$ at $t = t_{n+1} = t_n + \Delta t_n$. Our focus will be on the first order forward Euler time discretization for now. For the brevity of the presentation, we only describe the procedure to update $u_h^{n+1,C}$, as the one for $u_h^{n+1,D}$ is similar.

To obtain $u_h^{n+1,C}$, we apply to (1.2) the central DG method of [12] in space and the forward Euler method in time. That is, to look for $u_h^{n+1,C} \in \mathcal{W}_h^{C,i,j}$ such that $\forall v \in \mathcal{W}_h^{C,i,j}$ with any $i$ and $j$,

$$
\int_{C_{ij}} u_h^{n+1,C} v dx dy = \int_{C_{ij}} \left( \theta_n u_h^{n,D} + (1 - \theta_n) u_h^{n,C} \right) v dx dy + \Delta t_n \int_{C_{ij}} \left[ F(u_h^{n,D}) \cdot v_x + G(u_h^{n,D}) \cdot v_y \right] dx dy - \Delta t_n \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \left[ F(u_h^{n,D}(x_{i+\frac{1}{2}}, y)) \cdot v(x_{i+\frac{1}{2}}, y) - F(u_h^{n,D}(x_{i-\frac{1}{2}}, y)) \cdot v(x_{i-\frac{1}{2}}, y) \right] dy - \Delta t_n \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left[ G(u_h^{n,D}(x, y_{j+\frac{1}{2}})) \cdot v(x, y_{j+\frac{1}{2}}) - G(u_h^{n,D}(x, y_{j-\frac{1}{2}})) \cdot v(x, y_{j-\frac{1}{2}}) \right] dx.
$$

(2.9)

Here $\theta_n = \Delta t_n/\tau_n \in [0, 1]$, with $\tau_n$ being the maximal time step allowed by the CFL restriction at $t_n$. Similar as in one dimension, the numerical scheme (2.9) in general does not satisfy the maximum principle.

#### 2.2.1. Maximum-principle-satisfying central DG method

In this section, we extend the maximum-principle-satisfying central DG method in Section 2.1.1 to solve the two-dimensional conservation law. We consider the scheme satisfied by the
cell averages of the central DG solution with the forward Euler time discretization, which can be obtained by taking the test function \( v = 1 \) in (2.9),

\[
\bar{u}_{h,ij}^{n+1,C} = (1 - \theta_n)\bar{u}_{h,ij}^n + \frac{\theta_n \Delta t_n}{\Delta x \Delta y} \int_{C_{ij}} u_h^{n,D} \, dx \, dy
\]

- \[
\Delta t_n \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \left[ F(u_h^{n,D}(x, \frac{y_j + 1}{2}, y)) - F(u_h^{n,D}(x, \frac{y_j - 1}{2}, y)) \right] \, dy
\]

\[
\frac{\Delta t_n}{\Delta x \Delta y} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \left[ G(u_h^{n,D}(x, y, \frac{j+1}{2})) - G(u_h^{n,D}(x, y, \frac{j-1}{2})) \right] \, dx.
\]

Here \( \bar{u}_{h,ij}^n \) denotes the cell average of \( u_h^n \) on \( C_{ij} \) at time \( t_n \).

Assume \( \bar{u}_{h,ij}^n, u_h^{n,D} \in [m, M], \forall i, j \), we want to come up with a sufficient condition to ensure \( \bar{u}_{h,ij}^{n+1,C}, u_h^{n+1,D} \in [m, M], \forall i, j \). Let \( \tilde{L}_{i}^{L,x} = \{ \tilde{x}^{1,\beta}_i, \beta = 1, 2, ..., \tilde{N}_l \} \) and \( \tilde{L}_{i}^{L,y} = \{ \tilde{y}^{1,\beta}_i, \beta = 1, 2, ..., \tilde{N}_l \} \) be the Legendre Gauss-Lobatto quadrature points on \( [x_{i-\frac{1}{2}}, x_i] \) and \( [x_i, x_{i+\frac{1}{2}}] \), respectively, \( \tilde{L}_{j}^{G,y} = \{ \tilde{y}^{1,\alpha}_j, \alpha = 1, 2, ..., \tilde{N}_y \} \) and \( \tilde{L}_{j}^{G,x} = \{ \tilde{x}^{1,\alpha}_j, \alpha = 1, 2, ..., \tilde{N}_x \} \) be the quadrature points on \( [y_{j-\frac{1}{2}}, y_j] \) and \( [y_j, y_{j+\frac{1}{2}}] \), respectively. The corresponding quadrature weights \( \tilde{\omega}_{\beta, \beta}, \beta = 1, 2, ..., \tilde{N}_l \) are on the interval \( [-\frac{1}{2}, \frac{1}{2}] \) and \( \tilde{N} \) is chosen such that \( 2 \tilde{N} - 3 \geq k \). In addition, let \( L_i^{L,x} = \{ x^{1,\alpha}_i, \alpha = 1, 2, ..., N \} \) and \( L_j^{L,y} = \{ y^{1,\alpha}_j, \alpha = 1, 2, ..., N \} \) denote the Gaussian quadrature points on \( [x_{i-\frac{1}{2}}, x_i] \) and \( [x_i, x_{i+\frac{1}{2}}] \), respectively, \( L_j^{G,y} = \{ y^{1,\alpha}_j, \alpha = 1, 2, ..., N \} \) and \( L_j^{G,x} = \{ x^{1,\alpha}_j, \alpha = 1, 2, ..., N \} \) be the Gaussian quadrature points on \( [y_{j-\frac{1}{2}}, y_j] \) and \( [y_j, y_{j+\frac{1}{2}}] \), respectively. The corresponding quadrature weights \( \omega_{\alpha, \alpha}, \alpha = 1, 2, ..., N \) are on the interval \( [-\frac{1}{2}, \frac{1}{2}] \) and \( N \) is chosen such that the Gaussian quadrature is exact for polynomials of degree up to \( 2k + 1 \). Define \( L_{i,m} = (L_i^{L,x} \otimes \tilde{L}_m^{G,y}) \cup (\tilde{L}_i^{L,y} \otimes L_m^{G,x}) \) with \( l, m = 1, 2 \). We further approximate the boundary integrals in (2.10) using the Gaussian quadrature rule we just described. For the resulting scheme that is still referred to as (2.10), the following theorem holds.

**Theorem 2.2.** For the scheme (2.10) and its counter part for \( u_h^{n+1,D} \), assume \( u_h^{n,C}, u_h^{n,D} \in [m, M], \forall i, j \). If \( u_h^{C}(x, y, \{ t_n \}, t_n), u_h^{D}(x, y, \{ t_n \}) \in [m, M], \forall \{ t_n \} \in L_{i,m}, \forall i, j \) and \( l, m = 1, 2 \), then \( u_h^{n+1,C} \) and \( u_h^{n+1,D} \) will belong to \([m, M], \forall i, j \), under the CFL condition

\[
\lambda_x a_x + \lambda_y a_y \leq \frac{\theta_n}{2} \tilde{\omega}_{1,1}.
\]

where \( \lambda_x = \frac{\Delta t_n}{\Delta x}, \lambda_y = \frac{\Delta t_n}{\Delta y}, a_x = \max(||F'(u_h^{C}(\cdot, \cdot, t_n))||_{\infty}, ||F'(u_h^{D}(\cdot, \cdot, t_n))||_{\infty}), and a_y = \max(||G'(u_h^{C}(\cdot, \cdot, t_n))||_{\infty}, ||G'(u_h^{D}(\cdot, \cdot, t_n))||_{\infty}).
\]

The proof is essentially a combination of that for Theorem 2.1, and some (now standard) techniques in [16] for two dimensions. It is omitted here.

Next, we will give a maximum-principle-satisfying limiter which modifies the central DG solution \( u_h^C \) and \( u_h^D \) into \( \bar{u}_h^C \) and \( \bar{u}_h^D \) such that the modified solutions will satisfy the sufficient condition given in Theorem 2. The limiter is in the same form as that in one dimension, as long as the notation \( K \) and \( \tilde{L}_K \) are reinterpreted as follows:

On the primal mesh, \( K \) denotes a mesh element \( C_{ij} \) and \( \tilde{L}_K \) represents the set of relevant quadrature points in \( K \), namely \( \tilde{L}_K = \tilde{L}_{i,m} \otimes L_{l,m} \cdot \tilde{L}_K \). On the dual mesh, \( K \)
denotes a mesh element \( D_{ij} \) and \( \tilde{L}_K \) represents the set of relevant quadrature points
in \( K \), namely \( \hat{L}_K = L_{i,j}^{1,1} \cup L_{i,j-1}^{1,2} \cup L_{i-1,j}^{2,1} \cup L_{i-1,j-1}^{2,2} \). At time \( t_n \), the numerical solution on \( K \) is denoted by \( u_K \).

### 2.3. High order time discretizations.

To achieve better accuracy in time, strong stability preserving (SSP) high order time discretizations will be used [4]. Such discretizations can be written as a convex combination of the forward Euler method, and therefore the proposed schemes with a high order SSP time discretization are still maximum-principle-satisfying.

### 3. Positivity-preserving central DG method for compressible Euler equations.

#### 3.1. One-dimensional case.

In this section, we consider the following compressible Euler equations in one dimension,

\[(3.1) \quad U_t + F(U)_x = 0,\]

with \( U = (\rho, \rho u, \rho E)^T \) and \( F(U) = \left( \rho u, \rho u^2 + p, \rho Eu + pu \right)^T \), where \( \rho \) is the density, \( u \) is the velocity, \( E = \frac{1}{2}u^2 + e \) is the total energy with \( e \) being the internal energy, and \( p \) is the pressure. For the closure of the system (3.1), the following equation of state is used,

\[(3.2) \quad p = (\gamma - 1)\rho e,\]

where \( \gamma > 1 \) is the specific heat ratio (\( \gamma = 1.4 \) for air). For this nonlinear hyperbolic system, we want to design a positivity-preserving central DG method which preserves the positivity of density and pressure.

As in Section 2.1, we introduce two discrete function spaces associated with two overlapping meshes \( \{I_j\}_j \) and \( \{I_{j+\frac{1}{2}}\}_j \),

\( V_h^C = V_h^{C,k} = \{ v : v|_{I_j} \in [P^k(I_j)]^3, \forall j \} \),

\( V_h^D = V_h^{D,k} = \{ v : v|_{I_{j+\frac{1}{2}}} \in [P^k(I_{j+\frac{1}{2}})]^3, \forall j \} \),

where \( [P^k(I)]^3 = \{ v = (v_1, v_2, v_3)^T : v_i \in P^k(I), i = 1, 2, 3 \} \) is the vector version of \( P^k(I) \). Assume two copies of the numerical solutions are available at \( t = t_n \), denoted by \( U_h^{n,*} = (\rho_h^{n,*}, (\rho u_h^{n,*}, (\rho E_h^{n,*})^\top)^\top \in V_h^C \), and we want to find the solutions at \( t = t_{n+1} = t_n + \Delta t_n \) using central DG methods. For brevity of the presentation, we only describe the procedure to update \( U_h^{n+1,C} = (\rho_h^{n,C}, (\rho u_h^{n,C}, (\rho E_h^{n,C})^\top)^\top \).

To obtain \( U_h^{n+1,C} \), we apply to (3.1) the central DG method of [12] in space and the forward Euler method in time. That is, to look for \( U_h^{n+1,C} \in V_h^C \) such that for any \( V \in V_h^C \) with any \( j \),

\[
\int_{I_j} U_h^{n+1,C} \cdot V dx = \int_{I_j} \left( \theta_n U_h^{n,D} + (1 - \theta_n) U_h^{n,C} \right) \cdot V dx
+ \Delta t_n \int_{I_j} F(U_h^{n,D}) \cdot V dx
- \Delta t_n \left[ F(U_h^{n,D}(x_{j+\frac{1}{2}})) \cdot V(x_+^{j+\frac{1}{2}})
- F(U_h^{n,D}(x_{j-\frac{1}{2}})) \cdot V(x_-^{j-\frac{1}{2}}) \right].
\]
Here \( \theta_n = \Delta t_n / \tau_n \in [0, 1] \), with \( \tau_n \) being the maximal time step allowed by the CFL restriction at \( t_n \).

For problems with low density or pressure, the numerical scheme (3.3) may produce negative values for density or pressure. This can result in numerical instability and the breakdown of the algorithm.

### 3.1.1. Positivity-preserving central DG method

We start with an admissible set

\[
H = \{ U = (\rho, \rho u, \rho E)^\top : \rho > 0, \ p(U) = (\gamma - 1)(\rho E - \frac{1}{2}\rho u^2) > 0 \},
\]

which defines a convex set [15]. For the standard central DG method in (3.3), we want to propose a positivity-preserving limiter, with which the cell averages of the numerical solution on each mesh belong to the admissible set \( H \) at each time \( t_n \). To achieve this, we consider the first order central DG method using the forward Euler time discretization, given as

\[
U_{j+\frac{1}{2}}^{n+1} = (1 - \theta_n) U_j^n + \theta_n \frac{U_{j-\frac{1}{2}}^n + U_{j+\frac{1}{2}}^n}{2} - \lambda_x \left( F(U_{j+\frac{1}{2}}^n) - F(U_{j-\frac{1}{2}}^n) \right),
\]

where \( \lambda_x = \frac{\Delta x}{\Delta t} \), \( U_j^n \) denotes the cell average of \( U_j^C \) on \( I_j \) at time \( t_n \), and \( U_{j+\frac{1}{2}}^D \) denotes the cell average of \( U_j^D \) on \( I_{j+\frac{1}{2}} \) at time \( t_n \).

**Lemma 3.1.** The first order central DG method in (3.5) and its counter part for \( U_{j+\frac{1}{2}}^{n+1} \) is positivity-preserving under the CFL condition

\[
\lambda_x a_x \leq \frac{1}{2} \theta_n.
\]

That is, if \( U_j^n, U_{j-\frac{1}{2}}^n \in H, \forall j \), then \( U_{j+\frac{1}{2}}^{n+1}, U_{j+\frac{1}{2}}^{n+1} \in H, \forall j \). Here \( a_x = \max(||u_j^C||_\infty, ||C^D||_\infty) \), with \( c^* = \sqrt{\gamma \rho h / \rho} \) being the sound speed.

**Proof.** Equation (3.5) can be written as

\[
U_{j+\frac{1}{2}}^{n+1} = (1 - \theta_n) U_j^n + \frac{\theta_n}{2} \left( U_{j-\frac{1}{2}}^n + U_{j+\frac{1}{2}}^n \right) - \frac{2 \lambda_x}{\theta_n} \left( U_{j+\frac{1}{2}}^n - U_{j-\frac{1}{2}}^n \right) F(U_{j+\frac{1}{2}}^n).
\]

\[
(3.7)
\]

Notice that \( U_{j+\frac{1}{2}}^{n+1} \) is a convex combination of three vectors, \( U_j^n, U_{j-\frac{1}{2}}^n \), and \( U_{j+\frac{1}{2}}^n \), and \( U_{j+\frac{1}{2}}^n - 2 \lambda_x / \theta_n F(U_{j+\frac{1}{2}}^n) \). Since \( U_j^n, U_{j-\frac{1}{2}}^n \), and \( U_{j+\frac{1}{2}}^n \) are in the set \( H \), which is convex, we only need to show \( U_{j+\frac{1}{2}}^n - 2 \lambda_x / \theta_n F(U_{j+\frac{1}{2}}^n) \) are both in \( H \). For simplicity, we drop the subscripts and superscripts, and prove that if \( U \in H \), then \( U \pm 2 \lambda_x / \theta_n F(U) \in H \) under the CFL condition (3.6). For density, we have

\[
\rho \left( U \pm 2 \lambda_x / \theta_n F(U) \right) = \rho \pm 2 \lambda_x / \theta_n \rho u = \rho \left( 1 \pm 2 \lambda_x / \theta_n u \right)
\]

\[
> \rho \left( 1 - 2 \lambda_x / \theta_n a_x \right) \geq 0.
\]
For pressure,

\[
\hat{p} \left( U \pm \frac{2\lambda_x}{\theta_n} F(U) \right) = \hat{p} \left( \left( 1 \pm \frac{2\lambda_x}{\theta_n} u \right) \rho, \left( 1 \pm \frac{2\lambda_x}{\theta_n} u \right) \rho u \pm \frac{2\lambda_x}{\theta_n} p, \right) \\
= \left( 1 \pm \frac{2\lambda_x}{\theta_n} u \right) \left( 1 - \frac{p(\gamma - 1)}{2\rho(\theta_n/2\lambda_x \pm u)^2} \right) p.
\]

Given \(1 \pm \frac{2\lambda_x}{\theta_n} u > 0\) and \(p > 0\), one can see

\[
\hat{p} \left( U \pm \frac{2\lambda_x}{\theta_n} F(U) \right) > 0 \iff \frac{p(\gamma - 1)}{2\rho(\theta_n/2\lambda_x \pm u)^2} < 1 \\
\iff \frac{\gamma p}{\rho} < \frac{2\gamma}{\gamma - 1} \left( \frac{\theta_n}{2\lambda_x} \pm u \right)^2.
\]

The fact \(\gamma > 1\) gives \(\frac{2\gamma}{\gamma - 1} > 1\). In addition, the CFL condition (3.6) implies \(\left| \frac{\theta_n}{2\lambda_x} \pm u \right| \geq a_x - u \geq c = \sqrt{\frac{2\rho}{p}}\) thus \(\hat{p} \left( U \pm \frac{2\lambda_x}{\theta_n} F(U) \right) > 0\). Similarly, we can prove \(U^{n+1, D}_j \in H\), \(\forall j\).

Lemma 3.1 is for the first order central DG method. We are now ready to discuss central DG methods of general accuracy. We consider the scheme satisfied by the cell averages of the central DG solution with the forward Euler method in time, given as

\[
\hat{U}^{n+1, C}_{h,j} = (1 - \theta_n) \hat{U}^{n, C}_{h,j} + \frac{\theta_n}{\Delta x} \int_{I_j} U^{n, D}_{h,j} \, dx \\
- \lambda_x \left[ F(U^{n, D}_{h,j + \frac{1}{2}}) - F(U^{n, D}_{h,j - \frac{1}{2}}) \right].
\]

(3.8)

Here \(\hat{U}^{n, C}_{h,j}\) denotes the cell average of \(U^C_h\) on \(I_j\) at time \(t_n\), and \(\hat{U}^{n, D}_{h,j + \frac{1}{2}}\) denotes the cell average of \(U^D_h\) at time \(t_n\).

**Theorem 3.2.** For the scheme (3.8) and its counter part for \(\hat{U}^{n+1, D}_{h,j}\), assume \(\hat{U}^{n+1, C}_{h,j}, \hat{U}^{n+1, D}_{h,j - \frac{1}{2}} \in H, \forall j\). If \(U^C_h(x_o, t_n), U^D_h(x_o, t_n) \in H, \forall x_o \in \bar{I}_{I_j}^l, \forall j\) and \(l = 1, 2, 3\), then \(\hat{U}^{n+1, C}_{h,j}\) and \(\hat{U}^{n+1, D}_{h,j - \frac{1}{2}}\) will belong to \(H, \forall j\), under the CFL condition

\[
\lambda_x a_x \leq \frac{\theta_n}{2} \omega_1.
\]

(3.9)

Here \(a_x\) is defined in Lemma 3.1.

**Proof.** Using the Legendre Gauss-Lobatto quadrature rule, one has

\[
\frac{1}{\Delta x} \int_{I_j} U^{n, D}_{h} \, dx = \frac{1}{2} \left( \sum_{\beta = 1}^{N} \omega_{\beta} U^{n, D}_{1,\beta} + \sum_{\beta = 1}^{N} \omega_{\beta} U^{n, D}_{2,\beta} \right).
\]

(3.10)
Here \( U_{t,\beta}^{n,D} = U_h^n,\beta (\hat{x}_j^1) \). Now, substituting (3.10) into (3.8) yields

\[
\bar{U}_{h,j}^{n+1,C} = (1 - \theta_n)\bar{U}_{h,j}^{n,C} + \frac{\theta_n}{2} \left( \sum_{\beta=1}^{N} \omega_{\beta} U_{1,\beta}^{n,D} + \sum_{\beta=1}^{N} \omega_{\beta} U_{2,\beta}^{n,D} \right)
\]

(3.11)

\[
- \lambda_x \left[ F(U_{h,j+\frac{1}{2}}^{n,D}) - F(U_{h,j-\frac{1}{2}}^{n,D}) \right]
\]

\[
= (1 - \theta_n)\bar{U}_{h,j}^{n,C} + \frac{\theta_n}{2} \left( \sum_{\beta=2}^{N} \omega_{\beta} U_{1,\beta}^{n,D} + \sum_{\beta=1}^{N-1} \omega_{\beta} U_{2,\beta}^{n,D} \right) + \theta_n \hat{\omega}_1 \hat{U}_j
\]

where

\[
\hat{U}_j = \frac{U_{h,j+\frac{1}{2}}^{n,D} + U_{h,j-\frac{1}{2}}^{n,D}}{2} - \frac{\lambda_x}{\theta_n \hat{\omega}_1} \left( F(U_{h,j+\frac{1}{2}}^{n,D}) - F(U_{h,j-\frac{1}{2}}^{n,D}) \right).
\]

We here used \( \hat{\omega}_1 = \hat{\omega}_N, U_{1,1}^{n,D} = U_{2,N}^{n,D} \), and \( U_{2,N}^{n,D} = U_{h,j+\frac{1}{2}}^{n,D} \). Based on Lemma 3.1, we know \( \hat{U}_j \in H \) as long as \( \frac{\lambda_x}{\theta_n \hat{\omega}_1} a_x \leq \frac{1}{2} \), or equivalently \( \lambda_x a_x \leq \frac{1}{2} \theta_n \hat{\omega}_1 \). Note that \( \bar{U}_{h,j}^{n+1,C} \in H \) if \( \beta = 1, 2, \ldots, N-1 \) and \( \hat{U}_j \), which all belong to the convex admissible set \( H \), therefore \( \bar{U}_{h,j}^{n+1,C} \in H, \forall j \). Similarly, one can show \( \bar{U}_{h,j+\frac{1}{2}}^{n+1,D} \in H, \forall j \).

Next, we will give a positivity-preserving limiter which modifies the central DG solution \( U_K^C \) and \( U_h^D \) at time \( t_n \), with the cell averages in the set \( H \), into \( \hat{U}_K^C \) and \( \hat{U}_h^D \), such that the modified solutions will satisfy the sufficient condition in Theorem 3, while maintaining accuracy and local conservation (see [16]).

Let \( K \) denote a mesh element, with the numerical solution on \( K \) being \( U_K = (\rho, \rho u, \rho E) \), and let \( \hat{L}_K \) represent the set of relevant quadrature points in \( K \). Following [14, 1], the positivity-preserving limiter is given as follows.

(a) First, we enforce the positivity of density. In each element \( K \), we modify the density \( \rho \) into \( \hat{\rho} \),

\[
\hat{\rho} = \alpha_K (\rho - \bar{\rho}) + \bar{\rho}, \quad \alpha_K = \min_{x \in \hat{L}_K} \{ 1, |(\hat{\rho} - \bar{\rho})|/(\rho - \bar{\rho}) \}.
\]

Here \( \varepsilon \) is a small number such that \( \bar{\rho} \geq \varepsilon \) for all \( K \). In our simulation, \( \varepsilon = 10^{-13} \) is taken. We now define \( \hat{U}_K = (\hat{\rho}, \rho u, \rho E) \).

(b) Second, we enforce the positivity of pressure. For each \( x \in \hat{L}_K \), we define

\[
\delta_x = \begin{cases} 
\frac{1}{p(\bar{U}_K)} & \text{if } \hat{p}(\hat{U}_K) > 0 \\
\frac{\hat{p}(\hat{U}_K)}{p(\bar{U}_K) - p(\hat{U}_K)} & \text{otherwise}.
\end{cases}
\]

With this, the newly limited numerical solution is given as

\[
\hat{U}_K = \beta_K (\hat{U}_K - \bar{U}_K) + \bar{U}_K, \quad \beta_K = \min_{x \in \hat{L}_K} (\delta_x).
\]

### 3.2. Two-dimensional case.
In this section, we will extend the positivity-preserving central DG method in Section 3.1 to the two-dimensional compressible Euler equations,

\[
U_t + F(U)_x + G(U)_y = 0,
\]

where

\[
U = (\rho, \rho u, \rho v, \rho E)^\top,
\]
Here $\rho$ denotes the density, $u$ and $v$ are the velocity components in the $x$ and $y$ direction, respectively, $p = (\gamma - 1) \rho e$ is the pressure with $e$ being the internal energy, and $E = e + \frac{1}{2}(u^2 + v^2)$ is the total energy.

Following the same notation in Section 2.2 for overlapping meshes $\{C_{ij}\}_{ij}$ and $\{D_{ij}\}_{ij}$, we define two discrete spaces,

$$W_h^C = \{v : v|_{C_{ij}} \in [P^k(C_{ij})]^d, \forall i,j\},$$
$$W_h^D = \{v : v|_{D_{ij}} \in [P^k(D_{ij})]^d, \forall i,j\}.$$

Assume two copies of numerical solutions are available at $t = t_n$, denoted by $U_h^{n,*} = (\rho_h^{n,*}, (\rho u)_h^{n,*}, (\rho v)_h^{n,*}, (\rho E)_h^{n,*})^T \in W_h$, we want to find the solutions at $t = t_{n+1} = t_n + \Delta t_n$. Again, only the procedure to update $U_h^{n+1,C}$ is presented.

To obtain $U_h^{n+1,C}$, we apply to (3.12) the central DG method of [12] in space and the forward Euler method in time. That is, to look for $U_h^{n+1,C} \in W_h^C$ such that for any $V \in W_h^C$ with any $i$ and $j$,

$$\int_{C_{ij}} U_h^{n+1,C} \cdot V \, dx dy + \Delta t_n \int_{C_{ij}} \left[ F(U_h^{n,D}) \cdot V_x + G(U_h^{n,D}) \cdot V_y \right] dx dy - \Delta t_n \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \left[ F(U_h^{n,D}(x_{i+\frac{1}{2}}, y)) \cdot V(x_{i+\frac{1}{2}}, y) - F(U_h^{n,D}(x_{i-\frac{1}{2}}, y)) \cdot V(x_{i-\frac{1}{2}}, y) \right] dy - G(U_h^{n,D}(x, y_{j+\frac{1}{2}})) \cdot V(x, y_{j+\frac{1}{2}}) \cdot V(x, y_{j-\frac{1}{2}}) \right] dx = 0.$$ (3.13)

We define an admissible set

$$H = \left\{ U = (\rho, \rho u, \rho v, \rho E)^T : \rho > 0, \dot{\rho}(U) = (\gamma - 1) \left( \rho E - \frac{1}{2} \rho (u^2 + v^2) \right) > 0 \right\},$$

which is known to be a convex set [15]. Similar to the one-dimensional case, we consider the scheme satisfied by the cell averages of the central DG solution with the forward Euler method in time, given as

$$U_{h,ij}^{n+1,C} = (1 - \theta_n) U_{h,ij}^{n,C} + \frac{\theta_n}{\Delta x \Delta y} \int_{C_{ij}} U_h^{n,D} \, dx dy - \Delta t_n \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \left[ F(U_h^{n,D}(x_{i+\frac{1}{2}}, y)) - F(U_h^{n,D}(x_{i-\frac{1}{2}}, y)) \right] dy - \Delta t_n \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left[ G(U_h^{n,D}(x, y_{j+\frac{1}{2}})) - G(U_h^{n,D}(x, y_{j-\frac{1}{2}})) \right] dx.$$ (3.14)
where \( \bar{U}_{h,ij}^{n,C} \) denotes the cell average of the central DG solution \( U_h^C \) on \( C_{ij} \) at time \( t_n \). We further approximate the boundary integrals in (3.14) using the Gaussian quadrature rule given in Section 2.2.1. For the resulting scheme that is still referred to as (3.14), the following theorem holds.

**Theorem 3.3.** For the scheme (3.14) and its counter part for \( \bar{U}_{h,ij}^{n+1,D} \), assume \( \bar{U}_{h,ij}^{n,C}, \bar{U}_{h,ij}^{n,D} \in H, \forall i, j \). If \( U_h^C(x_o, y_o, t_n), U_h^D(x_o, y_o, t_n) \in H, \forall(x_o, y_o) \in L_{i,j}^{1,m}, \forall i, j \) and \( l, m = 1, 2 \), then \( \bar{U}_{h,ij}^{n+1,C} \) and \( \bar{U}_{h,ij}^{n+1,D} \) will belong to \( H, \forall i, j \), under the CFL condition

\[
\lambda_x a_x + \lambda_y a_y \leq \frac{\theta_n}{2} \tilde{\omega}_1.
\]

where \( \lambda_x = \frac{\Delta y}{\Delta x}, \lambda_y = \frac{\Delta x}{\Delta y}, a_x = \max(||u_h^C(\cdot, \cdot, t_n)|| + c^C(\cdot, \cdot, t_n)||u_h^D(\cdot, \cdot, t_n)|| + c^D(\cdot, \cdot, t_n)||, \lambda_y = \max(||v_h^C(\cdot, \cdot, t_n)|| + c^C(\cdot, \cdot, t_n)||v_h^D(\cdot, \cdot, t_n)|| + c^D(\cdot, \cdot, t_n)||)|| ), \)

with \( c^* = \sqrt{\gamma p_h/\rho_h} \) being the sound speed.

To enforce the sufficient condition in Theorem 3.3, we will apply a two-dimensional version of the positivity-preserving limiter in Section 3.1, and the detail will be omitted.

Similar to the discussions in Section 2.3, for higher order temporal accuracy, high order SSP time discretizations will be used [4]. With their intrinsic structure of being a convex combination of the Euler forward method, and the admissible set \( H \) being convex, SSP temporal discretizations will keep the positivity-preserving property of the overall methods. When the solutions contains non-smooth structures such as strong shocks, nonlinear limiters are often needed to further improve the numerical stability. In our numerical experiments, a total variation bounded (TVB) corrected minmod slope limiter [2] is applied for some examples. When the nonlinear limiter is used, it is implemented in local characteristic fields and is applied before the maximum-principle-satisfying limiter.

**4. Numerical examples.** In this section, numerical experiments are presented to demonstrate the performance of the proposed methods. We first examine in Section 4.1 the maximum-principle-satisfying central DG methods applied to scalar conservation laws. In Section 4.2, we test positivity-preserving central DG methods to solve compressible Euler equations, the parameter \( \gamma \) is taken to be 1.4 for all tests. In addition, a third order TVD Runge-Kutta method is used as the time discretization [4], with the time step dynamically determined by

\[
\Delta t_n = \frac{C_{cfl}}{a_x} (1D \text{ case}), \quad \Delta t_n = \frac{C_{cfl}}{a_x + a_y} (2D \text{ case}).
\]

Although \( \theta_n \) can be chosen from \([0, 1]\), we take \( \theta_n = 1 \) in all simulations for better computational efficiency. According to the theoretical results, we take \( N = 2 \) for \( k = 1 \) with \( \tilde{\omega}_1 = 1/2 \), and \( N = 3 \) for \( k = 2 \) with \( \tilde{\omega}_1 = 1/6 \). Together with \( \theta_n = 1 \), the requirement on the CFL number becomes \( C_{cfl} \leq 1/4 \) for \( k = 1 \) and \( C_{cfl} \leq 1/12 \) for \( k = 2 \). In this paper, we use \( C_{cfl} = 1/4 \) for \( P^1 \) approximation and \( C_{cfl} = 1/12 \) for \( P^2 \) approximation in our numerical experiments. The numerical results shown in all figures are based on \( P^2 \) approximation unless otherwise stated.

**4.1. Scalar conservation laws.** In this section, we examine the numerical performance of the maximum-principle-satisfying central DG methods for scalar conservation laws.
4.1.1. The 1D linear equation. We consider the linear advection equation in one-dimensional space

\[ u_t + u_x = 0 \]  

with a smooth initial condition \( u(x, 0) = \sin(2\pi x) \) and periodic boundary condition.

The computational domain is \([0, 1]\). In Table 1, we present \( L^1 \), \( L^2 \) and \( L^\infty \) errors and orders of accuracy for \( u \) at \( t = 0.1 \). Even though the errors in \( L^1 \) norm show optimal \((k + 1)\)-st order accuracy, the accuracy degeneracy can be seen especially in \( L^\infty \) errors. This was also observed in the numerical solutions of the finite volume and the DG methods in [15] with the same Runge-Kutta temporal discretization, and was attributed in [15] to the less accurate approximations from the inner stages of such multi-stage time discretizations.

For the linear equation (4.1), we now consider a discontinuous initial data

\[ u(x, 0) = \begin{cases} 
1, & -1 < x \leq 0, \\
-1, & 0 < x \leq 1 
\end{cases} \]

with periodic boundary condition. The computational domain is taken as \([-1, 1]\] with 160 uniform elements. In Figure 1, we compare the results from the central DG methods with and without using maximum-principle-satisfying limiter. It is observed that the numerical solution with the maximum-principle-satisfying limiter maintains maximum principle and has much more satisfactory resolution around the discontinuity, while overshoots and undershoots are observed near the discontinuity when such limiter is not applied.

It turns out the predicted bound for CFL number is not sharp. For instance, using the discontinuous initial data and \( P^2 \) approximation, we tested the proposed method with \( C_{\text{cfl}} = 1/12, 0.1, 0.105, 0.108, 0.109, 0.11, 0.15, 0.2 \). The numerical results show that the method can maintain maximum principle when \( C_{\text{cfl}} \leq 0.108 \), yet it cannot when \( C_{\text{cfl}} \geq 0.109 \). We did not test the CFL number when it is between 0.108 and 0.109.

<table>
<thead>
<tr>
<th>Mesh</th>
<th>( L^1 ) error</th>
<th>Order</th>
<th>( L^2 ) error</th>
<th>Order</th>
<th>( L^\infty ) error</th>
<th>Order</th>
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<td>20</td>
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<td>2.09</td>
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<td>5.53E-05</td>
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<td>9.48E-05</td>
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<td>1.02E-07</td>
<td>2.39</td>
<td>1.22E-06</td>
<td>1.95</td>
</tr>
</tbody>
</table>
4.1.2. The Buckley-Leverett equation. In this test, we solve the scalar conservation law (1.1) with the nonconvex Buckley-Leverett flux

\[ F(u) = \frac{4u^2}{4u^2 + (1 - u)^2} \]

on a computational domain \([-1, 1]\). The initial condition is given by

\[ u(x, 0) = \begin{cases} 1, & -0.5 < x \leq 0, \\ 0, & \text{otherwise}, \end{cases} \]

and outflow boundary condition is used. The numerical solutions on 160 and 2560 uniform elements at \(t = 0.4\) are plotted in Figure 2. The convergence to the entropy solutions is observed for this non-standard hyperbolic equation, and the numerical solutions stay in the range of \([0, 1]\) and compare well with that in [15].

4.1.3. A traffic flow model. In this test, we solve the scalar conservation law (1.1) with the flux

\[ F(u) = \begin{cases} -0.4u^2 + 100u, & 0 \leq u \leq 50, \\ -0.1u^2 + 15u + 3500, & 50 \leq u \leq 100, \\ -0.024u^2 - 5.2u + 4760, & 100 \leq u \leq 350, \end{cases} \]

which provides a traffic flow model. Here \(u \in [0, 350]\) denotes the density of the vehicles on a homogeneous highway, \(F(u)\) is the traffic flow flux function.
Fig. 3. Numerical results at \( t = 0.5 \) (left) and 1.5 (right) for the traffic flow model. Solid line: exact solution; circles: numerical solution.

In the computation, the initial and boundary conditions are taken from [15]. The computational domain is \([0, 20]\) with 800 uniform elements. Figure 3 shows the numerical solutions at \( t = 0.5 \) and 1.5, as well as the exact solutions. The computed solutions stay in the range of \([0, 350]\) and compare well with the exact ones.

**4.1.4. The 2D Burgers’ equation.** We consider the two-dimensional Burgers’ equation
\[ u_t + \left( \frac{u^2}{2} \right)_x + \left( \frac{u^2}{2} \right)_y = 0 \]
with a smooth initial condition \( u(x, 0) = 0.5 + \sin(\pi(x + y)) \) and periodic boundary condition. The computational domain is \([-1, 1] \times [-1, 1]\). In Table 2, we report the \( L^1 \), \( L^2 \) and \( L^\infty \) errors and orders of accuracy for \( u \) at \( t = 0.05 \) when the solution is still smooth. Numerical solutions at \( t = 0.23 \) and \( t = 0.6 \) along the diagonal of the domain, namely \( y = x \), are plotted in Figure 4 and compare quite well with the exact solutions.

**Table 2**
The \( L^1 \), \( L^2 \) and \( L^\infty \) errors and orders of accuracy of \( u \) at \( t = 0.05 \) for the 2D Burgers’ equation.

<table>
<thead>
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<th>( L^1 ) error</th>
<th>Order</th>
<th>( L^2 ) error</th>
<th>Order</th>
<th>( L^\infty ) error</th>
<th>Order</th>
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<td>20 ( \times ) 20</td>
<td>3.02E-02</td>
<td>—</td>
<td>2.33E-02</td>
<td>—</td>
<td>7.98E-02</td>
<td>—</td>
</tr>
<tr>
<td>40 ( \times ) 40</td>
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<tr>
<td>80 ( \times ) 80</td>
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<td>1.22E-03</td>
<td>2.14</td>
<td>3.41E-03</td>
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<tr>
<td>160 ( \times ) 160</td>
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<td>2.09</td>
<td>2.87E-04</td>
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<td>8.30E-04</td>
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<tr>
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<td>320 ( \times ) 320</td>
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<td>3.95E-06</td>
<td>2.91</td>
<td>2.36E-05</td>
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</table>

**4.2. Compressible Euler equations.** In this section, we examine the numerical performance of the positivity-preserving central DG methods for compressible
Euler equations. Except for example 4.2.1 where shocks are not present, the TVB minmod slope limiter is applied to the local characteristic fields before the positivity-preserving limiter is used. This slope limiter involves a parameter $M$, which is tuned for each example.

### 4.2.1. Accuracy test

We start with a two-dimensional low density example to demonstrate the accuracy of the proposed methods. The initial condition is given by

$$\rho(x, y, 0) = 1 + 0.99 \sin(2\pi(x + y)), u(x, y, 0) = v(x, y, 0) = 1, p(x, y, 0) = 1,$$

with periodic boundary condition. The computational domain is $[0, 1] \times [0, 1]$. In Table 3, we report $L^1$, $L^2$ and $L^\infty$ errors and orders of accuracy for $\rho$ at $t = 0.1$. We want to mention that the positivity-preserving limiter does work on coarser meshes. For this example, the numerical solutions show optimal $(k+1)$-st order accuracy with $k = 1, 2$, and no accuracy degeneracy is observed.

<table>
<thead>
<tr>
<th>Mesh</th>
<th>$L^1$ error</th>
<th>Order</th>
<th>$L^2$ error</th>
<th>Order</th>
<th>$L^\infty$ error</th>
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<td>8.47E-05</td>
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<td>1.32E-06</td>
<td>3.00</td>
<td>7.94E-06</td>
<td>3.00</td>
</tr>
</tbody>
</table>
4.2.2. Leblanc shock tube problem. In this test, we consider a one-dimensional Leblanc shock tube problem, with the initial condition given by
\[
(\rho, u, p) = \begin{cases} 
(2, 0, 10^9), & x \leq 0, \\
(0.001, 0, 1), & x > 0
\end{cases}
\]
on the computational domain \([-10, 10]\). Note that the discontinuity in pressure is fairly large. The limiter parameter \(M\) is set as \(10^{10}, 10^{10}, 10^{10}\) for variables \(\rho, \rho u, \rho E\), respectively. In Figure 5, we plot the numerical density, velocity, and pressure at \(t = 0.0001\) on 800 uniform meshes. The results are quite stable, and they overall match well with the exact solution.

4.2.3. The Sedov blast waves. Here, we test the Sedov point-blown wave in one-dimensional space and in two-dimensional space, respectively. The Sedov point-blow wave is a typical low density problem involving shocks.

For the initial condition in one-dimensional space, the density is 1, the velocity is 0, and the total energy is \(10^{-12}\) everywhere except that in the center cell the energy is set as \(\frac{3200000}{\Delta x}\). The computational domain is \([-2, 2]\), and the numerical solutions at \(t = 0.001\) are shown in Figure 6. For the initial condition in two-dimensional space, the density is 1, the velocity is 0, and the total energy is \(10^{-12}\) everywhere except that in the lower left corner cell the energy is set as \(\frac{0.244816}{\Delta x \Delta y}\). The computational domain is \([0, 1.1] \times [0, 1.1]\) with 160 \times 160 uniform elements. The numerical boundary treatment follows that in [16]. The numerical solutions at \(t = 1\) are shown in Figure 7.

In both cases, shocks are well captured, and the results are comparable to the computed solutions by the positivity-preserving DG methods, see [16].

4.2.4. Shock diffraction problem. In this test, we consider a shock diffraction problem where shock passing a backward facing corner is modeled [2, 16]. Standard numerical methods often result in negative density and/or negative pressure, and it is important to utilize positivity-preserving methods to simulate this example.
Fig. 6. Numerical results at $t = 0.001$ for the Sedov blast wave in one-dimensional space. Solid line: results on 3200 elements; circles: results on 800 elements.

Fig. 7. Numerical results at $t = 1$ for the Sedov blast wave in two-dimensional space. Left: numerical result (circles) on 160 elements compared with the exact solution (solid line); right: pressure.

The initial condition is given by

$$(\rho, u, v, p) = \begin{cases} 
(7.04113, 4.07795, 0, 30.05945), & x \leq 0.5, \\
(1.4, 0, 0, 1), & x > 0.5 \end{cases}$$

on an $L$-shape computational domain $[0, 1] \times [6, 11] \cup [1, 13] \times [0, 11]$. We use inflow boundary condition at $\{x = 0, y \in [6, 11]\}$, reflective boundary condition at $\{x \in [0, 1], y = 6\}$ and $\{x = 1, y \in [0, 6]\}$, and outflow boundary condition elsewhere. The mesh size is set as $1/16$ in both $x$ and $y$ directions. The limiter parameter $M$ is equal to 100. In Figure 8, the computed density and pressure at $t = 2.3$ are plotted, and no negative pressure or density is encountered during the simulation.
Fig. 8. Numerical results at $t = 2.3$ for shock diffraction problem. Left: density; right: pressure.

5. Conclusions. In this paper, we have developed and analyzed high order maximum-principle-satisfying central DG methods for scalar conservation laws as well as high order positivity-preserving central DG methods for compressible Euler equations in one- and two-dimensional spaces on Cartesian meshes. Based on the standard central DG methods, the proposed methods call a maximum-principle-satisfying or positivity-preserving limiter at each discrete time and also at each inner stage if multi-stage SSP time discretizations are used. Such limiters are easy to implement, and they do not increase significantly the computational cost.

Though the mathematical analysis for these limiters requires a CFL number smaller than that used in standard central DG methods, in practice, one can adopt a dynamical strategy, by starting with the usual CFL number (for central DG methods), until the computed solution fails to belong to $[m, M]$ or $H$. When this happens, the simulation returns to the previous discrete time and a smaller CFL number hence time step is taken based on the theorems.

REFERENCES

[9] M. Li and A. Chen, High order central discontinuous Galerkin-finite element methods for the


