HYBRIDIZATION AND POSTPROCESSING TECHNIQUES FOR MIXED EIGENFUNCTIONS

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Abstract. We introduce hybridization and postprocessing techniques for the Raviart-Thomas approximation of second-order elliptic eigenvalue problems. Hybridization reduces the Raviart-Thomas approximation to a *condensed* eigenproblem. The condensed eigenproblem is nonlinear, but is smaller than the original mixed approximation. We derive multiple iterative algorithms for solving the condensed eigenproblem and examine their interrelationships and convergence rates. An element-by-element postprocessing technique to improve accuracy of computed eigenfunctions is also presented. We prove that a projection of the error in the eigenspace approximation by the mixed method (of any order) superconverges and that the postprocessed eigenfunction approximations converge faster for smooth eigenfunctions. Numerical experiments using a square and an L-shaped domain illustrate the theoretical results.

1. Introduction

The subject of this paper is the Raviart-Thomas mixed approximation to the following eigenproblem: Find eigenvalues λ in $\mathbb{R}$ satisfying

$$-\nabla \cdot (\alpha \vec{\nabla} u) = \lambda u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

for some nontrivial function u. While this problem has been extensively studied by many authors [3, 7, 9, 17], the aim of the present paper is an investigation of its facets hitherto left largely untouched, namely computation by hybridization, postprocessing, and superconvergence of mixed eigenfunctions. Notational definitions and assumptions on the matrix-valued function α and the domain Ω appear later.

The main features of this work are as follows:

(1) We develop a hybridization technique to "condense" the mixed eigenvalue problem to lower dimensions. The condensed eigenproblem is nonlinear, but has significantly fewer degrees of freedom than the original mixed approximation.
(2) We show that the mixed eigenfunctions can be postprocessed locally to obtain more accurate eigenfunction approximations. We also prove that a projection of the error in the eigenspace approximation by the mixed method superconverges.
(3) We derive iterative algorithms for numerically solving the mixed eigenproblem by two different ways: (*i*) hybridization followed by linearization, and (*ii*) linearization

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followed by hybridization. We show that the two algorithms are mathematically equivalent in the sense that they yield the same approximate eigenpairs at every iteration. We also give an algorithm which exhibits cubic convergence numerically.

Hybridization [1, 5] is now a well-known technique for dimensional reduction in the finite element context. It achieves reduction in the number of globally coupled unknowns by condensing out interior unknowns, thus essentially discretizing a three-dimensional boundary value problem on a two-dimensional manifold (the union of mesh faces) if Ω is three dimensional. This results in efficient numerical methods, especially when finite elements of high polynomial degree are used. Hybridization techniques, thoroughly studied for source problems, poses interesting questions when applied to eigenvalue problems. The primary motivation to consider hybridization of eigenvalue problems is to achieve the same reduction in size for the eigenproblem that one achieves for the source problem. However, as we shall see, when this dimensional reduction is performed on the *linear* discrete eigenproblem, we obtain a *nonlinear* discrete eigenproblem.

Other examples, where dimensional reduction converts linear eigenproblems to nonlinear ones, can be found in computational chemistry. Here one approximates the spectra of a linear Schrödinger operator in high (thousand) space dimensions by a reduced eigenproblem in three space dimensions, obtained e.g. via the density functional theory [14, 4]. While such drastic dimensional reduction poses serious theoretical challenges, our simple dimensional reduction via hybridization offers an example for rigorous study. In this example, we reduce a (mixed) linear eigenproblem in n space dimensions to a (hybridized) nonlinear eigenproblem in $n-1$ space dimensions. We show that despite this dimensional reduction, we can capture all the relevant low energy modes.

To briefly review the background literature on application of mixed finite elements to eigenproblems, we recall that the first paper to state a result on the convergence of the Raviart-Thomas eigenproblem is [17]. This paper uses the abstract theory of spectral approximations developed by Osborn [18]. The results of [17] were further clarified and expanded upon in [3].

More recently, the superconvergence of (a projection of) mixed eigenfunctions has attracted the attention of researchers [3, 10]. Considering that for the mixed approximation of the source problem, such superconvergence results are well known [1, 22], it is natural to ask if a similar result can be found for the mixed eigenfunction approximations as well. However, technical difficulties have obscured a clear understanding of this issue so far, except in the case of the lowest order Raviart-Thomas method. The fact that the lowest order method is equivalent to a non-conforming method [16] was utilized in the eigenvalue context by [3]. In [10], Gardini used techniques similar to those in [3], to prove a superconvergence result for lowest order Raviart-Thomas eigenfunctions.

However, such techniques do not extend to the higher order case. In this paper, we lay out a new approach for proving such superconvergence properties for eigenfunctions. We first analyze a postprocessing operator, prove that it yields eigenfunctions of enhanced accuracy, and as a corollary to this analysis, derive the superconvergence properties. (In the known techniques for the source problem, one usually proceeds in the reverse order.)

In the next section, we begin with the preliminaries on the hybridized Raviart-Thomas method for both source and eigenvalue problems. In Section 3, we present the nonlinear eigenproblem resulting from hybridization of the mixed eigenproblem, as well as a "close-by" condensed linear eigenproblem. Section 4 is devoted to the study of a postprocessing

scheme and superconvergence of the eigenfunctions. Iterative algorithms for the numerical solution of the hybridized eigenproblem are described in Section 5. Finally, we present numerical results in Section 6.

2. The hybridized Raviart-Thomas method

In this preliminary section, we recall several well known features of the hybridized Raviart-Thomas (HRT) mixed method [1, 5, 20].

2.1. **The source problem.** Given any "source" f in $L^2(\Omega)$, this problem is to find the flux $\vec{q}^f$ and solution u^f, satisfying

$$\vec{q}^f + \alpha\, \vec{\nabla}\, u^f = 0 \qquad \text{on } \Omega, \tag{2a}$$

$$\nabla \cdot \vec{q}^f = f \qquad \text{on } \Omega, \tag{2b}$$

$$u^f = 0 \qquad \text{on } \partial\Omega. \tag{2c}$$

All functions are real-valued in this paper. Throughout, $\Omega \subset \mathbb{R}^n$ is a polyhedral domain ($n \geq 2$), $\alpha : \Omega \mapsto \mathbb{R}^{n\times n}$ denotes a variable matrix valued coefficient, which we assume to be symmetric and positive definite on all points in Ω. To facilitate our analysis, we introduce notation for the "solution operator" $T : L^2(\Omega) \mapsto L^2(\Omega)$, which is defined simply by $Tf = u^f$. It is well known that T is compact and self-adjoint. Its spectrum, denoted by $\sigma(T)$, consists of isolated points on $\mathbb{R}$ accumulating at zero. Clearly, μ is an eigenvalue of T if and only if $\mu = 1/\lambda$ for some λ satisfying (1).

Consider the standard finite element setting where the domain Ω is subdivided into simplices forming a mesh $\mathscr{T}_h$ satisfying the usual finite element (geometrical conformity) conditions. We also assume that $\mathscr{T}_h$ is shape regular. The collection of interior mesh faces (i.e., the intersections of two adjacent simplices) is denote by $\mathscr{E}_h$. Let k be a non-negative integer. Define

$$\begin{aligned} V_h &= \{\vec{v} \big|\ \text{ for every mesh element } K, \vec{v}|_K \in P_k(K)^n + \vec{x}P_k(K)\}, \\ W_h &= \{w \big|\ \text{ for every mesh element } K, w|_K \in P_k(K)\}, \\ M_h &= \{\mu \big|\ \text{ for every interior mesh face } e, \mu|_e \in P_k(e), \text{ and } \mu|_{\partial\Omega} = 0\}. \end{aligned}$$

Given f in $L^2(\Omega)$, the HRT approximations to $\vec{q}^f$ and u^f satisfying (2), are given as follows: $\vec{q}_h^f, u_h^f$, and in addition η_h^f (a variable approximating the trace of u^f on element interfaces), are functions in V_h, W_h and M_h, respectively, satisfying

$$(c\,\vec{q}_h^f, \vec{r})_{\mathscr{T}_h} - (u_h^f, \nabla\cdot\vec{r})_{\mathscr{T}_h} + \langle \eta_h^f, \vec{r}\cdot\vec{n}\rangle_{\partial\mathscr{T}_h} = 0, \qquad \forall \vec{r} \in V_h, \tag{3a}$$

$$(\nabla\cdot\vec{q}_h^f, w)_{\mathscr{T}_h} = (f, w)_{\mathscr{T}_h}, \quad \forall w \in W_h, \tag{3b}$$

$$\langle \mu, \vec{q}_h^f\cdot\vec{n}\rangle_{\partial\mathscr{T}_h} = 0, \qquad \forall \mu \in M_h, \tag{3c}$$

where $c = \alpha^{-1}$ and $\vec{n}$ denotes the unit outward normal on element boundaries. The differential operators above must be applied element by element. This, and the fact that functions (such as $\vec{n}$) in (3) can have unequal traces from either side on the element interfaces, motivates the notations therein, namely

$$(v, w)_{\mathscr{T}_h} = \sum_{K\in\mathscr{T}_h} (v, w)_K \qquad \text{and} \qquad \langle v, w\rangle_{\partial\mathscr{T}_h} = \sum_{K\in\mathscr{T}_h} \langle v, w\rangle_{\partial K},$$

where $(u,v)_D = \int_D uv\,dx$ whenever D is a domain of $\mathbb{R}^n$, whereas whenever D is an $n-1$ dimensional domain, the same is denoted by $\langle u,v\rangle_D$. When no confusion can arise, we omit the subscript $\mathscr{T}_h$ to simplify notation. A general version of the method (3) is considered in [6] wherein it is also proved that the above method is uniquely solvable for all the three variables. In analogy with T, we define the discrete solution operator T_h and the discrete flux operator Q_h by $T_h f = u_h^f$. and $Q_h f = \vec{q}_h^f$. Here u_h^f and $\vec{q}_h^f$ solve (3).

The hybridized formulation (3) is attractive because it yields a "reduced" system. To state it, we need more notation. Define $\mathcal{A} : V_h \mapsto V_h$, $\mathcal{B} : V_h \mapsto W_h$, and $\mathcal{C} : V_h \mapsto M_h$, by

$$(\mathcal{A}\vec{p},\vec{r})_\Omega = (c\,\vec{p},\vec{r})_{\mathscr{T}_h}, \qquad (\mathcal{B}\vec{p},v)_\Omega = -(v,\nabla\cdot\vec{p})_{\mathscr{T}_h}, \qquad \langle \mathcal{C}\vec{p},\mu\rangle_{\partial\mathscr{T}_h} = \langle \mu,\vec{p}\cdot\vec{n}\rangle_{\partial\mathscr{T}_h} \tag{4}$$

for all $\vec{p}$, $\vec{r} \in V_h$, $v \in W_h$, and $\mu \in M_h$. Additionally, we need *local solution operators* $\mathcal{Q} : M_h \mapsto V_h$, $\mathcal{U} : M_h \mapsto W_h$, $\mathcal{Q}_W : L^2(\Omega) \mapsto V_h$, $\mathcal{U}_W : L^2(\Omega) \mapsto W_h$. These operators are defined using the solution of the following systems:

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}^t \\ \mathcal{B} & 0 \end{pmatrix} \begin{pmatrix} \mathcal{Q}\mu \\ \mathcal{U}\mu \end{pmatrix} = \begin{pmatrix} -\mathcal{C}^t\mu \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} \mathcal{A} & \mathcal{B}^t \\ \mathcal{B} & 0 \end{pmatrix} \begin{pmatrix} \mathcal{Q}_W f \\ \mathcal{U}_W f \end{pmatrix} = \begin{pmatrix} 0 \\ -P_h^W f \end{pmatrix}, \tag{5}$$

for any $\mu \in M_h$ and $f \in L^2(\Omega)$. Here, and throughout, P_h^W denotes the $L^2(\Omega)$-orthogonal projection into W_h. The locality and other properties of these operators are discussed at length in [5, 6], where we also find the following theorem.

Theorem 2.1 (The reduced system [5, 6])**.** *The functions $\vec{q}_h^f, u_h^f, \eta_h^f$ in V_h, W_h and M_h, respectively, satisfy* (3) *if and only if η_h^f is the unique function in M_h satisfying*

$$a_h(\eta_h^f,\mu) = b_h(\mu) \qquad \forall \mu \in M_h, \tag{6}$$

$$\vec{q}_h^f = \mathcal{Q}\eta_h^f + \mathcal{Q}_W f, \qquad \textit{and} \tag{7}$$

$$u_h^f = \mathcal{U}\eta_h^f + \mathcal{U}_W f, \tag{8}$$

where $a_h(\mu_1,\mu_2) = (c\,\mathcal{Q}\mu_1, \mathcal{Q}\mu_2)$ and $b_h(\mu) = (f, \mathcal{U}\mu)$.

We will need one more result. Denote the norm in X by $\|\cdot\|_X$, the $L^2(\Omega)$-norm by simply $\|\cdot\|$, and set $h = \max\{\mathrm{diam}(K) : K \in \mathscr{T}_h\}$. Let Π_h^{RT} denote the Raviart-Thomas projection [20]. Then we have the following *superconvergence* result for the source problem.

Theorem 2.2. [1, 6, 8] *Suppose the solution u^f of* (2) *and its flux $\vec{q}^f$ satisfies*

$$\|u^f\|_{H^s(\Omega)} + \|\vec{q}^f\|_{H^s(\Omega)} \le C\|f\|, \tag{9}$$

for some $1/2 < s \le 1$ for all f in $L^2(\Omega)$. Then

$$\|u_h^f - P_h^W u^f\| \le C h^{\min(s,1)} \|\vec{q}^f - \Pi_h^{RT}\vec{q}^f\|_{H(\mathrm{div},\Omega)}.$$

Although this theorem is often stated with $s = 1$ only, the proof in [6] applies for any s for which one can apply Π_h^{RT} to $\vec{q}$. For instance, the assumed condition that $s > 1/2$ is sufficient for $\Pi_h^{RT}\vec{q}$ to be well-defined. Above, and in the remainder in the paper, C will be used to denote a generic constant (whose value at different occurrences may vary) independent of mesh sizes, but possibly dependent on the shape regularity of the mesh and polynomial degrees.

2.2. **The eigenproblem.** While (3) represents the *source problem*, our primary interest in this paper is the *eigenproblem*. This is to determine a nontrivial $(\vec{q}_h, u_h, \eta_h)$ in $V_h \times W_h \times M_h$, and a number λ_h in $\mathbb{R}$, satisfying

$$(c\,\vec{q}_h, \vec{r})_{\mathscr{T}_h} - (u_h, \nabla\cdot\vec{r})_{\mathscr{T}_h} + \langle \eta_h, \vec{r}\cdot\vec{n}\rangle_{\partial\mathscr{T}_h} = 0, \qquad \forall \vec{r} \in V_h, \tag{10a}$$

$$(\nabla\cdot\vec{q}_h, w)_{\mathscr{T}_h} = \lambda_h (u_h, w)_{\mathscr{T}_h}, \quad \forall w \in W_h, \tag{10b}$$

$$\langle \mu, \vec{q}_h\cdot\vec{n}\rangle_{\partial\mathscr{T}_h} = 0, \qquad \forall \mu \in M_h, \tag{10c}$$

or equivalently,

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}^t & \mathcal{C}^t \\ \mathcal{B} & 0 & 0 \\ \mathcal{C} & 0 & 0 \end{pmatrix} \begin{pmatrix} \vec{q}_h \\ u_h \\ \eta_h \end{pmatrix} = -\lambda_h \begin{pmatrix} 0 \\ u_h \\ 0 \end{pmatrix}. \tag{11}$$

This is a generalized eigenvalue problem of the type $\mathtt{Ax} = \lambda_h \mathtt{Bx}$, but is nonstandard because of the large kernel of $\mathtt{B}$. Such eigenvalue problems have been considered previously in [2] where preconditioned iterative techniques are suggested. Our aim here is to reformulate it into a smaller eigenproblem via hybridization.

Equation (11) can be recast as a standard eigenvalue problem for T_h. Indeed, it is easy to see that $T_h u_h = \frac{1}{\lambda_h} u_h$ if and only if λ_h and u_h satisfy (11). Furthermore, note that although we defined T_h as an operator on $L^2(\Omega)$, by the definition of the HRT method,

$$T_h P_h^W = T_h \tag{12}$$

where P_h^W, as before, denotes the $L^2(\Omega)$-orthogonal projection into W_h. Hence the nonzero part of the spectrum of T_h is the same as that of $T_h|_{W_h}$.

Recall that T_h is a self-adjoint operator. This follows from the easy identity $(f, T_h g) = (c\,Q_h g, Q_h f)$ which holds for any $f, g \in L^2(\Omega)$. Moreover, $T_h|_{W_h}$ is positive definite because if $T_h f = 0$ then by the above equation, we find that $Q_h f = 0$, which in turn implies that $f = 0$ by (3b) whenever f is in W_h. Hence, the mixed eigenvalues λ_h are all positive. Since the domain and range of $T_h|_{W_h}$ equal W_h, the numbers $\{1/\lambda_h\}$ are eigenvalues of a square matrix of dimension $\dim(W_h)$. Therefore, the number of mixed eigenvalues, counting according to multiplicity, is exactly $\dim(W_h)$.

Finally, we recall that the problem of convergence of the mixed eigenvalues and eigenspaces has been studied by several authors [3, 17]. In particular, it is known that the elements of the discrete spectrum $\sigma(T_h)$ converge to the corresponding exact eigenvalues in $\sigma(T)$. In fact, given any neighborhood (no matter how small) of $1/\lambda \in \sigma(T)$ containing no other eigenvalue of T, there is an $h_0 > 0$ such that for all $h < h_0$, there are m eigenvalues of T_h, denoted by $1/\lambda_h^{(1)}, 1/\lambda_h^{(2)}, \ldots, 1/\lambda_h^{(m)}$ (counting according to multiplicity) in the same neighborhood. Here, m is the multiplicity of $1/\lambda$. Moreover, the following theorem on the rate of convergence is also known [3, 17] (although it is not stated in this form in these references). Throughout this paper, we let E_λ denote the eigenspace of T corresponding to eigenvalue $1/\lambda$ while we use $E_{\lambda,h}$ to denote the direct sum of the eigenspaces of T_h corresponding to $1/\lambda_h^{(i)}$ for all $i = 1, 2, \ldots, m$. Whenever we use these notations, it is tacitly understood that h has been made "sufficiently small" so that quantities such as $1/\lambda_h^{(i)}$ can be identified.

Theorem 2.3. [3, 17] *Suppose $1/\lambda \in \sigma(T)$ and s_λ is the largest positive number such that*

$$\|\vec{q}^f\|_{H^{s_\lambda}(\Omega)} + \|u^f\|_{H^{s_\lambda+1}(\Omega)} \le C_{\mathrm{reg}} \|f\|_{L^2(\Omega)} \quad \forall f \in E_\lambda. \tag{13}$$

Assume $s_\lambda > 1/2$. Then there are positive constants h_0 and C_λ (both depending on λ) such that for all $h < h_0$,

$$|\lambda - \lambda_h^{(i)}| \le C_\lambda\, h^{2\min(s_\lambda, k+1)} \tag{14}$$

$$\delta(E_\lambda, E_{\lambda,h}) \le C_\lambda\, h^{\min(s_\lambda, k+1)} \tag{15}$$

where $\delta(E_\lambda, E_{\lambda,h})$ is the gap between E_λ and $E_{\lambda,h}$ as subspaces of $L^2(\Omega)$.

The above mentioned "gap" between two subspaces X and Y of $L^2(\Omega)$, denoted by $\delta(X, Y)$, is the number given by

$$\delta(X, Y) = \sup_{x \in X} \frac{\operatorname{dist}(x, Y)}{\|x\|} = \sup_{y \in Y} \frac{\operatorname{dist}(y, X)}{\|y\|}. \tag{16}$$

Above, we have used the simplified definition of the gap in Hilbert spaces [13], since $L^2(\Omega)$ is Hilbert. Another remark regarding Theorem 2.3 is that the condition $s_\lambda > 1/2$ is required only because the proof uses the Raviart-Thomas projection $\Pi_h^{RT}\vec{q}$ into V_h which is well defined as soon as the components of $\vec{q}$ are in $H^s(\Omega)$ for $s > 1/2$.

One of the aims of this paper is to prove that better eigenspace approximations (with faster convergence rates than in (15)) can be found by postprocessing the computed basis for $E_{\lambda,h}$. We will return to this issue in Section 4. But before that, let us develop a hybridization technique for the eigenproblem.

3. Hybridization of the eigenproblem

In the previous section we recalled that the main advantage of hybridization for the source problem is that all components of the solution can be recovered by means of a reduced, or "condensed" system, namely

$$a_h(\eta_h^f, \mu) = (f, \mathcal{U}\mu) \qquad \forall \mu \in M_h, \tag{17}$$

given by Theorem 2.1. It is natural to ask if such a technique can be designed for the eigenvalue problem. In particular, since the source problem condenses to (17), one may hazard a guess that the eigenvalue problem may be related to finding $\tilde{\lambda}_h$ and $\tilde{\eta}_h \not\equiv 0$ satisfying

$$a_h(\tilde{\eta}_h, \mu) = \tilde{\lambda}_h(\mathcal{U}\tilde{\eta}_h, \mathcal{U}\mu) \qquad \forall \mu \in M_h. \tag{18}$$

A few immediate questions then arise: First, what is the relationship between the mixed eigenvalues λ_h of (11) with the above $\tilde{\lambda}_h$? Are they the same? On closer inspection, we see that (18) is a generalized matrix eigenvalue problem of size $\dim(M_h)$, so the number of $\tilde{\lambda}_h$'s, counting according to multiplicity, is $\dim(M_h)$. On the other hand, as we have already seen (in § 2.2), the number of λ_h's equal $\dim(W_h)$. Since $\dim(M_h)$ is increasingly smaller than $\dim(W_h)$ as the polynomial degree k increases, condensed systems like (18) can be expected to lose more and more eigenmodes as k increases. Have we lost any of the physically important low energy modes? The purpose of this section is to answer such questions.

3.1. Reduction to a nonlinear eigenvalue problem.

Our first result towards answering the questions raised above is the next theorem. Let $h_K = \operatorname{diam}(K)$ for any element K in the mesh, $h = \max\{h_K : K \in \mathscr{T}_h\}$, and $\|\cdot\|_{\ell^2}$ denote the Euclidean norm as well as the norm it induces on $n \times n$ matrices.

Theorem 3.1. *There exists a constant C_*, independent of the polynomial degree k and the element sizes $\{h_K\}$, such that any number*

$$\lambda_h \leq \frac{C_*}{h^2} \tag{19}$$

satisfies

$$a_h(\eta_h, \mu) = \lambda_h \left((I - \lambda_h \mathfrak{U}_W)^{-1} \mathfrak{U}\eta_h, \mathfrak{U}\mu\right), \qquad \forall \mu \in M_h, \tag{20}$$

with some nontrivial η_h in M_h if and only if the number λ_h and the functions

$$\eta_h, \quad u_h = (I - \lambda_h \mathfrak{U}_W)^{-1} \mathfrak{U}\eta_h, \quad \text{and} \quad q_h = \mathfrak{Q}\eta_h + \lambda_h \mathfrak{Q}_W u_h, \tag{21}$$

together satisfy (11)*. We may choose C_* to be any constant satisfying*

$$C_* < \frac{9}{4\, c_{\max}}$$

where $c_{\max}$ denotes the maximum of $\|c(x)\|_{\ell^2}$ for all x in Ω. Above, the operator I denotes the identity on W_h and the inverse in (20) *exists whenever* (19) *holds.*

The implications of this theorem are as follows. First, the condensed form $a_h(\cdot,\cdot)$ does not lose the low energy modes, as the lower eigenvalues satisfy (19). For the source problem, we know that the condensed form is very useful for high degrees k, as the dimensional reduction lowers the number of globally coupled unknowns from $O(k^n)$ to $O(k^{n-1})$. Theorem 3.1 shows that the condensed form retains this advantage for the eigenproblem.

Second, while (20) is indeed *smaller* than the original system (11), it presents a *nonlinear* eigenvalue problem, for which there are fewer algorithms than standard eigenvalue problems. We will discuss our algorithmic options in Section 5.

Third, consider a fixed mesh and let the polynomial degree k increase. We know that the extent of the spectrum increases. The theorem indicates that since C_* remains fixed independent of k, the condensed form (20) may miss the oscillatory eigenfunctions at the high end of the spectrum. But the theorem guarantees that the lower end of the spectrum can be recovered. High k computations are commonly used for capturing (the smoother) low energy modes with high accuracy. These are the modes that the formulation (20) does not miss.

Finally, the theorem also tells us that since (20) and (18) are not identical, we do not expect $\tilde{\lambda}_h$ and λ_h to coincide in general. Nonetheless, (20) opens an avenue to compare $\tilde{\lambda}_h$ with λ_h. We shall do so in § 3.2.

In the remainder of this subsection, we prove Theorem 3.1. First recall from (4) that the operator $\mathcal{B}^t : W_h \mapsto V_h$ is the L^2-adjoint of the divergence map from V_h to W_h, i.e.,

$$(\mathcal{B}^t w, \vec{r})_K = (w, \nabla \cdot \vec{r})_K \quad \forall w \in W_h,\ \forall \vec{r} \in V_h,\ \forall K \in \mathscr{T}_h. \tag{22}$$

We need the following lemma. Below, the notation $\|\cdot\|_D$ denotes the $L^2(D)$-norm.

Lemma 3.1. *For all w in W_h,*

$$\|w\|_K \leq \frac{2}{3}\, h_K \|\mathcal{B}^t w\|_K.$$

Proof. Consider the right inverse of the divergence map, denoted by $\mathcal{D} : L^2(K) \mapsto H(\mathrm{div}, K)$, analyzed in [12]. It is proved in [12, Lemma 2.1] that

$$\nabla \cdot (\mathcal{D}\psi) = \psi \tag{23}$$

$$\|\mathcal{D}\psi\|_K \leq \frac{2}{3} h_K \|\psi\|_K \tag{24}$$

for all ψ in $L^2(K)$, and furthermore that, if $\psi \in P_k(K)$, then $\mathcal{D}\psi$ is in $\vec{x}P_k(K)$. Therefore, given any $w \in W_h$, we can choose $\vec{r} = \mathcal{D}w|_K$ in (22) to get

$$(\mathcal{B}^t w, \mathcal{D}w)_K = (w, \nabla \cdot \mathcal{D}w)_K = \|w\|_K^2$$

by (23). Thus, applying Cauchy-Schwarz inequality to the left hand side and using (24), we finish the proof. □

Note that the inequality $\|w\|_K \leq \tilde{C} h_K \|\mathcal{B}^t w\|_K$ can easily be derived from a simple scaling argument, but the constant $\tilde{C}$ so derived may depend on the polynomial degree. The use of the $\mathcal{D}$ operator in the above proof gives us the k independent constant of Lemma 3.1.

Lemma 3.2. *Let K be any mesh element and $f, g \in L^2(K)$. Then*

$$(\mathcal{U}_W f, g)_K = (c\, \mathcal{Q}_W f, \mathcal{Q}_W g)_K, \tag{25}$$

$$\|\mathcal{U}_W f\|_K \leq c_{\max}^K \frac{4}{9} h_K^2 \|f\|_K, \tag{26}$$

where $c_{\max}^K$ denotes the maximum of $\|c(x)\|_{\ell^2}$ for all x in K.

Proof. Recall from (5) that the local solution operators $\mathcal{Q}_W f$ and $\mathcal{U}_W f$ satisfy

$$(c\, \mathcal{Q}_W f, \vec{r})_K - (\mathcal{U}_W f, \nabla \cdot \vec{r})_K = 0, \tag{27}$$

$$(w, \nabla \cdot \mathcal{Q}_W f)_K = (f, w)_K, \tag{28}$$

for all $\vec{r}$ in V_h and all w in W_h. The proof of (25) follows immediately from the above equations:

$$\begin{aligned} (f, \mathcal{U}_W g)_K &= (\mathcal{U}_W g, \nabla \cdot \mathcal{Q}_W f)_K && \text{by (28)}, \\ &= (c\, \mathcal{Q}_W g, \mathcal{Q}_W f)_K && \text{by (27)}. \end{aligned}$$

To prove (26), first note that since $\mathcal{B}^t \mathcal{U}_W f = -\mathcal{A}\mathcal{Q}_W f$, using Lemma 3.1,

$$\begin{aligned} \|\mathcal{U}_W f\|_K^2 &\leq \left(\frac{2}{3} h_K \|\mathcal{B}^t \mathcal{U}_W f\|_K\right)^2 = \frac{4}{9} h_K^2 \|\mathcal{A}\mathcal{Q}_W f\|_K^2 \\ &\leq \frac{4}{9} h_K^2 c_{\max}^K (c\, \mathcal{Q}_W f, \mathcal{Q}_W f)_K = \frac{4}{9} h_K^2 c_{\max}^K (\mathcal{U}_W f, f)_K \end{aligned}$$

by (25). Thus, an application of Cauchy-Schwarz inequality proves (26). □

Proof of Theorem 3.1. Suppose $\lambda_h, u_h, \vec{q}_h$, and η_h satisfy (10). Then, set $f = \lambda_h u_h$ and apply Theorem 2.1 to get $u_h = \mathcal{U}\eta_h + \mathcal{U}_W f = \mathcal{U}\eta_h + \mathcal{U}_W(\lambda_h u_h)$. Here η_h is nontrivial as it satisfies $a_h(\eta_h, \mu) = (f, \mathcal{U}\mu)$ for all μ with a nonzero f. Now we can recursively apply this identity, *ad infinitum*:

$$\begin{aligned} u_h &= \mathcal{U}\eta_h + \mathcal{U}_W(\lambda_h u_h) = \mathcal{U}\eta_h + \lambda_h(\mathcal{U}_W u_h) \\ &= \mathcal{U}\eta_h + \lambda_h \mathcal{U}_W(\mathcal{U}\eta_h + \mathcal{U}_W(\lambda_h u_h)) \\ &= \big(I + (\lambda_h \mathcal{U}_W) + (\lambda_h \mathcal{U}_W)^2 + (\lambda_h \mathcal{U}_W)^3 + \cdots\big)\mathcal{U}\eta_h \end{aligned} \tag{29}$$

The series in (29) converges in norm, as we now show. By Lemma 3.2, $\|\lambda_h \mathcal{U}_W f\|_K \leq \lambda_h c^K_{\max} \frac{4}{9}\, h_K^2 \|f\|_K$. Hence, whenever

$$\lambda_h c^K_{\max} \frac{4}{9}\, h_K^2 < 1 \tag{30}$$

the $L^2(K)$-operator norm of $\lambda_h \mathcal{U}_W(\cdot)$ is less than one and the series in (29) converges. Set C_* to be any constant satisfying

$$\frac{1}{C_*} > \frac{4}{9} \max\{c^K_{\max} : K \in \mathscr{T}_h\}.$$

Then whenever (19) holds, the inequality (30) holds and the series in (29) converges. The limiting sum of the series is obviously given by

$$(I - \lambda_h \mathcal{U}_W)^{-1} = I + (\lambda_h \mathcal{U}_W) + (\lambda_h \mathcal{U}_W)^2 + (\lambda_h \mathcal{U}_W)^3 + \cdots .$$

Hence, returning to (29), we find that

$$u_h = (I - \lambda_h \mathcal{U}_W)^{-1} \mathcal{U} \eta_h. \tag{31}$$

Applying Theorem 2.1 and setting $f = \lambda_h u_h$ with the above expression for u_h, we conclude that λ_h satisfies (20).

To prove the converse, suppose (20) holds for some nontrivial η_h and some number λ_h satisfying (19), with the above defined C_*. Then, as we have shown above, the inverse in (31) exists. Set u_h by (31) and $f = \lambda_h u_h$. Multiplying (31) by $I - \lambda_h \mathcal{U}_W$ and rearranging, we obtain

$$u_h = \mathcal{U} \eta_h + \mathcal{U}_W f. \tag{32}$$

Next, set

$$\vec{q}_h = \mathcal{Q} \eta_h + \mathcal{Q}_W f. \tag{33}$$

Also, (20) is the same as

$$a_h(\eta_h, \mu) = (f, \mathcal{U}\mu). \tag{34}$$

Equations (32),(33) and (34) imply, by virtue of Theorem 2.1, that the functions $\eta_h, \vec{q}_h, u_h$, and $f = \lambda_h u_h$ satisfy (10). □

3.2. **The perturbed eigenvalue problem.** This subsection is devoted to comparing the mixed eigenvalues λ_h with the eigenvalues $\tilde{\lambda}_h$ of (18). Clearly, $\tilde{\lambda}_h$ can be computed by solving a standard symmetric generalized eigenproblem, for which the algorithmic state of the art is well developed. On the other hand, the mixed eigenvalues λ_h satisfy the nonlinear eigenvalue system (20). We will now show that the easily computable $\tilde{\lambda}_h$ provide good approximations for λ_h, in the lower range of the spectrum. In particular, they can be used as initial guesses in various algorithms to compute λ_h (discussed later in Section 5).

Theorem 3.2. *Suppose λ_h is an eigenvalue of* (10) *satisfying* (19). *Then there is an $h_0 > 0$ (depending on λ_h) and a C_1 (independent of λ_h) such that for all $h < h_0$, there is an eigenvalue $\tilde{\lambda}_h$ of* (18) *satisfying*

$$\frac{|\lambda_h - \tilde{\lambda}_h|}{\lambda_h} \leq C_1\, \lambda_h \tilde{\lambda}_h\, h^2.$$

Proof. This proof proceeds by identifying two nearby operators for which λ_h and $\tilde{\lambda}_h$ are eigenvalues. First, define an operator $S_h : M_h \mapsto M_h$ by

$$a_h(S_h\mu, \gamma) = (\mathcal{U}\mu, \mathcal{U}\gamma) \qquad \forall \gamma \in M_h.$$

Then (18) implies that the reciprocals of $\{\tilde{\lambda}_h\}$ form the spectrum $\sigma(S_h)$, i.e.,

$$S_h\tilde{\eta}_h = \frac{1}{\tilde{\lambda}_h}\tilde{\eta}_h.$$

Next, for any number $\kappa > 0$ such that $I - \kappa\mathcal{U}_W$ is invertible, define another operator $S_h^\kappa : M_h \mapsto M_h$ by

$$a_h(S_h^\kappa\mu, \gamma) = ((I - \kappa\mathcal{U}_W)^{-1}\mathcal{U}\mu, \mathcal{U}\gamma) \qquad \forall \gamma \in M_h.$$

The eigenvalues of S_h^κ are functions of κ and we enumerate all of them by $\{1/\Lambda_h^{(i)}(\kappa)\}$. We know from Theorem 3.1 that if we set κ to any of the mixed eigenvalues λ_h satisfying (19), we have

$$S_h^\kappa\eta_h = \frac{1}{\lambda_h}\eta_h \qquad \text{when } \kappa = \lambda_h,$$

where η_h is the nonlinear eigenfunction corresponding to λ_h. In other words, there is an index ℓ such that

$$\Lambda_h^{(\ell)}(\lambda_h) = \lambda_h. \tag{35}$$

The index ℓ may depend on λ_h, but for every eigenvalue λ_h satisfying (19), there is such an ℓ.

As a next step, we observe that both S_h and S_h^κ are self-adjoint in the $a_h(\cdot,\cdot)$-innerproduct. While the self-adjointness of S_h is obvious, to conclude that of S_h^κ, first note that $\mathcal{U}_W$ is self-adjoint in the $L^2(\Omega)$-innerproduct. This is because of (25) of Lemma 3.2. Consequently, so is $(I - \kappa\mathcal{U}_W)^{-1}$. Thus

$$a_h(S_h^\kappa\mu, \gamma) = ((I - \kappa\mathcal{U}_W)^{-1}\mathcal{U}\mu, \mathcal{U}\gamma) = (\mathcal{U}\mu, (I - \kappa\mathcal{U}_W)^{-1}\mathcal{U}\gamma) = a_h(\mu, S_h^\kappa\gamma),$$

and the self-adjointness of S_h^κ follows. Let $N = \dim(M_h)$ and let us enumerate the eigenvalues of S_h and S_h^κ monotonically by

$$\begin{aligned} \Lambda_h^{(1)}(\kappa) \le \Lambda_h^{(2)}(\kappa) &\le \cdots \le \Lambda_h^{(N)}(\kappa), \\ \tilde{\lambda}_h^{(1)} \le \quad \tilde{\lambda}_h^{(2)} \quad &\le \cdots \le \tilde{\lambda}_h^{(N)}. \end{aligned}$$

Applying Weyl's theorem [23] on eigenvalues of self-adjoint operators, we conclude that

$$\left| \frac{1}{\Lambda_h^{(i)}(\kappa)} - \frac{1}{\tilde{\lambda}_h^{(i)}} \right| \le \|S_h^\kappa - S_h\|_a, \tag{36}$$

where the norm

$$\|S_h^\kappa - S_h\|_a = \sup_{\gamma,\mu \in M_h} \frac{a_h((S_h^\kappa - S_h)\gamma, \mu)}{a_h(\gamma,\gamma)^{1/2} a_h(\mu,\mu)^{1/2}}$$

is the operator norm induced by $a_h(\cdot,\cdot)$.

The final step of this proof consists of estimating the above operator norm. Subtracting the defining equation of S_h from that of S_h^κ,

$$\begin{aligned} a_h((S_h^\kappa - S_h)\gamma, \mu) &= ((I - \kappa\mathcal{U}_W)^{-1}\mathcal{U}\gamma - \mathcal{U}\gamma, \mathcal{U}\mu) \\ &= (\kappa\mathcal{U}_W(I - \kappa\mathcal{U}_W)^{-1}\mathcal{U}\gamma, \mathcal{U}\mu). \end{aligned}$$

By Lemma 3.2,

$$\|\mathcal{U}_W(I-\kappa\mathcal{U}_W)^{-1}\mathcal{U}\gamma\|_K \le \frac{C_1' h_K^2}{1-\kappa C_1' h_K^2}\|\mathcal{U}\gamma\|_K$$

where $C_1' = 4c_{\max}^K/9$. Choosing h_0 to be sufficiently small, we can find a $\theta > 1$ such that $(1-\kappa C_1' h_K^2)^{-1} \le \theta$, for all $h < h_0$. Hence,

$$a_h((S_h^\kappa - S_h)\gamma, \mu) \le \theta C_1' \kappa\, h^2 \|\mathcal{U}\gamma\|\, \|\mathcal{U}\mu\|. \tag{37}$$

To bound the right hand side appropriately, we now recall two inequalities. The first, proved in [6, Lemma 3.3], states that

$$\|\mathcal{U}\mu\| \le C_2\|\mu\|_h, \qquad \text{where} \quad \|\mu\|_h^2 = \sum_{K\in\mathscr{T}_h} h_K \|\mu\|_{L^2(\partial K)}^2, \tag{38}$$

for all μ in M_h. The second is the Poincaré type inequality

$$C_3^2\|\mu\|_h^2 \le a_h(\mu,\mu), \qquad \forall \mu \in M_h, \tag{39}$$

which is established in [11] (see proof of [11, Theorem 2.3]). (The constants C_2, C_3 are independent of h, but may depend on the shape regularity of the mesh and k.) These two inequalities, when applied to the right hand side of (37), imply that

$$a_h((S_h^\kappa - S_h)\gamma, \mu) \le C_1\kappa\, h^2 a_h(\gamma,\gamma)^{1/2} a_h(\mu,\mu)^{1/2}$$

for all $\gamma, \mu \in M_h$, with $C_1 = \theta C_1' C_2^2/C_3^2$. Hence $\|S_h^\kappa - S_h\|_a \le C_1\kappa h^2$.

To conclude, we return to (36), which now implies that

$$\left|\Lambda_h^{(i)}(\kappa) - \tilde\lambda_h^{(i)}\right| \le C_1\kappa\, \tilde\lambda_h^{(i)}\Lambda_h^{(i)}(\kappa)\, h^2.$$

We apply this inequality with $\kappa = \lambda_h$. In view of (35), this means that for every λ_h, we have a $\tilde\lambda_h^{(\ell)}$ satisfying $|\lambda_h - \tilde\lambda_h^{(\ell)}| \le C\tilde\lambda_h^{(\ell)}\lambda_h^2\, h^2$. □

In Section 6, we verify numerically that a few values at the lower end of $\sigma(S_h)$ are indeed $O(h^2)$-approximations of the corresponding eigenvalues in $\sigma(T_h)$. Therefore we conjecture that the convergence rate with respect to h given in Theorem 3.2 cannot be improved in general. We conclude this subsection by noting that in the lowest order case $k = 0$, the mixed eigenvalues λ_h, as well as the perturbed eigenvalues $\tilde\lambda_h$ of S_h converge to the exact eigenvalue at the same rate of $O(h^2)$, assuming the eigenfunctions are smooth enough. It pays to be wary of this coincidence, as it holds only in the lowest order case. Indeed, to recover the full rate of approximation in the higher order case, we must compute λ_h, not $\tilde\lambda_h$.

4. Superconvergence and postprocessing

It has long been known that the solution obtained by the mixed method for the source problem, can be postprocessed to obtain new solutions of enhanced accuracy. This was first shown in [1]. Better postprocessing procedures were obtained later in [22]. In this section, our goal is to generalize the latter to the eigenproblem.

Before we embark on this, let us note a major difference in the analysis of postprocessing between the source and eigenvalue problems. The efficacy of postprocessing for the source problem works is proved using the superconvergence result of Theorem 2.2. However, for the eigenproblem, such a superconvergence result is not yet available (except in the lowest order case [10]). Therefore, we first present a technique to analyze the postprocessing scheme directly, without any knowledge of superconvergence of eigenfunctions. Furthermore, we

show afterward that this postprocessing result implies a superconvergence property for the eigenfunction error.

Let us first define the local postprocessing operator $\mathcal{L}_\ell(u', \vec{q}')$, following [22]. Given a pair of functions $u', \vec{q}'$, the operator gives a function $u'' \equiv \mathcal{L}_\ell(u', \vec{q}')$ in $P_{k+1}(K)$, defined element by element as follows:

$$(\vec{\nabla} u'', \vec{\nabla} w_{k+1})_K = -(c\,\vec{q}', \vec{\nabla} w_{k+1})_K \qquad \forall w_{k+1} \in P_{k+1}^{\perp,\ell}(K), \tag{40a}$$

$$(u'', v_\ell) = (u', v_\ell) \qquad \forall v_\ell \in P_\ell(K), \tag{40b}$$

for all elements $K \in \mathscr{T}_h$. Here $P_{k+1}^{\perp,\ell}(K)$ denotes the $L^2(K)$-orthogonal complement of $P_\ell(K)$ in $P_{k+1}(K)$. The following theorem is essentially contained in [22]. The estimate of the theorem can be proved by a local scaling argument.

Theorem 4.1. [22] *The system* (40) *uniquely defines u'' in $P_{k+1}(K)$ for all $0 \le \ell \le k$, given u' and $\vec{q}'$ in $L^2(K)$. Furthermore, for any u in $H^t(\Omega)$, setting $\vec{q} = -\alpha\,\vec{\nabla} u$, we have*

$$\|u'' - u\| \le C\big(h^t |u|_{H^t(\Omega)} + h\|\vec{q}' - \vec{q}\| + \|P_h^W(u' - u)\| \big)$$

for all $0 \le t \le k+2$ and all $0 \le \ell \le k$.

The postprocessed eigenfunctions are obtained by first computing a mixed eigenfunction u_h and its corresponding flux $\vec{q}_h$ (see (10) for their definitions) and then applying $\mathcal{L}_\ell$ to this pair:

$$\hat{u}_{h,\ell} = \mathcal{L}_\ell(\vec{q}_h, u_h). \tag{41}$$

To describe the corresponding eigenspace precisely, recall the notations and assertions of Theorem 2.3. For sufficiently small h, we know that the direct sum of the eigenspaces corresponding to all the eigenvalues $\lambda_h^{(i)}$ approximating λ, namely $E_{\lambda,h}$, approaches the exact eigenspace E_λ. If m is the multiplicity of λ, then there are m linearly independent eigenfunctions $u_h^{(i)}$, $i = 1, 2, \ldots, m$ of T_h, each corresponding to the eigenvalue $\lambda_h^{(i)}$. Let $\vec{q}_h^{(i)}$ denote the flux of $u_h^{(i)}$. Then the *postprocessed eigenspace* is defined by

$$\hat{E}_{\lambda,h}^\ell = \operatorname{span}\{\hat{u}_{h,\ell}^{(1)}, \hat{u}_{h,\ell}^{(2)}, \ldots, \hat{u}_{h,\ell}^{(m)}\}, \quad \text{where} \quad \hat{u}_{h,\ell}^{(i)} = \mathcal{L}_\ell(\vec{q}_h^{(i)}, u_h^{(i)}). \tag{42}$$

The following theorem shows that the postprocessed eigenfunctions converges at a higher rate than in Theorem 2.3, for sufficiently smooth eigenfunctions.

Theorem 4.2. *Suppose s_λ is the largest positive number such that*

$$\|\vec{q}^f\|_{H^{s_\lambda}(\Omega)} + \|u^f\|_{H^{s_\lambda+1}(\Omega)} \le C_\lambda^{\text{reg}} \|f\|_{L^2(\Omega)} \tag{43}$$

holds for all f in E_λ. Assume that $s_\lambda > 1/2$. Then, there are positive constants h_0 and C, depending on λ, such that all $h < h_0$, the postprocessed eigenspace satisfies

$$\delta(E_\lambda, \hat{E}_{\lambda,h}^\ell) \le C h^{\min(s_\lambda,1)} h^{\min(s_\lambda,k+1)}, \tag{44}$$

for all $0 \le \ell \le k$. We also have the superconvergence estimate

$$\delta(P_h^W E_\lambda, E_{\lambda,h}) \le C h^{\min(s_\lambda,1)} h^{\min(s_\lambda,k+1)}. \tag{45}$$

In the case of a simple eigenvalue λ, for small enough h, there is just one element of the spectrum $\sigma(T_h)$ approximating λ as $h \to 0$. In this case, $E_{\lambda,h}$ is the one-dimensional

eigenspace of that eigenvalue. If (43) holds with $s_\lambda \geq k+1$, and $u \in E_\lambda$, $u_h \in E_{\lambda,h}$, and $\hat{u}_h \in \hat{E}^\ell_{\lambda,h}$ are all functions of unit $L^2(\Omega)$-norm, then Theorem 4.2 implies that

$$\|u - (\pm)\hat{u}_h\| \leq Ch^{k+2}$$
$$\|P_h^W u - (\pm)u_h\| \leq Ch^{k+2}$$

where the notation $\|u - (\pm)v\| = \min(\|u - v\|, \|u + v\|)$ is used to disambiguate any directional mismatch between the eigenfunction and its approximation. For multidimensional eigenspaces, we must of course, use the more general notion of the gap defined by $\delta(\cdot,\cdot)$.

Condition (43) is an assumption on the regularity of eigenfunctions. It is known to hold with s_λ depending on the angles of reentrant corners as well as symmetries in Ω, when α is smooth. For example, suppose $\alpha \equiv c \equiv 1$ and Ω is a polygon having a vertex formed by edges meeting at an angle π/ω measured from within Ω. If ω is an integer, then the eigenfunction is infinitely smooth near that vertex. If not, it is of the form $Cr^\omega \sin(\omega\theta)$ near the vertex [15] (with r, θ being the local polar coordinates), which limits the number s_λ in assumption (43). In the case of an L-shaped domain in $\mathbb{R}^2$, numerical experiments with the lowest order Raviart-Thomas elements are reported in [10]. The eigenfunction u_h computed there approximates an eigenfunction u in $H^{s+1}(\Omega)$ (with its flux $\vec{q} \in H^s(\Omega)$) with s arbitrarily close to 2/3. The observed rate of convergence for $\|P_h^W u - (\pm)u_h\|$ reported in [10] is approximately 4/3, which is in accordance with Theorem 4.2. We will report further numerical experiments in Section 6.

The remainder of this section is devoted to proving Theorem 4.2. The proof relies on the properties of the operator $\hat{T}_h^{(\ell)} : L^2(\Omega) \mapsto L^2(\Omega)$ defined by

$$\hat{T}_h^{(\ell)} f = \mathcal{L}_\ell(Q_h f, T_h f). \tag{46}$$

The following lemma establishes the important properties of this operator when $\ell = k$.

Lemma 4.1. *The nonzero eigenvalues of $\hat{T}_h^{(k)}$ coincide with the nonzero eigenvalues of T_h. Furthermore, if u_h is an eigenfunction of T_h such that*

$$T_h u_h = \beta\, u_h \tag{47}$$

for some $\beta > 0$, then,

$$\hat{T}_h^{(k)}\, \hat{u}_h = \beta\, \hat{u}_h \tag{48}$$

where $\hat{u}_h = \mathcal{L}_k(\vec{q}_h, u_h)$ and $\vec{q}_h$ is the flux of u_h. The multiplicity of β, as an eigenvalue of T_h or $\hat{T}_h^{(k)}$, is the same.

Proof. Let $W_h^\perp$ denote the orthogonal complement of W_h in $L^2(\Omega)$. Since the right hand side of the equations of the method, specifically (3b), vanishes if we set f to any $w^\perp$ in $W_h^\perp$, we find that $T_h w^\perp = Q_h w^\perp = 0$, which implies that

$$\hat{T}_h^{(k)} w^\perp = 0, \qquad \forall\, w^\perp \in W_h^\perp.$$

Therefore $\hat{T}_h^{(k)}$ can have at most $\dim(W_h)$ nonzero eigenvalues, counting according to multiplicity.

Let us now prove that each eigenvalue of T_h is also an eigenvalue of $\hat{T}_h^{(k)}$. Let u_h satisfy (47) and set $\hat{u}_h = \mathcal{L}_k(\vec{q}_h, u_h)$. Then

$$\begin{aligned}
\hat{T}_h^{(k)}\hat{u}_h &= \mathcal{L}_k(Q_h\hat{u}_h, T_h\hat{u}_h) && \text{by (46)},\\
&= \mathcal{L}_k(Q_hP_h^W\hat{u}_h, T_hP_h^W\hat{u}_h) && \text{by (12)},\\
&= \mathcal{L}_k(Q_hu_h, T_hu_h) && \text{by (40b)},\\
&= \mathcal{L}_k(\beta\vec{q}_h, \beta u_h) = \beta\hat{u}_h,
\end{aligned}$$

so we have shown (48).

That the multiplicity of β is unaltered follows from the injectivity property that if $0 = \hat{u}_h = \mathcal{L}_k(\vec{q}_h, u_h)$, then $u_h = 0$, an obvious consequence of (40b) when $\ell = k$. □

Proof of Theorem 4.2. As a first step, we prove (44) when $\ell = k$. In this case, by Lemma 4.1, the postprocessed functions are eigenfunctions of the operators $\hat{T}_h^{(k)}$. Hence the distance between their span and the exact eigenspace can be bounded using the abstract theory of eigenvalue approximations of [18, Theorem 1], yielding

$$\delta(E_\lambda, \hat{E}^k_{\lambda,h}) \le C \sup_{f\in E_\lambda} \frac{\|Tf - \hat{T}_h^{(k)}f\|}{\|f\|}. \tag{49}$$

To bound the numerator above, we use Theorem 4.1 to get

$$\|u^f - \hat{u}_h^f\| \le C\big(\, h^{t+2}|u^f|_{H^{t+2}(\Omega)} + h\|\vec{q}_h^f - \vec{q}^f\| + \|u_h^f - P_h^W u^f\|\,\big) \tag{50}$$

where $\hat{u}_h^f = \hat{T}_h^{(k)}f$ and $t \le k$. Since we have assumed (43), we know that (9) holds with $s = s_\lambda$, so Theorem 2.2 implies

$$\begin{aligned}
\|u_h^f - P_h^W u^f\| &\le Ch^{\min(s_\lambda,1)}\big(\|\vec{q}^f - \Pi_h^{RT}\vec{q}^f\| + \|\nabla\cdot(\vec{q}^f - \Pi_h^{RT}\vec{q}^f)\|\big)\\
&\le Ch^{\min(s_\lambda,1)}(h^{r+1}|\vec{q}^f|_{H^{r+1}(\Omega)} + h^{r+1}|\nabla\cdot\vec{q}^f|_{H^{r+1}(\Omega)})
\end{aligned}$$

with $r+1 = \min(s_\lambda, k+1)$. Note that $\nabla\cdot\vec{q}^f = f = \lambda u^f$ for all f in E_λ, so the higher order norm on $\nabla\cdot\vec{q}^f$ can be bounded using higher norms of u^f. We bound the right hand side of (50) using the above, as well as the well known estimate

$$\|\vec{q}_h^f - \vec{q}^f\| \le C\|\vec{q}^f - \Pi_h^{RT}\vec{q}^f\|.$$

Then we obtain

$$\|u^f - \hat{u}_h^f\| \le Ch^{t+2}|u^f|_{H^{t+2}(\Omega)} + Ch^{\min(s_\lambda,1)}(h^{r+1}|\vec{q}^f|_{H^{r+1}(\Omega)} + h^{r+1}|\lambda u^f|_{H^{r+1}(\Omega)}).$$

We set $t = r$ so that $t+2 = \min(s_\lambda+1, k+2)$ and the regularity estimate (43) can be used to bound the higher order norms on the right hand side. Thus,

$$\begin{aligned}
\|Tf - \hat{T}_h^{(k)}f\| = \|u^f - \hat{u}_h^f\| &\le C(h^{\min(s_\lambda+1,k+2)} + h^{\min(s_\lambda,1)+r+1})\|f\|\\
&\le Ch^{\min(s_\lambda,1)+\min(s_\lambda,k+1)}\|f\|,
\end{aligned}$$

for all f in E_λ. Using this in (49), we prove (44) for the case $\ell = k$.

Our next step is to prove (45). By the definition of the gap (16),

$$\delta(P_h^W E_\lambda, E_{\lambda,h}) = \sup_{u\in E_\lambda} \frac{\operatorname{dist}(P_h^W u, E_{\lambda,h})}{\|u\|}.$$

By the definition of the postprocessing in the $\ell = k$ case, we know that $E_{\lambda,h} = P_h^W \hat{E}^k_{\lambda,h}$. Hence

$$\delta(P_h^W E_\lambda, E_{\lambda,h}) = \sup_{u\in E_\lambda} \frac{\operatorname{dist}(P_h^W u, P_h^W \hat{E}^k_{\lambda,h})}{\|u\|}.$$

Now, since

$$\operatorname{dist}(P_h^W u, P_h^W \hat{E}^k_{\lambda,h}) \le \operatorname{dist}(u, \hat{E}^k_{\lambda,h}),$$

we have

$$\delta(P_h^W E_\lambda, E_{\lambda,h}) \le \sup_{u\in E_\lambda} \frac{\operatorname{dist}(u, \hat{E}^k_{\lambda,h})}{\|u\|} = \delta(E_\lambda, \hat{E}^k_{\lambda,h}) \le Ch^\sigma, \tag{51}$$

by using (44) with $\ell = k$. Here $\sigma = \min(s_\lambda, 1) + \min(s_\lambda, k+1)$. This proves (45).

The final step to end the proof of the theorem involves proving (44) for $\ell \le k-1$. We know from (45) that

$$\sup_{u\in E_\lambda,\, \|u\|=1} \operatorname{dist}(P_h^W u, E_{\lambda,h}) \le Ch^\sigma,$$

with σ as in (51). Therefore, for all u in E_λ with unit norm, we have

$$\min_{v_h\in E_{\lambda,h}} \|P_h^W u - v_h\| \le Ch^\sigma. \tag{52}$$

This minimum is attained by the function u_h' in $E_{\lambda,h}$ satisfying

$$(P_h^W u - u_h', v_h) = 0, \qquad \forall v_h \in E_{\lambda,h}. \tag{53}$$

Expanding the above found u_h' in terms of the eigenfunctions $u_h^{(i)}$ that span $E_{\lambda,h}$,

$$u_h' = \sum_{i=1}^m \gamma_i\, u_h^{(i)},$$

for some numbers γ_i, we define a postprocessed function $\hat{u}_h'$ in $\hat{E}^\ell_{\lambda,h}$ by

$$\hat{u}_h' = \mathcal{L}_\ell(\vec{q}_h', u_h') \qquad \text{where} \qquad \vec{q}_h' = \sum_{i=1}^m \gamma_i\, \vec{q}_h^{(i)}.$$

By Theorem 4.1,

$$\|u - \hat{u}_h'\| \le C\big(\, h^{t+2}|u|_{H^{t+2}(\Omega)} + h\|\vec{q}_h' - \vec{q}\,\| + \|u_h' - P_h^W u\|\,\big) \tag{54}$$

with $t+2 = \min(s_\lambda + 1, k+2)$. To bound the flux error on the right hand side, note that

$$\begin{aligned}
(c\,(\vec{q} - \vec{q}_h'), \Pi_h^{RT}\vec{q} - \vec{q}_h') &= (u - u_h', \nabla\cdot(\Pi_h^{RT}\vec{q} - \vec{q}_h')) && \text{by (10a) and (10c)},\\
&= (P_h^W u - u_h', P_h^W\,\nabla\cdot\vec{q} - \nabla\cdot\vec{q}_h')) && \text{as } \nabla\cdot\Pi_h^{RT} = P_h^W\,\nabla\cdot,\\
&= (P_h^W u - u_h',\ \lambda P_h^W u - \sum_{i=1}^m \lambda_h^{(i)}\gamma_i\, u_h^{(i)}) && \text{by (10b)}.
\end{aligned}$$

Because of (53), we can replace the last sum by any function in $E_{\lambda,h}$, in particular by $\lambda u_h'$. Hence,

$$(c\,(\vec{q} - \vec{q}_h'), \Pi_h^{RT}\vec{q} - \vec{q}_h') = \lambda\|P_h^W u - u_h'\|^2,$$

and consequently,

$$\|\vec{q} - \vec{q}_h'\| \le C\big(\|P_h^W u - u_h'\| + \|\vec{q} - \Pi_h^{RT}\vec{q}\,\|\big).$$

Now returning to (54) and using this estimate there, together with the standard approximation estimate for Π_h^{RT}, we obtain

$$\begin{aligned}\|u - \hat{u}_h'\| &\le C\big(h^{t+2}|u|_{H^{t+2}(\Omega)} + h^{t+2}|\vec{q}|_{H^{t+1}(\Omega)} + \|u_h' - P_h^W u\|\big)\\ &\le Ch^{t+2} + C\|u_h' - P_h^W u\|\end{aligned}$$

Here used the regularity assumption (43). We could do so because $t+1 = \min(s_\lambda, k+1)$. We have also used the fact that $\|u\| = 1$. Since $\hat{u}_h'$ is in the postprocessed eigenspace $\hat{E}_{\lambda,h}^\ell$, the above estimate implies that

$$\operatorname{dist}(u, \hat{E}_{\lambda,h}^\ell) \le Ch^{t+2} + C\|u_h' - P_h^W u\|$$

The last term can be estimated using (52), because the minimum there is attained by u_h'. Moreover, since these arguments hold for every u in E_λ with unit norm, we have

$$\sup_{u\in E_\lambda, \|u\|=1} \operatorname{dist}(u, \hat{E}_{\lambda,h}^\ell) \le C(h^{t+2} + h^\sigma).$$

Since $t+2 \le \sigma$, we have thus proved (44) for all ℓ. □

Remark 4.1. That the dimensions of $\hat{E}_{\lambda,h}^\ell$ and E_λ coincide, was immediately clear for the $\ell = k$ case from Lemma 4.1. The spaces are of equal dimension even for other values of ℓ. This follows as a corollary of (44), by which we can conclude that the gap between the spaces becomes less than one for small enough h, and by standard results on the gap (see, e.g., [13, Lemma 221]). Thus, for sufficiently small h, there is no danger of two linearly independent eigenmodes being postprocessed into linearly dependent ones, even when $\ell = 0$.

5. Algorithmic strategies

The aim of this section is to discuss various algorithmic options for solution of the mixed eigenproblem. We begin by considering the nonlinear eigenproblem (20). Although it is not easy to solve a general nonlinear eigenproblem, we are in the fortunate situation of having very accurate initial approximations by solving one standard eigenproblem, namely the perturbed problem analyzed in § 3.2. Therefore, standard locally convergent iterations such as Newton's method are well suited for solving (20), as discussed in § 5.1. We can also solve the original mixed eigenproblem (11) directly by recasting it as a nonlinear system ready for Newton iteration. The linearized system needing solution in each Newton step can then be hybridized for efficiency. This process can be viewed as *linearization followed by hybridization*. We investigate this approach in § 5.2. In contrast, the above mentioned approach of § 5.1 consists of hybridizing (11) first to get a nonlinear eigenproblem and then applying a Newton iteration, i.e., it is *hybridization followed by linearization*. One of our results in this section (proved in § 5.3) is that both approaches yield the same algorithm. This is pictorially illustrated in the commuting diagram of Figure 1. In § 5.4, we derive an algorithm which exhibits cubic convergence numerically.

5.1. **Solving the nonlinear eigenproblem.** Hybridization of (11) gives rise to the nonlinear eigenproblem (20). Here, we will recast this as a problem of finding the zero of a differentiable function, and apply the Newton iteration. This is a standard approach to solve nonlinear eigenproblems [21]. First, define the operator $A : M_h \mapsto M_h$ by

$$\langle A\mu, \gamma\rangle = a_h(\mu, \gamma) \qquad \forall \mu, \gamma \in M_h.$$

Above, the notation $\langle\cdot,\cdot\rangle$ (without any subscript) denotes $L^2(\mathscr{E}_h)$-innerproduct, i.e., $\langle\mu,\gamma\rangle = \sum_{e\in\mathscr{E}_h}\langle\mu,\gamma\rangle_e$. Also define $M(\lambda)$ to be the operator valued function of λ given by

$$\langle M(\lambda)\mu,\gamma\rangle = ((I-\lambda\mathcal{U}_W)^{-1}\mathcal{U}\mu,\mathcal{U}\gamma).$$

The nonlinear eigenproblem then takes the following form: Find $\eta\in M_h$ and $\lambda>0$ satisfying

$$F(\eta,\lambda)\equiv\begin{pmatrix}A\eta-\lambda M(\lambda)\eta\\ \langle\eta,\eta\rangle-1\end{pmatrix}=0. \tag{55}$$

The first equation of the above system is the same as (20), while the second is a normalization condition. Other normalization conditions can also be used. We apply Newton's method to solve (55). Calculating the Fréchet derivative of F at an arbitrary (η,λ) and writing down the Newton iteration, we find that the next iterate (η',λ') is defined by

$$\begin{pmatrix}(A-\lambda M(\lambda))(\eta'-\eta)-(\lambda'-\lambda)N(\lambda)\eta\\ 2\langle\eta,\eta'-\eta\rangle\end{pmatrix}=-F(\eta,\lambda) \tag{56}$$

where $N(\lambda)=M(\lambda)+\lambda\,dM/d\lambda$. It is easy to see that

$$\langle N(\lambda)\mu,\gamma\rangle = ((I-\lambda\mathcal{U}_W)^{-2}\mathcal{U}\mu,\mathcal{U}\gamma).$$

Assuming that the initial approximation η satisfies $\langle\eta,\eta\rangle=1$, we can rewrite (56) as

$$(A-\lambda M(\lambda))\eta' = (\lambda'-\lambda)N(\lambda)\eta, \tag{57a}$$

$$\langle\eta,\eta'\rangle = 1. \tag{57b}$$

This is the basis of our first algorithm. Observe that the first equation implies that η' depends linearly on $\lambda'-\lambda$. Hence we can decouple the above system, and rearrange the computations, as stated in the next algorithm.

Algorithm 5.1 (Hybridization followed by linearization). To solve for a nonlinear eigenvalue and eigenfunction satisfying (20), proceed as follows:

(1) First obtain an initial approximation η_0 and λ_0 by solving the linear eigenproblem

$$A\eta_0=\lambda_0 M(0)\eta_0.$$

(2) For $n=0,1,2,\ldots,$ until convergence, perform the following steps:
   (a) Compute $\hat\eta$ by solving the linear system

$$(A-\lambda_n M(\lambda_n))\hat\eta = N(\lambda_n)\eta_n. \tag{58}$$

   (b) Set $\delta_\lambda = 1/\langle\hat\eta,\eta_n\rangle$.
   (c) Update eigenvalue: $\lambda_{n+1}=\lambda_n+\delta_\lambda$.
   (d) Update the nonlinear eigenfunction: $\eta_{n+1}=\delta_\lambda\hat\eta$.

Step 1 of the algorithm gives good initial approximations, as already established in Theorem 3.2. The value of δ_λ, equaling the difference of successive eigenvalue iterates, is determined by (57b).

5.2. **Linearization followed by hybridization.** Now we wish to investigate what happens if we perform hybridization *after* we apply Newton iteration on the original mixed eigenproblem. Let us first recast (11) as the problem of finding a zero of a nonlinear function on $V_h \times W_h \times M_h \times \mathbb{R}$ defined by

$$F(\vec{q}, u, \eta, \lambda) = \begin{pmatrix} \begin{pmatrix} \mathcal{A} & \mathcal{B}^t & \mathcal{C}^t \\ \mathcal{B} & 0 & 0 \\ \mathcal{C} & 0 & 0 \end{pmatrix} \begin{pmatrix} \vec{q} \\ u \\ \eta \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ u \\ 0 \end{pmatrix} \\ \langle \eta, \eta \rangle - 1 \end{pmatrix}.$$

A derivative calculation easily shows that the Newton iteration for this F defines a new iterate $(\vec{q}\,', u', \eta', \lambda')$, given an initial iterate $(\vec{q}, u, \eta, \lambda)$, by

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}^t & \mathcal{C}^t \\ \mathcal{B} & \mathcal{D} & 0 \\ \mathcal{C} & 0 & 0 \end{pmatrix} \begin{pmatrix} \vec{q}\,' \\ u' \\ \eta' \end{pmatrix} = -(\lambda' - \lambda) \begin{pmatrix} 0 \\ u \\ 0 \end{pmatrix}, \tag{59a}$$

$$\langle \eta', \eta \rangle = 1, \tag{59b}$$

where $\mathcal{D} : W_h \mapsto W_h$ is defined by

$$(\mathcal{D}u, v) = \lambda(u, v) \qquad \forall u, v \in W_h.$$

The next step is to hybridize (59a). Our aim is to obtain an iteration (after hybridization) in M_h, which has lesser degrees of freedom than W_h as k increases. Notice that, in anticipation of this possibility, we have chosen to normalize η in (59b), not u as one might typically do.

To hybridize (59a), we need local solution operators analogous to the earlier ones in (5). Define the operators $\mathcal{Q}^\lambda, \mathcal{U}^\lambda, \mathcal{Q}^\lambda_W$, and $\mathcal{U}^\lambda_W$ by

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}^t \\ \mathcal{B} & \mathcal{D} \end{pmatrix} \begin{pmatrix} \mathcal{Q}^\lambda \mu \\ \mathcal{U}^\lambda \mu \end{pmatrix} = \begin{pmatrix} -\mathcal{C}^t \mu \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} \mathcal{A} & \mathcal{B}^t \\ \mathcal{B} & \mathcal{D} \end{pmatrix} \begin{pmatrix} \mathcal{Q}^\lambda_W f \\ \mathcal{U}^\lambda_W f \end{pmatrix} = \begin{pmatrix} 0 \\ -P^W_h f \end{pmatrix}. \tag{60}$$

Now an important difference from the situation in (5) arises, namely, the invertibility of the above local operators depends on $\mathcal{D}$, i.e., on λ. From the identity

$$\begin{pmatrix} I & 0 \\ -\mathcal{B}\mathcal{A}^{-1} & I \end{pmatrix} \begin{pmatrix} \mathcal{A} & \mathcal{B}^t \\ \mathcal{B} & \mathcal{D} \end{pmatrix} = \begin{pmatrix} \mathcal{A} & \mathcal{B}^t \\ 0 & \mathcal{D} - \mathcal{B}\mathcal{A}^{-1}\mathcal{B}^t \end{pmatrix},$$

and the fact that $\mathcal{A}$ is invertible, it is clear that $\left(\begin{smallmatrix} \mathcal{A} & \mathcal{B}^t \\ \mathcal{B} & \mathcal{D} \end{smallmatrix}\right)$ is invertible if and only if $\mathcal{D} - \mathcal{B}\mathcal{A}^{-1}\mathcal{B}^t$ is invertible. Thus, a sufficient condition for the local solution operators in (60) to be well defined is that

$$\lambda(u, u) - (\mathcal{A}^{-1}\mathcal{B}^t u, \mathcal{B}^t u) > 0. \tag{61}$$

By Lemma 3.1, $\|u\|_K^2 \le c^K_{\max}(4/9)h_K^2(\mathcal{A}^{-1}\mathcal{B}^t u, \mathcal{B}^t u)_K$. Hence, whenever λ satisfies (19) with $C_* < 9/4c_{\max}$, the inequality (61) holds and consequently the local maps in (60) are well defined. Then, we have the following result. Its proof proceeds like the proof of Theorem 2.1 (see [6]), so we omit it. (In fact a lower order term is included in [6, Theorem A.1], albeit with a sign opposite to what we have here.)

$$\begin{tikzcd}
\text{Mixed eigenproblem} \arrow[r, "\text{hybridize}"] \arrow[d, "\text{linearize}"'] & \text{Nonlinear eigenproblem} \arrow[d, "\text{linearize}"] \\
\text{Newton iterations} \arrow[r, "\text{hybridize}"] & \text{Same algorithm}
\end{tikzcd}$$

FIGURE 1. Two strategies yielding the same algorithm

Proposition 5.1. *Suppose λ satisfies* (19)*. Then, the functions $\vec{q}', u', \eta'$ in V_h, W_h and M_h, respectively, satisfy* (59a) *if and only if η' satisfies*

$$\begin{aligned}
a_h^\lambda(\eta', \mu) &= (\lambda' - \lambda)(u, \mathcal{U}^\lambda \mu) && \forall \mu \in M_h, \\
\vec{q}' &= \mathcal{Q}^\lambda \eta' + (\lambda' - \lambda)\mathcal{Q}_W^\lambda u, && \textit{and} \\
u' &= \mathcal{U}^\lambda \eta' + (\lambda' - \lambda)\mathcal{U}_W^\lambda u,
\end{aligned}$$

where

$$a_h^\lambda(\mu_1, \mu_2) = (c\,\mathcal{Q}^\lambda \mu_1, \mathcal{Q}^\lambda \mu_2) - \lambda(\mathcal{U}^\lambda \mu_1, \mathcal{U}^\lambda \mu_2).$$

In view of this result, we have the following implementation of (59). Again, we exploit the linearity of the η' with respect to $\lambda' - \lambda$ to decouple the two equations of (59).

Algorithm 5.2 (Linearize & hybridize). To solve the mixed eigenproblem (11), follow:

(1) Set λ_0 and η_0 by solving the linear eigenproblem $A\eta_0 = \lambda_0 M(0)\eta_0$.
(2) Set $u_0 = (I - \lambda_0 \mathcal{U}_W)^{-1}\mathcal{U}\eta_0$.
(3) For $n = 0, 1, 2, \ldots,$ until convergence, perform the following steps:
  (a) Compute $\hat{\eta}$ by solving the linear system

$$a_h^{\lambda_n}(\hat{\eta}, \mu) = (u_n, \mathcal{U}^{\lambda_n}\mu) \qquad \forall \mu \in M_h. \tag{62}$$

  (b) Set $\delta_\lambda = 1/\langle \hat{\eta}, \eta_n \rangle$.
  (c) Update the eigenvalue by $\lambda_{n+1} = \lambda_n + \delta_\lambda$.
  (d) Update the approximation from M_h by $\eta_{n+1} = \delta_\lambda \hat{\eta}$.
  (e) Update eigenfunction u by $u_{n+1} = (I - \lambda_{n+1}\mathcal{U}_W)^{-1}\mathcal{U}\eta_{n+1}$.

Notice that this algorithm maintains an iterate u_n in W_h, in addition to the η_n's in M_h. The computation of u_n is local and inexpensive. The formula for updating u_n in step 3e is motivated by the form of exact eigenfunction u_h as seen from (21) of Theorem 3.1.

5.3. **Equivalence of the algorithms.** We are now in a position to precisely state what we indicated in Figure 1. We show that the two algorithms presented earlier are mathematically equivalent (assuming no round off) in the next theorem.

Theorem 5.1. *The iterates η_n and λ_n of Algorithms 5.1 and 5.2 are identical.*

Proof. Both algorithms start with the same initial approximation. Algorithm 5.1 updates the iterates after solving (58), while Algorithm 5.2 requires the solution of (62). These two

linear systems can be rewritten as

$$a_h(\hat{\eta}_1, \mu) - \lambda((I - \lambda \mathcal{U}_W)^{-1}\mathcal{U}\hat{\eta}_1, \mathcal{U}\mu) = ((I - \lambda \mathcal{U}_W)^{-2}\mathcal{U}\eta, \mathcal{U}\mu), \tag{63}$$

$$a_h^\lambda(\hat{\eta}_2, \mu) = ((I - \lambda \mathcal{U}_W)^{-1}\mathcal{U}\eta, \mathcal{U}^\lambda\mu) \tag{64}$$

where η and λ are the iterates at any stage. To distinguish the solutions of (58) and (62), we have denoted them $\hat{\eta}_1$ and $\hat{\eta}_2$, respectively.

To compare these systems, we first examine the difference between $\mathcal{Q}^\lambda, \mathcal{U}^\lambda$ and $\mathcal{Q}, \mathcal{U}$. Rewriting the definition (60) of $\mathcal{Q}^\lambda\mu, \mathcal{U}^\lambda\mu$ as

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}^t \\ \mathcal{B} & 0 \end{pmatrix} \begin{pmatrix} \mathcal{Q}^\lambda\mu \\ \mathcal{U}^\lambda\mu \end{pmatrix} = \begin{pmatrix} -\mathcal{C}^t\mu \\ -\lambda\mathcal{U}^\lambda\mu \end{pmatrix}, \tag{65}$$

we find, by linear superposition, that $\mathcal{U}^\lambda\mu = \mathcal{U}\mu + \mathcal{U}_W(\lambda\mathcal{U}^\lambda\mu)$, and $\mathcal{Q}^\lambda\mu = \mathcal{Q}\mu + \mathcal{Q}_W(\lambda\mathcal{U}^\lambda\mu)$. Therefore, we can express $\mathcal{Q}^\lambda\mu, \mathcal{U}^\lambda\mu$ using the original local solvers in (5) by

$$\mathcal{U}^\lambda\mu = (I - \lambda\mathcal{U}_W)^{-1}\mathcal{U}\mu, \tag{66}$$

$$\mathcal{Q}^\lambda\mu = \mathcal{Q}\mu + \mathcal{Q}_W(\lambda(I - \lambda\mathcal{U}_W)^{-1}\mathcal{U}\mu). \tag{67}$$

Equation (66) already yields that the right hand sides of (63) and (64) coincide: Indeed,

$$\begin{aligned} ((I - \lambda\mathcal{U}_W)^{-1}\mathcal{U}\eta, \mathcal{U}^\lambda\mu) &= ((I - \lambda\mathcal{U}_W)^{-1}\mathcal{U}\eta, (I - \lambda\mathcal{U}_W)^{-1}\mathcal{U}\mu) \\ &= ((I - \lambda\mathcal{U}_W)^{-2}\mathcal{U}\eta, \mathcal{U}\mu), \end{aligned}$$

where we have used that $(I - \lambda\mathcal{U}_W)^{-1}$ is self-adjoint in $L^2(\Omega)$, a fact immediate from (25).

We will now show that the left hand sides of (63) and (64) also coincide:

$$\begin{aligned} a_h^\lambda(\eta, \mu) &= (c\,\mathcal{Q}^\lambda\eta, \mathcal{Q}^\lambda\mu) - \lambda(\mathcal{U}^\lambda\eta, \mathcal{U}^\lambda\mu) && \text{by Proposition 5.1,} \\ &= (c\,\mathcal{Q}^\lambda\eta, \mathcal{Q}^\lambda\mu) - \lambda((I - \lambda\mathcal{U}_W)^{-2}\mathcal{U}\eta, \mathcal{U}\mu) && \text{by (66).} \end{aligned} \tag{68}$$

Now we use (67). To simplify, we note that by [5, Lemma 2.2], $(c\,\mathcal{Q}\mu, \mathcal{Q}_W f) = 0$, so

$$\begin{aligned} (c\,\mathcal{Q}^\lambda\eta, \mathcal{Q}^\lambda\mu) &= (c\,\mathcal{Q}\eta, \mathcal{Q}\mu) + (c\,\mathcal{Q}_W(\lambda(I - \lambda\mathcal{U}_W)^{-1}\mathcal{U}\eta), \mathcal{Q}_W(\lambda(I - \lambda\mathcal{U}_W)^{-1}\mathcal{U}\mu)) \\ &= (c\,\mathcal{Q}\eta, \mathcal{Q}\mu) + (\lambda(I - \lambda\mathcal{U}_W)^{-1}\mathcal{U}\eta, \mathcal{U}_W(\lambda(I - \lambda\mathcal{U}_W)^{-1}\mathcal{U}\mu)) \end{aligned}$$

by (25) of Lemma 3.2. Using this in (68) and simplifying, we find that

$$a_h^\lambda(\eta, \mu) = a_h(\eta, \mu) - \lambda((I - \lambda\mathcal{U}_W)^{-1}\mathcal{U}\eta, \mathcal{U}\mu),$$

hence the solutions $\hat{\eta}_1$ and $\hat{\eta}_2$ of (63) and (64) coincide. Therefore, all the remaining quantities in both the algorithms coincide. □

5.4. **A variant of the algorithm.** We discuss one more algorithm, motivated by the Rayleigh quotient iteration [19], known to yield cubic convergence (while Newton iteration generally yields only quadratic convergence). To derive it we consider the iterates η_{n+1} of Algorithm 5.2. Since (62) is derived from (59), we know that η_{n+1} solves

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}^t & \mathcal{C}^t \\ \mathcal{B} & \mathcal{D}_n & 0 \\ \mathcal{C} & 0 & 0 \end{pmatrix} \begin{pmatrix} \vec{q}_{n+1} \\ \tilde{u}_{n+1} \\ \eta_{n+1} \end{pmatrix} = -\delta_\lambda \begin{pmatrix} 0 \\ u_n \\ 0 \end{pmatrix}, \tag{69}$$

where $\mathcal{D}_n v = \lambda_n v$, $u_n = (I - \lambda_n\mathcal{U}_W)^{-1}\mathcal{U}\eta_n$ and $\delta_\lambda = \lambda_{n+1} - \lambda_n$. Hence, in place of step 3c in Algorithm 5.2, the new algorithm updates λ_{n+1} as the Rayleigh quotient of the current

iterate, namely $\lambda_{n+1} = (c\,\vec{q}_{n+1}, \vec{q}_{n+1})/\|\tilde{u}_{n+1}\|^2$. Since $\tilde{u}_{n+1} = (I - \lambda_n \mathcal{U}_W)^{-1}(\mathcal{U}\eta_{n+1} + \mathcal{U}_W \delta_\lambda u_n)$ and $\vec{q}_{n+1} = \mathcal{Q}\eta_{n+1} + \mathcal{Q}_W(\lambda_n \tilde{u}_{n+1} + \delta_\lambda u_n)$ by Theorem 2.1, we have the following algorithm:

Algorithm 5.3 (Hybridized Rayleigh quotient iteration). To solve (11), follow:

(1) Set λ_0 and η_0 by solving the linear eigenproblem $A\eta_0 = \lambda_0 M(0)\eta_0$.
(2) Set $u_0 = (I - \lambda_0 \mathcal{U}_W)^{-1}\mathcal{U}\eta_0$.
(3) For $n = 0, 1, 2, \ldots$, until convergence, perform the following steps:
  (a) Compute $\hat{\eta}$ by solving (58), i.e., $(A - \lambda_n M(\lambda_n))\hat{\eta} = N(\lambda_n)\eta_n$.
  (b) Set $\delta_\lambda = 1/\langle \hat{\eta}, \eta_n \rangle$.
  (c) Update:

$$\begin{aligned}
\eta_{n+1} &= \delta_\lambda \hat{\eta}, \\
\tilde{u}_{n+1} &= (I - \lambda_n \mathcal{U}_W)^{-1}(\mathcal{U}\eta_{n+1} + \mathcal{U}_W \delta_\lambda u_n), \\
\vec{q}_{n+1} &= \mathcal{Q}\eta_{n+1} + \mathcal{Q}_W(\lambda_n \tilde{u}_{n+1} + \delta_\lambda u_n), \\
\lambda_{n+1} &= \frac{(c\,\vec{q}_{n+1}, \vec{q}_{n+1})}{\|\tilde{u}_{n+1}\|^2}.
\end{aligned}$$

In our numerical studies in Section 6, this algorithm yields the best performance.

6. Numerical results

In this section we present numerical examples to illustrate the theoretical results of Section 3 and 4. Moreover, we investigate the performance of various algorithms proposed in Section 5. Numerical results are presented for a square and an L-shaped domain.

6.1. **Square domain.** We consider the domain $\Omega = (0, \pi) \times (0, \pi)$. In this case, the exact eigenvalues and eigenfunctions are given by $\lambda^{mn} = m^2 + n^2$, and $u^{mn}(x, y) = \sin(mx)\sin(ny)$, respectively, for $m, n \in \mathbb{N}_+$. These eigenfunctions have infinite regularity.

We obtain an initial mesh of Ω by subdividing it into a uniform 4×4 grid of congruent squares ($h = \pi/4$) and splitting each square into two triangles by its positively sloped diagonal. Successively finer meshes are obtained by subdividing each triangle into four congruent subtriangles. The mesh of "level ℓ" ($h = \pi/2^{\ell+2}$) is obtained from the original mesh by ℓ refinements.

We first present the error and order of convergence in Table 1 for some eigenmodes. The approximate eigenvalues converge at order $2k + 2$, in accordance with Theorem 2.3. (For this example, we may choose s_λ in Theorem 2.3 to be as large as we wish due to the infinite regularity.) Moreover, we observed that the proposed algorithms correctly capture the multiplicity of the eigenvalue of the second and sixth eigenmodes.

Next, in Table 2, we report the difference between the mixed and perturbed eigenvalues (λ_h and $\tilde{\lambda}_h$) of § 3.2, as a function of h for $k = 0, 1, 2$. We observe that $|\lambda_h - \tilde{\lambda}_h|$ converges at rate $O(h^2)$ (irrespective of k), as predicted by Theorem 3.2.

In Table 3, we present the convergence of both the approximate and postprocessed eigenfunctions for the first and fourth eigenmodes. The numerical results indicate convergence of order $k + 2$ for the postprocessed eigenfunctions, in accordance with Theorem 4.2.

Finally, we report the performance of a few iterative algorithms considered in Section 5. In Figure 2, we plot δ_λ versus the iteration level n in log scale for the first eigenmode for the case $\ell = 2$ and $k = 1$. When we set the initial guess as the eigenpair of the linear discrete eigenproblem (18), all the proposed algorithms required at most three iterations to achieve

degree	mesh	first mode		second mode		fourth mode		sixth mode	
k	ℓ	error	order	error	order	error	order	error	order
	0	3.24e-2	−−	1.66e-1	−−	7.66e-2	−−	5.86e-1	−−
	1	8.45e-3	1.94	3.60e-2	2.20	1.19e-1	−0.63	1.85e-1	1.66
0	2	2.13e-3	1.98	8.83e-3	2.03	3.32e-2	1.84	4.84e-2	1.93
	3	5.35e-4	2.00	2.20e-3	2.01	8.50e-3	1.97	1.23e-2	1.98
	4	1.34e-4	2.00	5.49e-4	2.00	2.14e-3	1.99	3.08e-3	2.00
	0	1.78e-3	−−	1.13e-2	−−	8.99e-2	−−	7.34e-2	−−
	1	1.17e-4	3.93	7.32e-4	3.95	7.01e-3	3.68	5.96e-3	3.62
1	2	7.35e-6	3.99	4.58e-5	4.00	4.63e-4	3.92	3.88e-4	3.94
	3	4.60e-7	4.00	2.85e-6	4.01	2.93e-5	3.98	2.44e-5	3.99
	4	2.87e-8	4.00	1.78e-7	4.00	1.84e-6	4.00	1.52e-6	4.00
	0	2.78e-5	−−	3.11e-4	−−	5.91e-3	−−	7.59e-3	−−
	1	4.52e-7	5.94	5.94e-6	5.71	1.10e-4	5.74	1.45e-4	5.71
2	2	7.12e-9	5.99	9.73e-8	5.93	1.80e-6	5.94	2.39e-6	5.92
	3	1.10e-10	6.01	1.53e-9	5.99	2.85e-8	5.99	3.78e-8	5.98
	4	4.83e-12	4.52	2.51e-11	5.93	4.17e-10	6.09	6.37e-10	5.89

TABLE 1. Convergence of the eigenvalues λ_h.

degree	mesh	first mode		second mode		fourth mode		sixth mode	
k	ℓ	error	order	error	order	error	order	error	order
	0	7.33e-2	−−	4.37e-1	−−	1.30e-0	−−	1.81e-0	−−
	1	1.74e-2	2.07	1.08e-1	2.02	2.93e-1	2.15	4.31e-1	2.07
0	2	4.30e-3	2.02	2.68e-2	2.01	6.97e-2	2.07	1.07e-1	2.01
	3	1.07e-3	2.00	6.70e-3	2.00	1.72e-2	2.02	2.68e-2	2.00
	4	2.68e-4	2.00	1.67e-3	2.00	4.29e-3	2.00	6.69e-3	2.00
	0	5.11e-2	−−	3.19e-1	−−	8.20e-1	−−	1.24e-0	−−
	1	1.28e-2	1.99	8.01e-2	1.99	2.04e-1	2.00	3.18e-1	1.96
1	2	3.21e-3	2.00	2.01e-2	2.00	5.13e-2	1.99	8.01e-2	1.99
	3	8.03e-4	2.00	5.02e-3	2.00	1.28e-2	2.00	2.01e-2	2.00
	4	2.01e-4	2.00	1.25e-3	2.00	3.21e-3	2.00	5.02e-3	2.00
	0	3.34e-2	−−	2.11e-1	−−	5.52e-1	−−	8.73e-1	−−
	1	8.26e-3	2.02	5.18e-2	2.03	1.34e-1	2.05	2.09e-1	2.06
2	2	2.06e-3	2.00	1.29e-2	2.01	3.30e-2	2.02	5.16e-2	2.02
	3	5.14e-4	2.00	3.21e-3	2.00	8.23e-3	2.00	1.29e-2	2.01
	4	1.29e-4	2.00	8.03e-4	2.00	2.06e-3	2.00	3.21e-3	2.00

TABLE 2. The differences $|\lambda_h - \tilde{\lambda}_h|$ for various h and k.

δ_λ less than 10^{-12}. To better see the convergence rates, we repeated by setting the initial guess as a random perturbation of the solution of (18). The results are in Table 4, where we report δ_λ versus the iteration level n for $\ell = 1, 2, 3$ and $k = 1$. Algorithm 5.1 appears to converge quadratically, and Algorithm 5.3 cubically. Similar convergence behaviors were observed for many other eigenmodes on different meshes and polynomial degrees.

6.2. **L-shaped domain.** To study the limitations imposed by singularities of eigenfunctions, we consider the L-shaped domain $\Omega = \Omega_0 \backslash \Omega_1$, where $\Omega_0 \equiv (0, 2) \times (0, 2)$ and $\Omega_1 \equiv (1, 2) \times (1, 2)$ are the square domains. Since Ω has a reentrant corner at the point $(1, 1)$, the exact eigenfunctions are singular. Specifically, we may only expect (43) to hold with $s_\lambda = \frac{2}{3} - \varepsilon$ for an arbitrarily small $\varepsilon > 0$. We shall focus on the numerical approximation of the ground state. As before, we consider triangular meshes that are successive

eigenmode		first				fourth			
degree	mesh	$\|u-u_h\|$		$\|u-u_h^*\|$		$\|u-u_h\|$		$\|u-u_h^*\|$	
k	ℓ	error	order	error	order	error	order	error	order
0	0	2.43e-2	−−	5.90e-2	−−	5.56e-1	−−	5.79e-1	−−
	1	6.13e-3	1.99	1.50e-2	1.97	8.92e-2	2.64	9.79e-2	2.56
	2	1.53e-3	2.00	3.77e-3	1.99	1.99e-2	2.16	2.27e-2	2.11
	3	3.82e-4	2.00	9.44e-4	2.00	4.85e-3	2.04	5.58e-3	2.02
	4	9.55e-5	2.00	2.36e-4	2.00	1.21e-3	2.01	1.39e-3	2.01
1	0	1.64e-3	−−	6.10e-3	−−	2.39e-2	−−	5.09e-2	−−
	1	2.06e-4	1.98	7.66e-4	2.99	2.34e-3	1.94	6.28e-3	3.02
	2	2.60e-5	2.00	9.56e-5	3.00	2.35e-4	1.98	7.70e-4	3.03
	3	3.25e-6	2.00	1.19e-5	3.00	2.70e-5	2.00	9.56e-5	3.01
	4	4.07e-7	2.00	1.49e-6	3.00	3.29e-6	2.00	1.19e-5	3.00
2	0	1.09e-4	−−	5.84e-4	−−	3.05e-3	−−	9.09e-3	−−
	1	6.66e-6	2.98	3.71e-5	3.98	1.20e-4	2.92	5.86e-4	3.96
	2	4.18e-7	2.99	2.33e-6	3.99	6.83e-6	2.98	3.71e-5	3.98
	3	2.63e-8	3.00	1.46e-7	4.00	4.23e-7	2.99	2.33e-6	3.99
	4	2.56e-9	3.00	1.04e-8	3.81	3.68e-8	3.00	1.58e-7	3.88

TABLE 3. Convergence of the approximate (u_h) and postprocessed (u_h^*) eigenfunctions.

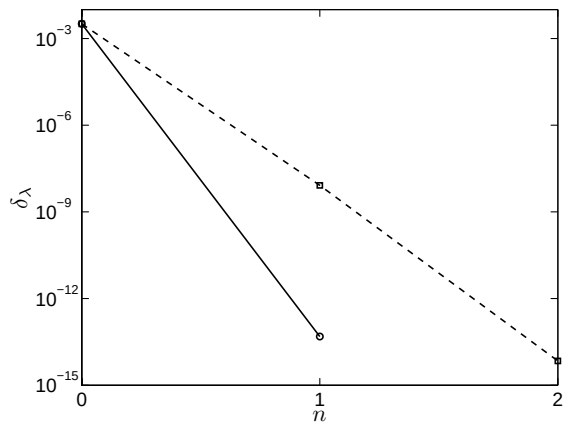

FIGURE 2. A plot of δ_λ versus n for Algorithm 5.1 (dashed line) and Algorithm 5.4 (solid line) for the first eigenmode (using $\ell = 2$ and $k = 1$).

Iter.	Algorithm 5.1			Algorithm 5.4		
n	$\ell=1$	$\ell=2$	$\ell=3$	$\ell=1$	$\ell=2$	$\ell=3$
0	2.13e-0	2.03e-0	2.03e-0	2.75e-0	3.09e-0	3.15e-0
1	1.30e-2	3.19e-2	3.29e-2	5.02e-2	6.28e-2	5.93e-2
2	1.62e-4	4.23e-4	4.92e-4	6.07e-6	1.45e-5	1.29e-5
3	2.62e-8	7.12e-8	1.05e-7	4.61e-15	9.85e-15	4.24e-15

TABLE 4. The value of δ_λ versus the iteration level n for $\ell = 1, 2, 3$ for the computation of the first eigenmode using $k = 1$.

uniform refinements of an initial uniform mesh. The initial mesh is obtained as in § 6.1 using a 4×4 uniform grid of Ω_0, except we now omit all triangles in Ω_1. Since the exact values are not known, errors are estimated using the approximate eigenvalue and postprocessed eigenfunction computed with degree $k = 2$ on the mesh level $\ell = 5$.

degree k	mesh ℓ	$\|\lambda - \lambda_h\|$ error	order	$\|u - u_h\|$ error	order	$\|u - u_h^*\|$ error	order
0	0	7.77e-1	−−	3.92e-1	−−	2.47e-1	−−
	1	3.18e-1	1.29	1.95e-1	1.01	7.41e-2	1.74
	2	1.30e-1	1.29	9.73e-2	1.01	2.38e-2	1.64
	3	5.22e-2	1.31	4.85e-2	1.00	8.26e-3	1.53
	4	2.06e-2	1.34	2.42e-2	1.00	3.01e-3	1.45
1	0	1.38e-1	−−	1.03e-1	−−	6.23e-2	−−
	1	5.89e-2	1.22	2.76e-2	1.90	1.67e-2	1.90
	2	2.32e-2	1.34	7.61e-3	1.86	4.92e-3	1.76
	3	8.85e-3	1.39	2.20e-3	1.79	1.56e-3	1.66
	4	3.15e-3	1.49	6.56e-4	1.75	4.98e-4	1.65

TABLE 5. Convergence history for the eigenpair (λ_h, u_h) and postprocessed eigenfunction u_h^* for the L-shaped domain.

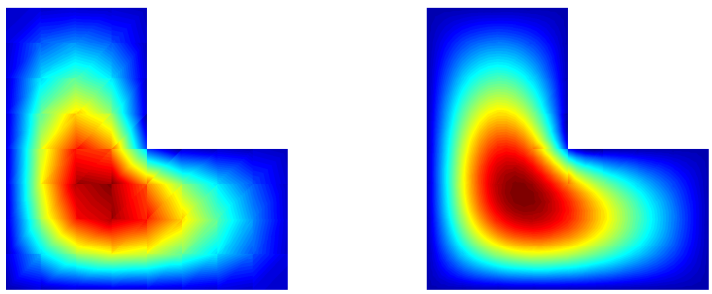

FIGURE 3. The approximate eigenfunction u_h (left) and postprocessed eigenfunction u_h^* (right) on the mesh level $\ell = 1$ for $k = 1$.

The apparent orders of convergence for the approximate and postprocessed eigenfunction are reported in Table 5 for $k = 0$ and $k = 1$. The convergence rates agree with Theorem 2.3. Furthermore, the postprocessed eigenfunction converges close to the order 4/3, in good agreement with Theorem 4.2. Figure 3 shows the approximate and postprocessed eigenfunctions on the mesh level $\ell = 1$ for $k = 1$. Clearly, the postprocessing technique visually improves the approximation of the eigenfunction even in this singular case. Note however that Table 5 shows that improvement obtained by postprocessing is limited by the regularity of the eigenfunction. In particular, the gain in accuracy after postprocessing is not as significant for the $k = 1$ case as the $k = 0$ case.

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