

# A NONCONFORMING FINITE ELEMENT METHOD FOR A TWO-DIMENSIONAL CURL-CURL AND GRAD-DIV PROBLEM

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ABSTRACT. A numerical method for a two-dimensional curl-curl and grad-div problem is studied in this paper. It is based on a discretization using weakly continuous  $P_1$  vector fields and includes two consistency terms involving the jumps of the vector fields across element boundaries. Optimal convergence rates (up to an arbitrary positive  $\epsilon$ ) in both the energy norm and the  $L_2$  norm are established on graded meshes. The theoretical results are confirmed by numerical experiments.

## 1. INTRODUCTION

Let  $\Omega \subset \mathbb{R}^2$  be a bounded polygonal domain. In this paper we consider the following curl-curl and grad-div problem:

Find  $\mathbf{u} \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$  such that

$$(1.1) \quad (\nabla \times \mathbf{u}, \nabla \times \mathbf{v}) + \gamma(\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}) + \alpha(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})$$

for all  $\mathbf{v} \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$ , where  $(\cdot, \cdot)$  denotes the inner product of  $L_2(\Omega)$  (or  $[L_2(\Omega)]^2$ ),  $\alpha \in \mathbb{R}$  and  $\gamma > 0$  are constants,  $\mathbf{f} \in [L_2(\Omega)]^2$ , and the spaces  $H_0(\text{curl}; \Omega)$  and  $H(\text{div}; \Omega)$  are defined as follows.

$$H(\text{curl}; \Omega) = \left\{ \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in [L_2(\Omega)]^2 : \nabla \times \mathbf{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \in L_2(\Omega) \right\},$$

$$H_0(\text{curl}; \Omega) = \{ \mathbf{v} \in H(\text{curl}; \Omega) : \mathbf{n} \times \mathbf{v} = 0 \text{ on } \partial\Omega \},$$

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where  $\mathbf{n}$  is the unit outer normal, and

$$H(\operatorname{div}; \Omega) = \left\{ \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in [L_2(\Omega)]^2 : \nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \in L_2(\Omega) \right\}.$$

Note that  $\mathbf{n} \times \mathbf{v} = 0$  on  $\partial\Omega$  is equivalent to  $\boldsymbol{\tau} \cdot \mathbf{v} = 0$  on  $\partial\Omega$ , where  $\boldsymbol{\tau}$  is a unit tangent along  $\partial\Omega$ .

For positive  $\alpha$ , the problem (1.1) is uniquely solvable by the Riesz representation theorem applied to the Hilbert space

$$X_N = H_0(\operatorname{curl}; \Omega) \cap H(\operatorname{div}; \Omega)$$

with the inner product

$$(\mathbf{v}, \mathbf{w})_{X_N} = (\nabla \times \mathbf{v}, \nabla \times \mathbf{w}) + (\nabla \cdot \mathbf{v}, \nabla \cdot \mathbf{w}) + (\mathbf{v}, \mathbf{w}).$$

Since  $H_0(\operatorname{curl}; \Omega) \cap H(\operatorname{div}; \Omega)$  is compactly embedded in  $[L_2(\Omega)]^2$  (cf. [41, 49, 23, 51, 42] and the discussion in Section 2 below), there exists a sequence of nonnegative numbers  $0 \leq \lambda_{\gamma,1} \leq \lambda_{\gamma,2} \leq \dots \rightarrow \infty$  such that the following eigenproblem has a nontrivial solution  $\mathbf{w} \in H_0(\operatorname{curl}; \Omega) \cap H(\operatorname{div}; \Omega)$ :

$$(1.2) \quad (\nabla \times \mathbf{w}, \nabla \times \mathbf{v}) + \gamma(\nabla \cdot \mathbf{w}, \nabla \cdot \mathbf{v}) = \lambda_{\gamma,j}(\mathbf{w}, \mathbf{v})$$

for all  $\mathbf{v} \in H_0(\operatorname{curl}; \Omega) \cap H(\operatorname{div}; \Omega)$ .

For  $\alpha \leq 0$ , the problem (1.1) is well-posed as long as  $\alpha \neq -\lambda_{\gamma,j}$  for  $j \geq 1$ , which we assume to be the case throughout the paper. In particular, in the case where  $\alpha = 0$  and  $\partial\Omega$  is connected, the problem (1.1) is uniquely solvable due to Friedrichs' inequality [42]:

$$\|\mathbf{v}\|_{L_2(\Omega)} \leq C(\|\nabla \times \mathbf{v}\|_{L_2(\Omega)} + \|\nabla \cdot \mathbf{v}\|_{L_2(\Omega)})$$

for all  $\mathbf{v} \in H_0(\operatorname{curl}; \Omega) \cap H(\operatorname{div}; \Omega)$ .

When  $\nabla \cdot \mathbf{f} = 0$  and (1.1) is well-posed, the solution  $\mathbf{u}$  of (1.1) belongs to the space  $H(\operatorname{div}^0; \Omega)$  defined by

$$H(\operatorname{div}^0; \Omega) = \{\mathbf{v} \in H(\operatorname{div}; \Omega) : \nabla \cdot \mathbf{v} = 0\},$$

and it is also a solution of the following curl-curl problem:

Find  $\mathbf{u} \in H_0(\operatorname{curl}; \Omega)$  such that

$$(1.3) \quad (\nabla \times \mathbf{u}, \nabla \times \mathbf{v}) + \alpha(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})$$

for all  $\mathbf{v} \in H_0(\operatorname{curl}; \Omega)$ .

The curl-curl problem (1.3) appears in semi-discretizations of electric fields in the time-dependent (time-domain) Maxwell equations when  $\alpha > 0$  and the time-harmonic (frequency-domain) Maxwell equations when  $\alpha \leq 0$ . When  $\alpha = 0$ , it is also related to electrostatic problems.

The numerical solution of (1.1) by finite element methods has an interesting history. It was realized early on [46, 29, 47, 50] that the

solution of the non-elliptic curl-curl problem (1.3) can be obtained by solving the elliptic curl-curl and grad-div problem (1.1) in the case where  $\nabla \cdot \mathbf{f} = 0$ . Since it is difficult to construct finite element subspaces for  $H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$ , the problem (1.1) was discretized by  $H^1$ -conforming vector nodal finite elements [29]. However, the space  $[H^1(\Omega)]^2 \cap X_N$  turns out to be a closed subspace of  $X_N$  [11, 24]. Therefore any  $H^1$ -conforming finite element method for (1.1) must fail if the solution  $\mathbf{u}$  does not belong to  $[H^1(\Omega)]^2$ , which happens when  $\Omega$  is non-convex [11, 6, 26]. Even worse, the solutions obtained by  $H^1$ -conforming finite element methods in such situations converge to the wrong solution (the projection of  $\mathbf{u}$  in  $[H^1(\Omega)]^2 \cap X_N$ ). Consequently the idea of solving (1.3) through (1.1) was abandoned. Instead, the curl-curl problem (1.3) is usually solved by  $H(\text{curl})$ -conforming edge elements [44, 45, 39, 42, 33, 14].

Nevertheless, the elliptic problem (1.1) remains an attractive alternative approach and successful schemes have been discovered in recent years that either solve (1.1) using nodal  $H^1$  vector finite elements complemented by singular vector fields [13, 5, 38, 3, 4], or solve a regularized version of (1.1) using standard nodal  $H^1$  vector finite elements [27, 28, 21].

In this paper we will show that (1.1) can also be solved by a non-conforming method using weakly continuous piecewise  $P_1$  vector fields, where optimal convergence rates (up to an arbitrarily small  $\epsilon > 0$ ) in both the energy norm and the  $L_2$  norm can be achieved on general polygonal domains, provided that two consistency terms involving the jumps of the vector fields across element boundaries are included in the discretization and properly graded meshes are used. This is a continuation of our previous work in [17, 16], which considerably facilitates the analysis of the new method.

Note that, since we are working in two-dimensions, the problem (1.1) is equivalent to the following problem:

Find  $\mathbf{u} \in H_0(\text{div}; \Omega) \cap H(\text{curl}; \Omega)$  such that

$$(1.4) \quad \gamma(\nabla \times \mathbf{u}, \nabla \times \mathbf{v}) + (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}) + \alpha(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})$$

for all  $\mathbf{v} \in H_0(\text{div}; \Omega) \cap H(\text{curl}; \Omega)$ , where

$$H_0(\text{div}; \Omega) = \{\mathbf{v} \in H(\text{div}; \Omega) : \mathbf{n} \cdot \mathbf{v} = 0 \text{ on } \partial\Omega\}.$$

Therefore all the results in this paper (after straight-forward modifications) hold for the problem (1.4), which appears in problems involving magnetic fields and also problems in fluid-structure interaction [37, 10, 9, 12].

The rest of the paper is organized as follows. We discuss the elliptic regularity of (1.1) in Section 2 and introduce the nonconforming finite element method in Section 3. The convergence analysis is given in Section 4, followed by the results of numerical experiments in Section 5. We end the paper with some concluding remarks in Section 6.

## 2. REGULARITY OF THE CURL-CURL AND GRAD-DIV PROBLEM

The regularity of (1.1) is closely related to the regularity of the Laplace operator with homogeneous Dirichlet or Neumann boundary conditions. To explain this connection, we begin by reviewing the relation between the space  $H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$  and Sobolev spaces. For simplicity, we first assume that  $\Omega$  is simply connected.

Let  $\mathbf{u} \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$ . By a well-known Helmholtz decomposition [34, 42], we have an orthogonal decomposition for  $\mathbf{u}$  in both  $[L_2(\Omega)]^2$  and  $H_0(\text{curl}; \Omega)$ :

$$(2.1) \quad \mathbf{u} = \hat{\mathbf{u}} + \nabla \phi,$$

where  $\hat{\mathbf{u}} \in H_0(\text{curl}; \Omega) \cap H(\text{div}^0; \Omega)$  and  $\phi \in H_0^1(\Omega)$ .

The function  $\phi \in H_0^1(\Omega)$  in (2.1) is the variational solution of the following Dirichlet boundary value problem:

$$(2.2a) \quad \Delta \phi = \nabla \cdot \mathbf{u} \quad \text{in } \Omega,$$

$$(2.2b) \quad \phi = 0 \quad \text{on } \partial\Omega.$$

Since we assume  $\Omega$  to be simply connected, there exists (cf. [34])  $\psi \in H^1(\Omega)$  such that

$$\nabla \times \psi = \hat{\mathbf{u}} \quad \text{and} \quad \int_{\Omega} \psi \, dx = 0,$$

and we can rewrite (2.1) as

$$(2.3) \quad \mathbf{u} = \nabla \times \psi + \nabla \phi.$$

Note that  $\psi$  is the unique variational solution with zero mean of the following Neumann boundary value problem:

$$(2.4a) \quad \Delta \psi = -(\nabla \times \mathbf{u}) \quad \text{in } \Omega,$$

$$(2.4b) \quad \frac{\partial \psi}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

Since the right-hand side of (2.2a) belongs to  $L_2(\Omega)$ , the elliptic regularity theory for polygonal domains [35, 32, 36, 43] provides a decomposition

$$(2.5) \quad \phi = \phi_R + \phi_S,$$

where the *regular* part  $\phi_R \in H^2(\Omega)$  and the *singular* part  $\phi_S$  is supported near the corners  $c_1, \dots, c_L$  of  $\Omega$ . More precisely, we can choose a small positive number  $\delta$  such that the neighborhoods

$$\mathcal{N}_{\ell, 2\delta} = \{x \in \Omega : |x - c_\ell| < 2\delta\}$$

are disjoint. Then we can write

$$(2.6) \quad \phi_S = \sum_{\ell=1}^L \chi_\ell(r_\ell) \sum_{\substack{j \in \mathbb{N} \\ j(\pi/\omega_\ell) \in (0,1)}} \kappa_{\ell,j} r_\ell^{j(\pi/\omega_\ell)} \sin(j(\pi/\omega_\ell)\theta_\ell),$$

where  $(r_\ell, \theta_\ell)$  are the polar coordinates at  $c_\ell$  so that the two edges of  $\Omega$  emanating from  $c_\ell$  are defined by  $\theta = 0$  and  $\theta = \omega_\ell$ ,  $\chi_\ell(t)$  is a smooth cut-off function that equals 1 for  $t < 3\delta/2$  and vanishes for  $t > 7\delta/4$ , and  $\kappa_{\ell,j}$  are constants. Furthermore, we have the following elliptic regularity estimate:

$$(2.7) \quad \|\phi_R\|_{H^2(\Omega)} + \sum_{\ell=1}^L \sum_{\substack{j \in \mathbb{N} \\ j(\pi/\omega_\ell) \in (0,1)}} |\kappa_{\ell,j}| \leq C \|\nabla \cdot \mathbf{u}\|_{L_2(\Omega)}.$$

Here and below we use  $C$  with or without subscripts to denote a generic positive constant independent of  $h$  that can take different values at different appearances.

Similarly, since the right-hand side of (2.4a) belongs to  $L_2(\Omega)$ , we have the following decomposition for  $\psi$ :

$$(2.8) \quad \psi = \psi_R + \psi_S,$$

where  $\psi_R \in H^2(\Omega)$ , and

$$(2.9) \quad \psi_S = \sum_{\ell=1}^L \chi_\ell(r_\ell) \sum_{\substack{j \in \mathbb{N} \\ j(\pi/\omega_\ell) \in (0,1)}} \varrho_{\ell,j} r_\ell^{j(\pi/\omega_\ell)} \cos(j(\pi/\omega_\ell)\theta_\ell).$$

Furthermore, the following analog of (2.7) holds:

$$(2.10) \quad \|\psi_R\|_{H^2(\Omega)} + \sum_{\ell=1}^L \sum_{\substack{j \in \mathbb{N} \\ j(\pi/\omega_\ell) \in (0,1)}} |\varrho_{\ell,j}| \leq C \|\nabla \times \mathbf{u}\|_{L_2(\Omega)}.$$

Combining (2.3) and (2.5)–(2.10), we have the following description of  $\mathbf{u}$ . First of all,  $\mathbf{u} \in [H^1(\Omega_\delta)]^2$ , where

$$\Omega_\delta = \{x \in \Omega : |x - c_\ell| > \delta \text{ for } 1 \leq \ell \leq L\},$$

and the following estimate holds:

$$(2.11) \quad \|\mathbf{u}\|_{H^1(\Omega_\delta)} \leq C (\|\nabla \cdot \mathbf{u}\|_{L_2(\Omega)} + \|\nabla \times \mathbf{u}\|_{L_2(\Omega)}).$$

Secondly, in the neighborhood  $\mathcal{N}_{\ell, 3\delta/2}$  of the corner  $c_\ell$ , we have

$$(2.12) \quad \mathbf{u} = \mathbf{u}_R + \mathbf{u}_S,$$

where  $\mathbf{u}_R \in [H^1(\mathcal{N}_{\ell, 3\delta/2})]^2$ ,

$$(2.13) \quad \mathbf{u}_S = \sum_{\substack{j \in \mathbb{N} \\ j(\pi/\omega) \in (0,1)}} \nu_{\ell,j} r_\ell^{j(\pi/\omega_\ell)-1} \begin{bmatrix} \sin(j(\pi/\omega_\ell) - 1)\theta_\ell \\ \cos(j(\pi/\omega_\ell) - 1)\theta_\ell \end{bmatrix},$$

and

$$(2.14) \quad \nu_{\ell,j} = j(\pi/\omega_\ell)(\kappa_{\ell,j} - \varrho_{\ell,j}).$$

Moreover, we have the following estimate:

$$(2.15) \quad \sum_{\ell=1}^L \left( \|\mathbf{u}_R\|_{H^1(\mathcal{N}_{\ell, 3\delta/2})} + \sum_{\substack{j \in \mathbb{N} \\ j(\pi/\omega) \in (0,1)}} |\nu_{\ell,j}| \right) \leq C(\|\nabla \times \mathbf{u}\|_{L_2(\Omega)} + \|\nabla \cdot \mathbf{u}\|_{L_2(\Omega)}).$$

In particular, it follows from (2.11)–(2.13) and (2.15) that  $\mathbf{u} \in [H^s(\Omega)]^2$  for any  $s \in (1/2, 1]$  such that  $s < \min_{1 \leq \ell \leq L} \pi/\omega_\ell$ , and

$$(2.16) \quad \|\mathbf{u}\|_{H^s(\Omega)} \leq C_s(\|\nabla \times \mathbf{u}\|_{L_2(\Omega)} + \|\nabla \cdot \mathbf{u}\|_{L_2(\Omega)}),$$

i.e.,  $H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$  can be embedded into  $[H^s(\Omega)]^2$ .

Now we turn to the regularity/singularity of the solution  $\mathbf{u}$  of (1.1). For simplicity we first discuss the case where  $\alpha > 0$ . From (1.1) we immediately see that

$$(2.17) \quad \|\mathbf{u}\|_{L_2(\Omega)} \leq \alpha^{-1} \|\mathbf{f}\|_{L_2(\Omega)},$$

$$(2.18) \quad \|\nabla \times \mathbf{u}\|_{L_2(\Omega)}^2 + \gamma \|\nabla \cdot \mathbf{u}\|_{L_2(\Omega)}^2 \leq \alpha^{-1} \|\mathbf{f}\|_{L_2(\Omega)}^2.$$

In view of (1.1), the divergence free part  $\dot{\mathbf{u}}$  in the Helmholtz decomposition (2.1) satisfies

$$(2.19) \quad (\nabla \times \dot{\mathbf{u}}, \nabla \times \mathbf{v}) + \alpha(\dot{\mathbf{u}}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})$$

for all  $\mathbf{v} \in H_0(\text{curl}; \Omega) \cap H(\text{div}^0; \Omega)$ , which implies

$$(2.20) \quad \nabla \times (\nabla \times \dot{\mathbf{u}}) + \alpha \dot{\mathbf{u}} = Q\mathbf{f},$$

where  $Q$  is the orthogonal projection from  $[L_2(\Omega)]^2$  onto  $H(\text{div}^0; \Omega)$ . Indeed, let  $\boldsymbol{\zeta} \in [C_0^\infty(\Omega)]^2$  be a test vector field. Then  $\boldsymbol{\zeta} \in H_0(\text{curl}; \Omega)$  and  $(\boldsymbol{\zeta} - Q\boldsymbol{\zeta}) \in \nabla H_0^1(\Omega) \subset H_0(\text{curl}; \Omega)$ , which imply that  $Q\boldsymbol{\zeta} \in H_0(\text{curl}; \Omega) \cap H(\text{div}^0; \Omega)$ . Hence it follows from (2.19) that

$$\begin{aligned} (\nabla \times \dot{\mathbf{u}}, \nabla \times \boldsymbol{\zeta}) + \alpha(\dot{\mathbf{u}}, \boldsymbol{\zeta}) &= (\nabla \times \dot{\mathbf{u}}, \nabla \times [Q\boldsymbol{\zeta} + (\boldsymbol{\zeta} - Q\boldsymbol{\zeta})]) + \alpha(\dot{\mathbf{u}}, Q\boldsymbol{\zeta}) \\ &= (\nabla \times \dot{\mathbf{u}}, \nabla \times Q\boldsymbol{\zeta}) + \alpha(\dot{\mathbf{u}}, Q\boldsymbol{\zeta}) = (\mathbf{f}, Q\boldsymbol{\zeta}) = (Q\mathbf{f}, \boldsymbol{\zeta}), \end{aligned}$$

which yields (2.20).

We deduce from (2.1), (2.17) and (2.20) that  $\nabla \times \mathbf{u} = \nabla \times \dot{\mathbf{u}} \in H^1(\Omega)$  and

$$(2.21) \quad |\nabla \times \mathbf{u}|_{H^1(\Omega)} = |\nabla \times \dot{\mathbf{u}}|_{H^1(\Omega)} = \|\mathbf{Q}\mathbf{f} - \alpha\dot{\mathbf{u}}\|_{L_2(\Omega)} \leq 2\|\mathbf{f}\|_{L_2(\Omega)},$$

which together with (1.1) implies that  $\nabla \cdot \mathbf{u} \in H^1(\Omega)$  and

$$(2.22) \quad \begin{aligned} |\nabla \cdot \mathbf{u}|_{H^1(\Omega)} &\leq \gamma^{-1} \|\mathbf{f} - \alpha\mathbf{u} - \nabla \times (\nabla \times \mathbf{u})\|_{L_2(\Omega)} \\ &\leq 4\gamma^{-1} \|\mathbf{f}\|_{L_2(\Omega)}. \end{aligned}$$

In particular, it follows from the regularity of  $\nabla \times \mathbf{u}$  and  $\nabla \cdot \mathbf{u}$  and the usual variational argument that the boundary value problem corresponding to (1.1) is

$$(2.23a) \quad \nabla \times (\nabla \times \mathbf{u}) - \gamma \nabla (\nabla \cdot \mathbf{u}) + \alpha \mathbf{u} = \mathbf{f} \quad \text{in } \Omega,$$

$$(2.23b) \quad \mathbf{n} \times \mathbf{u} = 0 \quad \text{on } \partial\Omega,$$

$$(2.23c) \quad \nabla \cdot \mathbf{u} = 0 \quad \text{on } \partial\Omega.$$

The regularity/singularity of  $\mathbf{u}$  can be derived through (2.2)–(2.4) and the elliptic regularity theory for polygonal domains. Since  $\nabla \cdot \mathbf{u} \in H^1(\Omega)$ , the regular part  $\phi_R$  in (2.5) now belongs to  $H^3(\Omega_\delta)$ , and  $\phi_R \in H^{3-\epsilon}(\mathcal{N}_{\ell,2\delta})$  for any  $\epsilon > 0$  and  $1 \leq \ell \leq L$ . The singular part  $\phi_S$  is now given by

$$(2.24) \quad \phi_S = \sum_{\ell=1}^L \chi_\ell(r_\ell) \sum_{\substack{j \in \mathbb{N} \\ j(\pi/\omega_\ell) \in (0,2) \setminus \{1\}}} \kappa_{\ell,j} r_\ell^{j(\pi/\omega_\ell)} \sin(j(\pi/\omega_\ell)\theta_\ell).$$

Furthermore, we have the following elliptic regularity estimates:

$$(2.25a) \quad \begin{aligned} \|\phi_R\|_{H^3(\Omega_\delta)} &\leq C \|\nabla \cdot \mathbf{u}\|_{H^1(\Omega)} \\ &\leq C\gamma^{-1/2}(\gamma^{-1/2} + \alpha^{-1/2}) \|\mathbf{f}\|_{L_2(\Omega)}, \end{aligned}$$

$$(2.25b) \quad \begin{aligned} \sum_{\ell=1}^L \|\phi_R\|_{H^{3-\epsilon}(\mathcal{N}_{\ell,2\delta})} &\leq C_\epsilon \|\nabla \cdot \mathbf{u}\|_{H^1(\Omega)} \\ &\leq C_\epsilon \gamma^{-1/2}(\gamma^{-1/2} + \alpha^{-1/2}) \|\mathbf{f}\|_{L_2(\Omega)}, \end{aligned}$$

$$(2.25c) \quad \begin{aligned} \sum_{\ell=1}^L \sum_{\substack{j \in \mathbb{N} \\ j(\pi/\omega_\ell) \in (0,2) \setminus \{1\}}} |\kappa_{\ell,j}| &\leq C \|\nabla \cdot \mathbf{u}\|_{H^1(\Omega)} \\ &\leq C\gamma^{-1/2}(\gamma^{-1/2} + \alpha^{-1/2}) \|\mathbf{f}\|_{L_2(\Omega)}, \end{aligned}$$

where we have used the estimates (2.18) and (2.22).

Similarly, since  $\nabla \times \mathbf{u} \in H^1(\Omega)$ , the regular part  $\psi_R$  in (2.8) now belongs to  $H^3(\Omega_\delta)$ , and  $\psi_R \in H^{3-\epsilon}(\mathcal{N}_{\ell,2\delta})$  for any  $\epsilon > 0$  and  $1 \leq \ell \leq L$ . The singular part  $\psi_S$  is now given by

$$(2.26) \quad \psi_S = \sum_{\ell=1}^L \chi_\ell(r_\ell) \sum_{\substack{j \in \mathbb{N} \\ j(\pi/\omega_\ell) \in (0,2) \setminus \{1\}}} \varrho_{\ell,j} r_\ell^{j(\pi/\omega_\ell)} \cos(j(\pi/\omega_\ell)\theta_\ell).$$

Furthermore, the following analog of (2.25) holds:

$$(2.27a) \quad \begin{aligned} \|\psi_R\|_{H^3(\Omega_\delta)} &\leq C \|\nabla \times \mathbf{u}\|_{H^1(\Omega)} \\ &\leq C(1 + \alpha^{-1/2}) \|\mathbf{f}\|_{L_2(\Omega)}, \end{aligned}$$

$$(2.27b) \quad \begin{aligned} \sum_{\ell=1}^L \|\psi_R\|_{H^{3-\epsilon}(\mathcal{N}_{\ell,2\delta})} &\leq C_\epsilon \|\nabla \times \mathbf{u}\|_{H^1(\Omega)} \\ &\leq C_\epsilon(1 + \alpha^{-1/2}) \|\mathbf{f}\|_{L_2(\Omega)}, \end{aligned}$$

$$(2.27c) \quad \begin{aligned} \sum_{\ell=1}^L \sum_{\substack{j \in \mathbb{N} \\ j(\pi/\omega_\ell) \in (0,2) \setminus \{1\}}} |\varrho_{\ell,j}| &\leq C \|\nabla \times \mathbf{u}\|_{H^1(\Omega)} \\ &\leq C(1 + \alpha^{-1/2}) \|\mathbf{f}\|_{L_2(\Omega)}, \end{aligned}$$

where we have used the estimates (2.18) and (2.21).

Combining (2.3), (2.5), (2.8) and (2.24)–(2.27), we can describe the regularity/singularity of the solution  $\mathbf{u}$  of (1.1) as follows. We have  $\mathbf{u} \in [H^2(\Omega_\delta)]^2$  and the following estimate is valid:

$$(2.28) \quad \|\mathbf{u}\|_{H^2(\Omega_\delta)} \leq C[1 + \gamma^{-1} + \alpha^{-1/2}(1 + \gamma^{-1/2})] \|\mathbf{f}\|_{L_2(\Omega)}.$$

In the neighborhood  $\mathcal{N}_{\ell,3\delta/2}$  of the corner  $c_\ell$ , we have

$$(2.29) \quad \mathbf{u} = \mathbf{u}_R + \mathbf{u}_S,$$

where  $\mathbf{u}_R \in [H^{2-\epsilon}(\mathcal{N}_{\ell,3\delta/2})]^2$  for any  $\epsilon > 0$ ,

$$(2.30) \quad \mathbf{u}_S = \sum_{\substack{j \in \mathbb{N} \\ j(\pi/\omega_\ell) \in (0,2) \setminus \{1\}}} \nu_{\ell,j} r_\ell^{j(\pi/\omega_\ell)-1} \begin{bmatrix} \sin(j(\pi/\omega_\ell) - 1)\theta_\ell \\ \cos(j(\pi/\omega_\ell) - 1)\theta_\ell \end{bmatrix},$$

and the constants  $\nu_{\ell,j}$  are related to  $\kappa_{\ell,j}$  and  $\varrho_{\ell,j}$  by (2.14). Moreover, we have the following corner regularity estimates:

$$(2.31a) \quad \begin{aligned} \sum_{\ell=1}^L \|\mathbf{u}_R\|_{H^{2-\epsilon}(\mathcal{N}_{\ell,3\delta/2})} \\ \leq C_\epsilon [1 + \gamma^{-1} + \alpha^{-1/2}(1 + \gamma^{-1/2})] \|\mathbf{f}\|_{L_2(\Omega)}, \end{aligned}$$

$$(2.31b) \quad \sum_{\ell=1}^L \sum_{\substack{j \in \mathbb{N} \\ j(\pi/\omega_\ell) \in (0,2) \setminus \{1\}}} |\nu_{\ell,j}| \\ \leq C[1 + \gamma^{-1} + \alpha^{-1/2}(1 + \gamma^{-1/2})] \|\mathbf{f}\|_{L_2(\Omega)}.$$

*Remark 2.1.* Note that the description of the regularity/singularity of the solution of the reduced time-harmonic Maxwell equations given in [17] is the same as (2.28)–(2.31), only with all  $\kappa_{\ell,j}$ 's equal to 0.

We have derived the regularity/singularity of  $\mathbf{u}$  under the assumption that  $\Omega$  is simply connected. Since the regularity/singularity is a local behavior, the preceding results remain valid for general polygonal domains by a standard partition of unity argument.

For  $\alpha \leq 0$ , the problem (1.1) is well-posed as long as  $\alpha \neq -\lambda_{\gamma,j}$ , where  $0 \leq \lambda_{\gamma,1} \leq \lambda_{\gamma,2} \leq \dots \rightarrow \infty$  are the eigenvalues defined by (1.2), in which case we can replace (2.17) and (2.18) by

$$(2.32) \quad \|\nabla \times \mathbf{u}\|_{L_2(\Omega)}^2 + \gamma \|\nabla \cdot \mathbf{u}\|_{L_2(\Omega)} + \|\mathbf{u}\|_{L_2(\Omega)}^2 \leq C_\alpha \|\mathbf{f}\|_{L_2(\Omega)}^2.$$

Hence the results for  $\alpha > 0$  remain valid for  $\alpha \leq 0$  provided  $\alpha \neq -\lambda_{\gamma,j}$  for  $j \geq 1$ , except that the dependence of the estimates on  $\alpha$  is no longer explicit.

### 3. THE NONCONFORMING FINITE ELEMENT METHOD

Let  $\mathcal{T}_h$  be a family of simplicial triangulations of  $\Omega$  with mesh-parameter  $h = \max_{T \in \mathcal{T}_h} h_T$ , where  $h_T$  is the diameter of the triangle  $T$ . To recover optimal *a priori* error estimates in the presence of singularities, the triangulation  $\mathcal{T}_h$  is graded around the corners  $c_1, \dots, c_L$  of  $\Omega$  with the property that

$$(3.1) \quad C_1 h_T \leq h \Phi_\mu(T) \leq C_2 h_T,$$

where

$$(3.2) \quad \Phi_\mu(T) = \prod_{\ell=1}^L |c_\ell - c_T|^{1-\mu_\ell}.$$

Here  $c_T$  is the center of  $T$  and the positive constants  $C_1$  and  $C_2$  are independent of  $h$ .

The grading parameters  $\mu_1, \dots, \mu_L$  are chosen according to

$$(3.3) \quad \begin{aligned} \mu_\ell = 1 & \quad \text{if } \omega_\ell \leq \frac{\pi}{2}, \\ \mu_\ell < \frac{\pi}{2\omega_\ell} & \quad \text{if } \omega_\ell > \frac{\pi}{2}. \end{aligned}$$

In other words, grading is needed around any corner whose angle is larger than a right angle, which is different from the grading strategy for the Laplace operator, where grading is needed only around re-entrant corners. This is due to the fact that the singularity of (1.1) is one order more severe than the singularity of the Laplace operator (cf. (2.6), (2.9) and (2.30)).

The construction of  $\mathcal{T}_h$  is described for example in [1, 2, 15, 8]. Note that  $\mathcal{T}_h$  satisfies the minimum angle condition for any given grading parameters.

Let  $V_h$  be the space of weakly continuous  $P_1$  vector fields associated with  $\mathcal{T}_h$  whose tangential components vanish at the midpoints of the boundary edges in  $\mathcal{T}_h$ . More precisely, let  $\mathcal{E}_h$  (resp.  $\mathcal{E}_h^b$  and  $\mathcal{E}_h^i$ ) be the set of the edges (resp. boundary edges and interior edges) of  $\mathcal{T}_h$ . Then

$$\begin{aligned} V_h = \{ & \mathbf{v} \in [L_2(\Omega)]^2 : \mathbf{v}_T = \mathbf{v}|_T \in [P_1(T)]^2 \quad \forall T \in \mathcal{T}_h, \\ & \mathbf{v} \text{ is continuous at the midpoint of any } e \in \mathcal{E}_h, \\ & \mathbf{n} \times \mathbf{v} \text{ vanishes at the midpoint of any } e \in \mathcal{E}_h^b. \}. \end{aligned}$$

The advantage of using weakly continuous  $P_1$  vector fields is due to the fact that it is easy to define, for any  $s > 1/2$ , a weak interpolation operator  $\Pi_T : [H^s(T)]^2 \rightarrow [P_1(T)]^2$  as follows:

$$(\Pi_T \boldsymbol{\zeta})(m_{e_j}) = \frac{1}{|e_j|} \int_{e_j} \boldsymbol{\zeta} ds \quad \text{for } 1 \leq j \leq 3,$$

where  $e_1$ ,  $e_2$  and  $e_3$  are the edges of  $T$ , and  $m_e$  and  $|e|$  denote the midpoint and length of the edge  $e$ . In view of the midpoint rule, we can also write

$$(3.4) \quad \int_{e_j} \Pi_T \boldsymbol{\zeta} ds = \int_{e_j} \boldsymbol{\zeta} ds \quad \text{for } 1 \leq j \leq 3.$$

Furthermore, the operator  $\Pi_T$  satisfies a standard error estimate [30]:

$$(3.5) \quad \|\boldsymbol{\zeta} - \Pi_T \boldsymbol{\zeta}\|_{L_2(T)} + h_T^{\min(s,1)} \|\boldsymbol{\zeta} - \Pi_T \boldsymbol{\zeta}\|_{H^{\min(s,1)}(T)} \leq C_T h_T^s \|\boldsymbol{\zeta}\|_{H^s(T)}$$

for all  $\boldsymbol{\zeta} \in [H^s(T)]^2$  and  $s \in (1/2, 2]$ , where the positive constant  $C_T$  depends on the minimum angle of  $T$  (and also on  $s$  when  $s$  tends to  $1/2$ ).

Since  $H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega) \subset [H^s(\Omega)]^2$  for some  $s > 1/2$ , we can define a global interpolation operator

$$\Pi_h : H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega) \rightarrow V_h$$

by piecing together the local interpolation operators:

$$(3.6) \quad (\Pi_h \mathbf{v})_T = \Pi_T \mathbf{v}_T \quad \forall T \in \mathcal{T}_h.$$

Let  $\nabla_h \times$  and  $\nabla_h \cdot$  be the piecewise curl and div operator defined by

$$(3.7) \quad (\nabla_h \times \mathbf{v})_T = \nabla \times (\mathbf{v}_T) \quad \forall T \in \mathcal{T}_h,$$

$$(3.8) \quad (\nabla_h \cdot \mathbf{v})_T = \nabla \cdot (\mathbf{v}_T) \quad \forall T \in \mathcal{T}_h.$$

Observe that (3.4) and Green's theorem imply, for any  $\mathbf{v} \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$  and  $T \in \mathcal{T}_h$ ,

$$\begin{aligned} \int_T \nabla \times (\Pi_T \mathbf{v}) \, dx &= \int_T \nabla \times \mathbf{v} \, dx, \\ \int_T \nabla \cdot (\Pi_T \mathbf{v}) \, dx &= \int_T \nabla \cdot \mathbf{v} \, dx, \end{aligned}$$

which, in view of (3.6)–(3.8), means that

$$(3.9) \quad \nabla_h \times (\Pi_h \mathbf{v}) = \Pi_0^h(\nabla \times \mathbf{v}) \quad \forall \mathbf{v} \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega),$$

$$(3.10) \quad \nabla_h \cdot (\Pi_h \mathbf{v}) = \Pi_0^h(\nabla \cdot \mathbf{v}) \quad \forall \mathbf{v} \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega),$$

where  $\Pi_0^h$  is the orthogonal projection from  $L_2(\Omega)$  onto the space of piecewise constant functions associated with  $\mathcal{T}_h$ . These commutative diagram relations indicate that we have good control over  $\nabla_h \times (\Pi_h \mathbf{u})$  and  $\nabla_h \cdot (\Pi_h \mathbf{u})$  simultaneously, which explains why weakly continuous  $P_1$  vector fields can be used to solve problems involving the space  $H(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$ .

*Remark 3.1.*  $\Pi_h$  is also the interpolation operator used in [17, 16].

Let  $e \in \mathcal{E}_h^i$  be shared by the two triangles  $T_{e,1}, T_{e,2} \in \mathcal{T}_h$  and  $\mathbf{n}_1$  (resp.  $\mathbf{n}_2$ ) be the unit normal of  $e$  pointing towards the outside of  $T_{e,1}$  (resp.  $T_{e,2}$ ). We define, on  $e$ ,

$$(3.11a) \quad \llbracket \mathbf{n} \times \mathbf{v} \rrbracket = \mathbf{n}_1 \times (\mathbf{v}_{T_{e,1}}|_e) + \mathbf{n}_2 \times (\mathbf{v}_{T_{e,2}}|_e),$$

$$(3.11b) \quad \llbracket \mathbf{n} \cdot \mathbf{v} \rrbracket = \mathbf{n}_1 \cdot (\mathbf{v}_{T_{e,1}}|_e) + \mathbf{n}_2 \cdot (\mathbf{v}_{T_{e,2}}|_e).$$

For an edge  $e \in \mathcal{E}_h^b$ , we take  $\mathbf{n}_e$  to be the unit normal of  $e$  pointing towards the outside of  $\Omega$  and define

$$(3.12) \quad \llbracket \mathbf{n} \times \mathbf{v} \rrbracket = \mathbf{n}_e \times (\mathbf{v}|_e).$$

The nonconforming finite element method for (1.1) is:

Find  $\mathbf{u}_h \in V_h$  such that

$$(3.13) \quad a_h(\mathbf{u}_h, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in V_h,$$

where

$$\begin{aligned}
(3.14) \quad a_h(\mathbf{w}, \mathbf{v}) &= (\nabla_h \times \mathbf{w}, \nabla_h \times \mathbf{v}) + \gamma(\nabla_h \cdot \mathbf{w}, \nabla_h \cdot \mathbf{v}) + \alpha(\mathbf{w}, \mathbf{v}) \\
&+ \sum_{e \in \mathcal{E}_h} \frac{[\Phi_\mu(e)]^2}{|e|} \int_e \llbracket \mathbf{n} \times \mathbf{w} \rrbracket \llbracket \mathbf{n} \times \mathbf{v} \rrbracket ds \\
&+ \sum_{e \in \mathcal{E}_h^i} \frac{[\Phi_\mu(e)]^2}{|e|} \int_e \llbracket \mathbf{n} \cdot \mathbf{w} \rrbracket \llbracket \mathbf{n} \cdot \mathbf{v} \rrbracket ds,
\end{aligned}$$

and the edge weight  $\Phi_\mu(e)$  is defined by

$$(3.15) \quad \Phi_\mu(e) = \prod_{\ell=1}^L |c_\ell - m_e|^{1-\mu_\ell}.$$

Note that, by comparing (3.2) and (3.15), we have

$$(3.16) \quad C_1 \Phi_\mu(e) \leq \Phi_\mu(T) \leq C_2 \Phi_\mu(e) \quad \text{if } e \subset \partial T,$$

where the positive constants  $C_1$  and  $C_2$  are independent of  $h$ . This relation is important for the derivation of optimal *a priori* error estimates.

*Remark 3.2.* The last two terms on the right-hand side of (3.14) involving the tangential and normal jumps of the weakly continuous  $P_1$  vector fields are crucial for the convergence of the scheme. Unlike the nonconforming  $P_1$  finite element method for the Stokes problem or the membrane problem, a naive discretization of (1.1) with only the first three terms does not converge (see the numerical results in Table 5.3 below). This is one of the reasons why classical nonconforming finite element methods have not been pursued in computational electromagnetics (see also the comments on page 200 of [42]). The crucial difference is that the piecewise  $H(\text{curl}) \cap H(\text{div})$  semi-norm, unlike the piecewise  $H^1$  semi-norm, is too weak to control the jumps even with the weak continuity of the vector fields in  $V_h$ . Hence the two terms involving the jumps must be included in the discretization to control the consistency error.

#### 4. CONVERGENCE ANALYSIS

We will measure the discretization error in the  $L_2$  norm and the mesh-dependent energy norm  $\|\cdot\|_h$  defined by

$$\begin{aligned}
(4.1) \quad \|\mathbf{v}\|_h^2 &= \|\nabla_h \times \mathbf{v}\|_{L_2(\Omega)}^2 + \gamma \|\nabla_h \cdot \mathbf{v}\|_{L_2(\Omega)}^2 + \|\mathbf{v}\|_{L_2(\Omega)}^2 \\
&+ \sum_{e \in \mathcal{E}_h} \frac{[\Phi_\mu(e)]^2}{|e|} \|\llbracket \mathbf{n} \times \mathbf{v} \rrbracket\|_{L_2(e)}^2
\end{aligned}$$

$$+ \sum_{e \in \mathcal{E}_h^i} \frac{[\Phi_\mu(e)]^2}{|e|} \|[\mathbf{n} \cdot \mathbf{v}]\|_{L_2(e)}^2.$$

Note that we have suppressed the dependence of the norm on  $\gamma$  to keep the notation simple.

Observe that  $a_h(\cdot, \cdot)$  is bounded by the energy norm, i.e.,

$$(4.2) \quad |a_h(\mathbf{w}, \mathbf{v})| \leq (|\alpha| + 1) \|\mathbf{w}\|_h \|\mathbf{v}\|_h$$

for all  $\mathbf{v}, \mathbf{w} \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega) + V_h$ .

For  $\alpha > 0$ ,  $a_h(\cdot, \cdot)$  is also coercive with respect to  $\|\cdot\|_h$ , i.e.,

$$a_h(\mathbf{v}, \mathbf{v}) \geq \min(1, \alpha) \|\mathbf{v}\|_h^2$$

for all  $\mathbf{v} \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega) + V_h$ . In this case the discrete problem is well-posed and we have the following abstract error estimate, whose proof is identical with the proof of Lemma 3.5 in [16].

**Lemma 4.1.** *Let  $\alpha$  be positive,  $\beta = \min(1, \alpha)$ ,  $\mathbf{u} \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$  be the solution of (1.1), and  $\mathbf{u}_h$  satisfy the discrete problem (3.13). It holds that*

$$(4.3) \quad \|\mathbf{u} - \mathbf{u}_h\|_h \leq \left( \frac{1 + \alpha + \beta}{\beta} \right) \inf_{\mathbf{v} \in V_h} \|\mathbf{u} - \mathbf{v}\|_h + \frac{1}{\beta} \sup_{\mathbf{w} \in V_h \setminus \{\mathbf{0}\}} \frac{a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{w})}{\|\mathbf{w}\|_h}.$$

For  $\alpha \leq 0$ , we have a Gårding (in)equality:

$$(4.4) \quad a_h(\mathbf{v}, \mathbf{v}) + (|\alpha| + 1)(\mathbf{v}, \mathbf{v}) = \|\mathbf{v}\|_h^2$$

for all  $\mathbf{v} \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega) + V_h$ . In this case the discrete problem is indefinite and the following lemma provides an abstract error estimate for the scheme (3.13) under the assumption that it has a solution. Its proof, which is based on (4.2) and (4.4), is identical with the proof of Lemma 3.6 in [16].

**Lemma 4.2.** *Let  $\alpha \leq 0$  and  $\mathbf{u} \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$  satisfy (1.1). Assume that the discrete problem (3.13) has a solution  $\mathbf{u}_h$ . Then we have*

$$(4.5) \quad \|\mathbf{u} - \mathbf{u}_h\|_h \leq (2|\alpha| + 3) \inf_{\mathbf{v} \in V_h} \|\mathbf{u} - \mathbf{v}\|_h + \sup_{\mathbf{w} \in V_h \setminus \{\mathbf{0}\}} \frac{a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{w})}{\|\mathbf{w}\|_h} + (|\alpha| + 1) \|\mathbf{u} - \mathbf{u}_h\|_{L_2(\Omega)}.$$

From here on we consider  $\alpha$  and  $\gamma$  to be fixed and drop the dependence on these constants in our estimates. We also assume in Lemmas 4.8 and 4.9 below that the discrete problem (3.13) has a solution  $\mathbf{u}_h$  when  $\alpha \leq 0$

*Remark 4.3.* The first term on the right-hand side of (4.3) and (4.5) measures the approximation property of  $V_h$  with respect to the energy norm. The second term measures the consistency error. The third term on the right-hand side of (4.5) addresses the indefiniteness of the problem when  $\alpha \leq 0$ .

Since the description of the regularity/singularity of the solution of the reduced time-harmonic Maxwell equations in [17] is identical with the description of the regularity/singularity of the solution  $\mathbf{u}$  of (1.1) and the interpolation operator  $\Pi_h$  defined in Section 3 is also the one employed in [17], we can use in our analysis the following two results from that paper (cf. Lemma 5.1 and Lemma 5.2 of [17]), which were obtained using (2.28)–(2.31), (3.1), (3.3), (3.5), (3.15) and (3.16).

**Lemma 4.4.** *Let  $\mathbf{u} \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$  be the solution of (1.1). We have the following interpolation error estimate:*

$$(4.6) \quad \|\mathbf{u} - \Pi_h \mathbf{u}\|_{L_2(\Omega)} \leq C_\epsilon h^{2-\epsilon} \|\mathbf{f}\|_{L_2(\Omega)} \quad \text{for any } \epsilon > 0.$$

**Lemma 4.5.** *Let  $\mathbf{u} \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$  be the solution of (1.1). We have the following interpolation error estimate:*

$$(4.7) \quad \sum_{e \in \mathcal{E}_h} \frac{[\Phi_\mu(e)]^2}{|e|} \|\llbracket \mathbf{u} - \Pi_h \mathbf{u} \rrbracket\|_{L_2(e)}^2 \leq C_\epsilon h^{2-\epsilon} \|\mathbf{f}\|_{L_2(\Omega)}^2$$

for any  $\epsilon > 0$ , where  $\llbracket \mathbf{u} - \Pi_h \mathbf{u} \rrbracket$  is the jump of  $\mathbf{u} - \Pi_h \mathbf{u}$  across the interior edges of  $\mathcal{T}_h$  and  $\llbracket \mathbf{u} - \Pi_h \mathbf{u} \rrbracket = \mathbf{u} - \Pi_h \mathbf{u}$  on the boundary edges of  $\mathcal{T}_h$ .

The following result gives the approximation property of  $V_h$ .

**Lemma 4.6.** *Let  $u \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$  be the solution of (1.1). For any  $\epsilon > 0$  there exists a positive constant  $C_\epsilon$  independent of  $h$  and  $\mathbf{f}$  such that:*

$$(4.8) \quad \inf_{\mathbf{v} \in V_h} \|\mathbf{u} - \mathbf{v}\|_h \leq \|\mathbf{u} - \Pi_h \mathbf{u}\|_h \leq C_\epsilon h^{1-\epsilon} \|\mathbf{f}\|_{L_2(\Omega)}.$$

*Proof.* According to (4.1), we have

$$\begin{aligned} \|\mathbf{u} - \Pi_h \mathbf{u}\|_h^2 &= \|\nabla_h \times (\mathbf{u} - \Pi_h \mathbf{u})\|_{L_2(\Omega)}^2 + \gamma \|\nabla_h \cdot (\mathbf{u} - \Pi_h \mathbf{u})\|_{L_2(\Omega)}^2 \\ &\quad + \|\mathbf{u} - \Pi_h \mathbf{u}\|_{L_2(\Omega)}^2 \end{aligned}$$

$$(4.9) \quad \begin{aligned} & + \sum_{e \in \mathcal{E}_h} \frac{[\Phi_\mu(e)]^2}{|e|} \|\llbracket \mathbf{n} \times (\mathbf{u} - \Pi_h \mathbf{u}) \rrbracket\|_{L_2(e)}^2 \\ & + \sum_{e \in \mathcal{E}_h^i} \frac{[\Phi_\mu(e)]^2}{|e|} \|\llbracket \mathbf{n} \cdot (\mathbf{u} - \Pi_h \mathbf{u}) \rrbracket\|_{L_2(e)}^2. \end{aligned}$$

The third term on the right-hand side of (4.9) has been estimated in Lemma 4.4, and the last two terms can be estimated using Lemma 4.5. Therefore it only remains to estimate the first two terms.

It follows from (2.21), (2.22), (3.9), (3.10) and a standard interpolation error estimate [22, 20] that

$$(4.10) \quad \begin{aligned} \|\nabla_h \times (\mathbf{u} - \Pi_h \mathbf{u})\|_{L_2(\Omega)}^2 &= \|\nabla \times \mathbf{u} - \Pi_h^0(\nabla \times \mathbf{u})\|_{L_2(\Omega)}^2 \\ &\leq Ch^2 \|\nabla \times \mathbf{u}\|_{H^1(\Omega)}^2 \leq Ch^2 \|\mathbf{f}\|_{L_2(\Omega)}^2, \end{aligned}$$

$$(4.11) \quad \begin{aligned} \gamma \|\nabla_h \cdot (\mathbf{u} - \Pi_h \mathbf{u})\|_{L_2(\Omega)}^2 &= \gamma \|\nabla \cdot \mathbf{u} - \Pi_h^0(\nabla \cdot \mathbf{u})\|_{L_2(\Omega)}^2 \\ &\leq Ch^2 \|\nabla \cdot \mathbf{u}\|_{H^1(\Omega)}^2 \leq Ch^2 \|\mathbf{f}\|_{L_2(\Omega)}^2. \end{aligned}$$

The estimate (4.8) follows from (4.9)–(4.11) and Lemmas 4.4–4.5.  $\square$

Next we turn to the consistency error. The following lemma, which is identical with Lemma 5.3 in [17], is useful for estimating terms involving the jumps of the weakly continuous  $P_1$  vector fields across edges.

**Lemma 4.7.** *It holds that*

$$\sum_{e \in \mathcal{E}_h} |e| [\Phi_\mu(e)]^{-2} \|\eta - \bar{\eta}_{T_e}\|_{L_2(e)}^2 \leq Ch^2 \|\eta\|_{H^1(\Omega)}^2 \quad \forall \eta \in H^1(\Omega),$$

where  $\bar{\eta}_{T_e} = \int_{T_e} \eta \, dx / |T_e|$  is the mean of  $\eta$  over  $T_e$ , one of the triangles in  $\mathcal{T}_h$  that has  $e$  as an edge.

The following result gives an optimal bound for the consistency error.

**Lemma 4.8.** *Let  $\mathbf{u} \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$  be the solution of (1.1), and  $\mathbf{u}_h \in V_h$  satisfy (3.13). Then we have*

$$(4.12) \quad \sup_{\mathbf{w} \in V_h \setminus \{0\}} \frac{a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{w})}{\|\mathbf{w}\|_h} \leq Ch \|\mathbf{f}\|_{L_2(\Omega)}.$$

*Proof.* Let  $\mathbf{w} \in V_h$  be arbitrary. Since the strong form of (1.1) is given by (2.23a), we have, by (3.11), (3.12), (3.14) and integration by parts,

$$\begin{aligned}
(4.13) \quad a_h(\mathbf{u}, \mathbf{w}) &= \sum_{T \in \mathcal{T}_h} \int_T (\nabla \times \mathbf{u})(\nabla \times \mathbf{w}) dx \\
&\quad + \sum_{T \in \mathcal{T}_h} \gamma \int_T (\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{w}) dx + \alpha(\mathbf{u}, \mathbf{w}) \\
&= (\mathbf{f}, \mathbf{w}) + \sum_{e \in \mathcal{E}_h} \int_e (\nabla \times \mathbf{u}) \llbracket \mathbf{n} \times \mathbf{w} \rrbracket ds + \sum_{e \in \mathcal{E}_h^i} \gamma \int_e (\nabla \cdot \mathbf{u}) \llbracket \mathbf{n} \cdot \mathbf{w} \rrbracket ds.
\end{aligned}$$

Note that the last sum on the right-hand side of (4.13) involves only the interior edges because of (2.23c).

Subtracting (3.13) from (4.13), we find

$$\begin{aligned}
(4.14) \quad a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{w}) &= \sum_{e \in \mathcal{E}_h} \int_e (\nabla \times \mathbf{u}) \llbracket \mathbf{n} \times \mathbf{w} \rrbracket ds \\
&\quad + \sum_{e \in \mathcal{E}_h^i} \gamma \int_e (\nabla \cdot \mathbf{u}) \llbracket \mathbf{n} \cdot \mathbf{w} \rrbracket ds.
\end{aligned}$$

Since  $\mathbf{n} \times \mathbf{w}$  is continuous at the midpoints of the interior edges and vanishes at the midpoints of the boundary edges, we can write, using the midpoint rule,

$$(4.15) \quad \sum_{e \in \mathcal{E}_h} \int_e (\nabla \times \mathbf{u}) \llbracket \mathbf{n} \times \mathbf{w} \rrbracket ds = \sum_{e \in \mathcal{E}_h} \int_e (\nabla \times \mathbf{u} - \overline{(\nabla \times \mathbf{u})}_{T_e}) \llbracket \mathbf{n} \times \mathbf{w} \rrbracket ds$$

where  $\overline{(\nabla \times \mathbf{u})}_{T_e}$  is the mean of  $\nabla \times \mathbf{u}$  on  $T_e$ , one of the triangles in  $\mathcal{T}_h$  that has  $e$  as an edge. It then follows from the Cauchy-Schwarz inequality, (2.21), (4.1) and Lemma 4.7 that

$$\begin{aligned}
(4.16) \quad &\sum_{e \in \mathcal{E}_h} \int_e (\nabla \times \mathbf{u}) \llbracket \mathbf{n} \times \mathbf{w} \rrbracket ds \\
&\leq \left\{ \sum_{e \in \mathcal{E}_h} |e| [\Phi_\mu(e)]^{-2} \|(\nabla \times \mathbf{u} - \overline{(\nabla \times \mathbf{u})}_{T_e})\|_{L_2(e)}^2 \right\}^{1/2} \\
&\quad \times \left\{ \sum_{e \in \mathcal{E}_h} \frac{[\Phi_\mu(e)]^2}{|e|} \|\llbracket \mathbf{n} \times \mathbf{w} \rrbracket\|_{L_2(e)}^2 \right\}^{1/2} \\
&\leq C(h|\nabla \times \mathbf{u}|_{H^1(\Omega)}) \|\mathbf{w}\|_h \leq Ch \|\mathbf{f}\|_{L_2(\Omega)} \|\mathbf{w}\|_h,
\end{aligned}$$

and similarly,

$$(4.17) \quad \sum_{e \in \mathcal{E}_h^i} \gamma \int_e (\nabla \cdot \mathbf{u}) \llbracket \mathbf{n} \cdot \mathbf{w} \rrbracket ds \leq Ch \|\mathbf{f}\|_{L_2(\Omega)} \|\mathbf{w}\|_h.$$

The estimate (4.12) follows from (4.14), (4.16) and (4.17).  $\square$

We now derive an  $L_2$  error estimate by a duality argument.

**Lemma 4.9.** *Let  $\mathbf{u} \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$  be the solution of (1.1) and  $\mathbf{u}_h \in V_h$  satisfy (3.13). Then we have*

$$(4.18) \quad \|\mathbf{u} - \mathbf{u}_h\|_{L_2(\Omega)} \leq C_\epsilon (h^{2-\epsilon} \|\mathbf{f}\|_{L_2(\Omega)} + h^{1-\epsilon} \|\mathbf{u} - \mathbf{u}_h\|_h)$$

for any  $\epsilon > 0$ .

*Proof.* Let  $\mathbf{z} \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$  satisfy

$$(4.19) \quad (\nabla \times \mathbf{v}, \nabla \times \mathbf{z}) + \gamma(\nabla \cdot \mathbf{v}, \nabla \cdot \mathbf{z}) + \alpha(\mathbf{v}, \mathbf{z}) = (\mathbf{v}, (\mathbf{u} - \mathbf{u}_h))$$

for all  $\mathbf{v} \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$ . Note that the strong form of (4.19) is

$$(4.20) \quad \nabla \times (\nabla \times \mathbf{z}) - \gamma \nabla(\nabla \cdot \mathbf{z}) + \alpha \mathbf{z} = \mathbf{u} - \mathbf{u}_h,$$

and we have the following analog of (2.21) and (2.22):

$$(4.21) \quad |\nabla \times \mathbf{z}|_{H^1(\Omega)} + |\nabla \cdot \mathbf{z}|_{H^1(\Omega)} \leq C \|\mathbf{u} - \mathbf{u}_h\|_{L_2(\Omega)}.$$

Furthermore we can write (4.19) as

$$(4.22) \quad a_h(\mathbf{v}, \mathbf{z}) = (\mathbf{v}, (\mathbf{u} - \mathbf{u}_h)) \quad \forall \mathbf{v} \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega).$$

It follows from (4.20), (4.22), and integration by parts that the following analog of (4.13) holds:

$$(4.23) \quad \begin{aligned} a_h(\mathbf{u}_h, \mathbf{z}) &= \sum_{T \in \mathcal{T}_h} \int_T (\nabla \times \mathbf{u}_h)(\nabla \times \mathbf{z}) dx \\ &\quad + \sum_{T \in \mathcal{T}_h} \gamma \int_T (\nabla \cdot \mathbf{u}_h)(\nabla \cdot \mathbf{z}) dx + \alpha(\mathbf{u}_h, \mathbf{z}) \\ &= (\mathbf{u}_h, (\mathbf{u} - \mathbf{u}_h)) + \sum_{e \in \mathcal{E}_h} \int_e [\mathbf{n} \times \mathbf{u}_h](\nabla \times \mathbf{z}) ds \\ &\quad + \sum_{e \in \mathcal{E}_h^i} \gamma \int_e [\mathbf{n} \cdot \mathbf{u}_h](\nabla \cdot \mathbf{z}) ds \end{aligned}$$

Combining (4.22) and (4.23), we find

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}_h\|_{L_2(\Omega)}^2 &= (\mathbf{u}, \mathbf{u} - \mathbf{u}_h) - (\mathbf{u}_h, \mathbf{u} - \mathbf{u}_h) \\
(4.24) \quad &= a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{z}) + \sum_{e \in \mathcal{E}_h} \int_e \llbracket \mathbf{n} \times \mathbf{u}_h \rrbracket (\nabla \times \mathbf{z}) ds \\
&\quad + \sum_{e \in \mathcal{E}_h^i} \gamma \int_e \llbracket \mathbf{n} \cdot \mathbf{u}_h \rrbracket (\nabla \cdot \mathbf{z}) ds,
\end{aligned}$$

and we will estimate the three terms on the right-hand side of (4.24) separately.

We can write the first term as

$$(4.25) \quad a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{z}) = a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{z} - \Pi_h \mathbf{z}) + a_h(\mathbf{u} - \mathbf{u}_h, \Pi_h \mathbf{z}).$$

From (4.2) and Lemma 4.6 (applied to  $\mathbf{z}$ ) we immediately have the following estimate:

$$\begin{aligned}
(4.26) \quad a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{z} - \Pi_h \mathbf{z}) &\leq C \|\mathbf{u} - \mathbf{u}_h\|_h \|\mathbf{z} - \Pi_h \mathbf{z}\|_h \\
&\leq C_\epsilon h^{1-\epsilon} \|\mathbf{u} - \mathbf{u}_h\|_h \|\mathbf{u} - \mathbf{u}_h\|_{L_2(\Omega)}.
\end{aligned}$$

Using (4.14) we can rewrite the second term on the right-hand side of (4.25) as

$$\begin{aligned}
(4.27) \quad a_h(\mathbf{u} - \mathbf{u}_h, \Pi_h \mathbf{z}) &= \sum_{e \in \mathcal{E}_h} \int_e (\nabla \times \mathbf{u}) \llbracket \mathbf{n} \times (\Pi_h \mathbf{z}) \rrbracket ds \\
&\quad + \sum_{e \in \mathcal{E}_h^i} \int_e \gamma (\nabla \cdot \mathbf{u}) \llbracket \mathbf{n} \cdot (\Pi_h \mathbf{z}) \rrbracket ds.
\end{aligned}$$

Following the notation introduced in (4.15), the first term on the right-hand side of (4.27) can be written as

$$\begin{aligned}
&\sum_{e \in \mathcal{E}_h} \int_e (\nabla \times \mathbf{u}) \llbracket \mathbf{n} \times (\Pi_h \mathbf{z}) \rrbracket ds \\
&= \sum_{e \in \mathcal{E}_h} \int_e (\nabla \times \mathbf{u} - \overline{(\nabla \times \mathbf{u})}_{T_e}) \llbracket \mathbf{n} \times (\Pi_h \mathbf{z}) \rrbracket ds \\
&= \sum_{e \in \mathcal{E}_h} \int_e (\nabla \times \mathbf{u} - \overline{(\nabla \times \mathbf{u})}_{T_e}) \llbracket \mathbf{n} \times (\Pi_h \mathbf{z} - \mathbf{z}) \rrbracket ds,
\end{aligned}$$

since  $\mathbf{n} \times (\Pi_h \mathbf{z})$  is continuous at the midpoint of any edge  $e \in \mathcal{E}_h^i$  and vanishes at the midpoint of any edge  $e \in \mathcal{E}_h^b$ , and  $\llbracket \mathbf{n} \times \mathbf{z} \rrbracket = 0$ . It then follows from the Cauchy-Schwarz inequality, (2.21), Lemma 4.5

(applied to  $\mathbf{z}$ ) and Lemma 4.7 that

$$\begin{aligned}
& \sum_{e \in \mathcal{E}_h} \int_e (\nabla \times \mathbf{u}) \llbracket \mathbf{n} \times (\Pi_h \mathbf{z}) \rrbracket ds \\
& \leq \left( \sum_{e \in \mathcal{E}_h} |e| [\Phi_\mu(e)]^{-2} \|(\nabla \times \mathbf{u} - \overline{(\nabla \times \mathbf{u})}_{T_e})\|_{L_2(e)}^2 \right)^{1/2} \\
(4.28) \quad & \quad \times \left( \sum_{e \in \mathcal{E}_h} \frac{[\Phi_\mu(e)]^2}{|e|} \|\llbracket \mathbf{n} \times (\Pi_h \mathbf{z} - \mathbf{z}) \rrbracket\|_{L_2(e)}^2 \right)^{1/2} \\
& \leq C_\epsilon (h \|\nabla \times \mathbf{u}\|_{H^1(\Omega)}) (h^{1-\epsilon} \|\mathbf{u} - \mathbf{u}_h\|_{L_2(\Omega)}) \\
& \leq C_\epsilon h^{2-\epsilon} \|\mathbf{f}\|_{L_2(\Omega)} \|\mathbf{u} - \mathbf{u}_h\|_{L_2(\Omega)}.
\end{aligned}$$

Similarly, the second term on the right-hand side of (4.27) satisfies the following estimate:

$$(4.29) \quad \sum_{e \in \mathcal{E}_h^i} \gamma \int_e (\nabla \cdot \mathbf{u}) \llbracket \mathbf{n} \cdot (\Pi_h \mathbf{z}) \rrbracket ds \leq C_\epsilon h^{2-\epsilon} \|\mathbf{f}\|_{L_2(\Omega)} \|\mathbf{u} - \mathbf{u}_h\|_{L_2(\Omega)}.$$

Combining (4.25)–(4.29), we have

$$\begin{aligned}
(4.30) \quad & a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{z}) \\
& \leq C_\epsilon (h^{2-\epsilon} \|\mathbf{f}\|_{L_2(\Omega)} + h^{1-\epsilon} \|\mathbf{u} - \mathbf{u}_h\|_h) \|\mathbf{u} - \mathbf{u}_h\|_{L_2(\Omega)}.
\end{aligned}$$

We now consider the second term on the right-hand side of (4.24). Since  $\mathbf{n} \times \mathbf{u}_h$  is continuous at the midpoints of interior edges and vanishes at the midpoints of boundary edges, and  $\llbracket \mathbf{n} \times \mathbf{u} \rrbracket = 0$ , we can write, following the notation introduced in (4.15),

$$\begin{aligned}
& \sum_{e \in \mathcal{E}_h} \int_e \llbracket \mathbf{n} \times \mathbf{u}_h \rrbracket (\nabla \times \mathbf{z}) ds \\
& = \sum_{e \in \mathcal{E}_h} \int_e \llbracket \mathbf{n} \times \mathbf{u}_h \rrbracket (\nabla \times \mathbf{z} - \overline{(\nabla \times \mathbf{z})}_{T_e}) ds \\
& = \sum_{e \in \mathcal{E}_h} \int_e \llbracket \mathbf{n} \times (\mathbf{u}_h - \mathbf{u}) \rrbracket (\nabla \times \mathbf{z} - \overline{(\nabla \times \mathbf{z})}_{T_e}) ds.
\end{aligned}$$

Using the Cauchy-Schwarz inequality, (4.1), (4.21) and Lemma 4.7, we obtain

$$\begin{aligned}
& \sum_{e \in \mathcal{E}_h} \int_e \llbracket \mathbf{n} \times \mathbf{u}_h \rrbracket (\nabla \times \mathbf{z}) ds \\
& \leq \left( \sum_{e \in \mathcal{E}_h} |e| [\Phi_\mu(e)]^{-2} \|\nabla \times \mathbf{z} - \overline{(\nabla \times \mathbf{z})}_{T_e}\|_{L_2(e)}^2 \right)^{1/2}
\end{aligned}$$

$$\begin{aligned}
(4.31) \quad & \times \left( \sum_{e \in \mathcal{E}_h} \frac{[\Phi_\mu(e)]^2}{|e|} \|\llbracket \mathbf{n} \times (\mathbf{u}_h - \mathbf{u}) \rrbracket\|_{L_2(e)}^2 \right)^{1/2} \\
& \leq C(h|\nabla \times \mathbf{z}|_{H^1(\Omega)}) \|\mathbf{u} - \mathbf{u}_h\|_h \\
& \leq Ch \|\mathbf{u} - \mathbf{u}_h\|_{L_2(\Omega)} \|\mathbf{u} - \mathbf{u}_h\|_h.
\end{aligned}$$

Similarly, we have the following bound on the third term on the right-hand side of (4.24):

$$(4.32) \quad \sum_{e \in \mathcal{E}_h^i} \gamma \int_e \llbracket \mathbf{n} \cdot (\Pi_h \mathbf{u}_h) \rrbracket (\nabla \cdot \mathbf{z}) ds \leq Ch \|\mathbf{u} - \mathbf{u}_h\|_{L_2(\Omega)} \|\mathbf{u} - \mathbf{u}_h\|_h.$$

The estimate (4.18) follows from (4.24) and (4.30)–(4.32).  $\square$

In the case where  $\alpha > 0$ , the following theorem is an immediate consequence of Lemma 4.1, Lemma 4.6, Lemma 4.8 and Lemma 4.9.

**Theorem 4.10.** *Let  $\alpha$  be positive. The following discretization error estimates hold for the solution  $\mathbf{u}_h$  of (3.13):*

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}_h\|_h &\leq C_\epsilon h^{1-\epsilon} \|\mathbf{f}\|_{L_2(\Omega)} && \text{for any } \epsilon > 0, \\
\|\mathbf{u} - \mathbf{u}_h\|_{L_2(\Omega)} &\leq C_\epsilon h^{2-\epsilon} \|\mathbf{f}\|_{L_2(\Omega)} && \text{for any } \epsilon > 0.
\end{aligned}$$

In the case where  $\alpha \leq 0$ , we have the following convergence theorem for the scheme (3.13). The proof, which is based on Lemma 4.2, Lemma 4.6, Lemma 4.8, Lemma 4.9 and the approach of Schatz for indefinite problems [48], is identical with the proof of Theorem 4.5 in [16].

**Theorem 4.11.** *Assume that  $-\alpha \geq 0$  is not one of the eigenvalues  $\lambda_{\gamma,j}$  defined by (1.2). There exists a positive number  $h_*$  such that the discrete problem (3.13) is uniquely solvable for all  $h \leq h_*$ , in which case the following discretization error estimates are valid:*

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}_h\|_h &\leq C_\epsilon h^{1-\epsilon} \|\mathbf{f}\|_{L_2(\Omega)} && \text{for any } \epsilon > 0, \\
\|\mathbf{u} - \mathbf{u}_h\|_{L_2(\Omega)} &\leq C_\epsilon h^{2-\epsilon} \|\mathbf{f}\|_{L_2(\Omega)} && \text{for any } \epsilon > 0.
\end{aligned}$$

## 5. NUMERICAL RESULTS

In this section we report the results of a series of numerical experiments that confirm our theoretical results. We take  $\gamma$  to be 1 in all the experiments. Besides the errors in the  $L_2$  norm  $\|\cdot\|_{L_2(\Omega)}$  and the energy norm  $\|\cdot\|_h$ , we also include the errors in the semi-norms  $|\cdot|_{\text{curl}}$  and  $|\cdot|_{\text{div}}$  defined by

$$|\mathbf{v}|_{\text{curl}} = \|\nabla_h \times \mathbf{v}\|_{L_2(\Omega)}, \quad |\mathbf{v}|_{\text{div}} = \|\nabla_h \cdot \mathbf{v}\|_{L_2(\Omega)}.$$

In the first experiment we examine the convergence behavior of our numerical scheme on the square domain  $(0, 0.5)^2$  with uniform meshes, where the exact solution  $\mathbf{u}$  is given by

$$(5.1) \quad \mathbf{u} = \begin{bmatrix} \left(\frac{x^3}{3} - \frac{x^2}{4}\right)(y^2 - 0.5y) \sin(ky) \\ \left(\frac{y^3}{3} - \frac{y^2}{4}\right)(x^2 - 0.5x) \cos(kx) \end{bmatrix}.$$

The results are tabulated in Table 5.1 for  $\alpha = k^2$  and  $k = 0, 1$  and  $10$ , and in Table 5.2 for  $\alpha = -k^2$  and for  $k = 1$  and  $10$ . They show that the scheme (3.13) is second order accurate in the  $L_2$  norm and first order accurate in the energy norm, which agrees with the error estimates in Theorem 4.10 and Theorem 4.11.

TABLE 5.1. Convergence of the scheme on the square  $(0, 0.5)^2$  for  $\alpha = k^2$ , with uniform meshes and exact solution given by (5.1)

$h$	$\frac{\ \mathbf{u} - \mathbf{u}_h\ _{L_2(\Omega)}}{\ \mathbf{u}\ _{L_2(\Omega)}}$	order	$\frac{\ \mathbf{u} - \mathbf{u}_h\ _h}{\ \mathbf{u}\ _h}$	order	$\frac{ \mathbf{u} - \mathbf{u}_h _{\text{curl}}}{ \mathbf{u} _{\text{curl}}}$	order	$\frac{ \mathbf{u} - \mathbf{u}_h _{\text{div}}}{ \mathbf{u} _{\text{div}}}$	order
$k = 0$								
1/10	6.45E-02	—	3.46E-01	—	1.89E-01	—	2.13E-01	—
1/20	1.38E-02	2.23	1.70E-01	1.03	9.49E-02	0.99	1.08E-01	0.99
1/40	3.20E-03	2.11	8.37E-02	1.01	4.75E-02	1.00	5.40E-02	1.00
1/80	7.73E-04	2.05	4.17E-02	1.01	2.37E-02	1.00	2.70E-02	1.00
$k = 1$								
1/10	5.49E-02	—	3.23E-01	—	1.74E-01	—	2.13E-01	—
1/20	1.20E-02	2.19	1.59E-01	1.02	8.71E-02	0.99	1.07E-01	0.99
1/40	2.83E-03	2.09	7.92E-02	1.01	4.36E-02	1.00	5.35E-02	1.00
1/80	6.87E-04	2.04	3.94E-02	1.01	2.18E-02	1.00	2.67E-02	1.00
$k = 10$								
1/10	1.77E-01	—	6.54E-01	—	3.90E-01	—	4.08E-01	—
1/20	3.93E-02	2.17	3.37E-01	0.96	1.98E-01	0.98	1.99E-01	1.04
1/40	8.90E-03	2.14	1.67E-01	1.01	9.92E-02	1.00	9.81E-02	1.02
1/80	2.12E-03	2.07	8.34E-02	1.01	4.96E-02	1.00	4.89E-02	1.01

In the second experiment we check the behavior of the scheme without the consistency terms (cf. Remark 3.2). The results in Table 5.3 show that these terms are necessary for the convergence of the proposed scheme.

The goal of the third experiment is to demonstrate the convergence behavior of our scheme on the  $L$ -shaped domain  $(-0.5, 0.5)^2 \setminus [0, 0.5]^2$ . The exact solution is chosen to be

$$(5.2) \quad \mathbf{u} = \nabla \times \left( r^{2/3} \cos\left(\frac{2}{3}\theta - \frac{\pi}{3}\right) \phi(r/0.5) \right),$$

TABLE 5.2. Convergence of the scheme on the square  $(0, 0.5)^2$  for  $\alpha = -k^2$ , with uniform meshes and exact solution given by (5.1)

$h$	$\frac{\ \mathbf{u} - \mathbf{u}_h\ _{L_2(\Omega)}}{\ \mathbf{u}\ _{L_2(\Omega)}}$	order	$\frac{\ \mathbf{u} - \mathbf{u}_h\ _h}{\ \mathbf{u}\ _h}$	order	$\frac{ \mathbf{u} - \mathbf{u}_h _{\text{curl}}}{ \mathbf{u} _{\text{curl}}}$	order	$\frac{ \mathbf{u} - \mathbf{u}_h _{\text{div}}}{ \mathbf{u} _{\text{div}}}$	order
$k = 1$								
1/10	5.59E-02	—	3.24E-01	—	1.74E-01	—	2.13E-01	—
1/20	1.21E-02	2.20	1.59E-01	1.02	8.71E-02	0.99	1.07E-01	0.99
1/40	2.86E-03	2.09	7.92E-02	1.01	4.36E-02	1.00	5.35E-02	1.00
1/80	6.94E-04	2.04	3.94E-02	1.01	2.18E-02	1.00	2.67E-02	1.00
$k = 10$								
1/10	4.42E-01	—	8.79E-01	—	4.10E-01	—	4.66E-01	—
1/20	5.94E-02	2.89	3.50E-01	1.33	1.99E-01	1.05	2.00E-01	1.22
1/40	1.26E-02	2.24	1.69E-01	1.05	9.92E-02	1.00	9.82E-02	1.03
1/80	2.96E-03	2.09	8.34E-02	1.02	4.96E-02	1.00	4.89E-02	1.01

TABLE 5.3. Errors of the scheme without the consistency terms on the square  $(0, 0.5)^2$ , with uniform meshes and exact solution given by (5.1) with  $k = 1$

$h$	$\frac{\ \mathbf{u} - \mathbf{u}_h\ _{L_2(\Omega)}}{\ \mathbf{u}\ _{L_2(\Omega)}}$	$\frac{ \mathbf{u} - \mathbf{u}_h _{\text{curl}}}{ \mathbf{u} _{\text{curl}}}$	$\frac{ \mathbf{u} - \mathbf{u}_h _{\text{div}}}{ \mathbf{u} _{\text{div}}}$
$\alpha = 1$			
1/10	4.14E+01	4.57E-01	5.53E-01
1/20	4.18E+01	4.41E-01	5.48E-01
1/40	4.18E+01	4.36E-01	5.46E-01
1/80	4.18E+01	4.35E-01	5.46E-01
$\alpha = -1$			
1/10	4.14E+01	4.57E-01	5.53E-01
1/20	4.18E+01	4.41E-01	5.48E-01
1/40	4.18E+01	4.36E-01	5.46E-01
1/80	4.18E+01	4.35E-01	5.46E-01

where  $(r, \theta)$  are the polar coordinates at the origin and the cut-off function is given by

$$\phi(r) = \begin{cases} 1 & r \leq 0.25 \\ -16(r - 0.75)^3 \\ \quad \times [5 + 15(r - 0.75) + 12(r - 0.75)^2] & 0.25 \leq r \leq 0.75 \\ 0 & r \geq 0.75 \end{cases} .$$

The meshes are graded around the re-entrant corner with the grading parameter  $1/3$ . The results are tabulated in Table 5.4 and they agree with the error estimates for our scheme. That is, the scheme is second order accurate in the  $L_2$  norm and first order accurate in the energy norm. Since the divergence of the exact solution is zero, the absolute errors instead of the relative errors in divergence are included in Table 5.4.

TABLE 5.4. Convergence of the scheme on the  $L$ -shaped domain  $(-0.5, 0.5)^2 \setminus [0, 0.5]^2$ , with graded meshes and the exact solution given by (5.2)

$h$	$\frac{\ \mathbf{u} - \mathbf{u}_h\ _{L_2(\Omega)}}{\ \mathbf{u}\ _{L_2(\Omega)}}$	order	$\frac{\ \mathbf{u} - \mathbf{u}_h\ _h}{\ \mathbf{u}\ _h}$	order	$\frac{ \mathbf{u} - \mathbf{u}_h _{\text{curl}}}{ \mathbf{u} _{\text{curl}}}$	order	$ \mathbf{u} - \mathbf{u}_h _{\text{div}}$	order
$\alpha = 0$								
1/4	9.93E+01	—	1.32E+01	—	4.93E+00	—	1.84E-02	—
1/8	3.24E+01	1.62	6.70E-00	0.97	3.59E-00	0.46	1.85E-02	—
1/16	3.29E-00	3.30	2.24E-00	1.58	7.79E-01	2.20	3.61E-03	2.36
1/32	6.91E-01	2.25	1.11E-00	1.01	4.33E-01	0.84	2.14E-03	0.76
1/64	1.71E-01	2.01	5.54E-01	1.00	2.34E-01	0.90	6.49E-04	1.72
$\alpha = 1$								
1/4	7.57E+01	—	1.01E+01	—	3.52E-00	—	1.65E-02	—
1/8	2.82E+01	1.43	6.07E-00	0.74	3.05E-00	0.20	1.80E-02	—
1/16	3.23E-00	3.13	2.21E-00	1.46	7.73E-01	1.98	3.59E-03	2.33
1/32	6.84E-01	2.23	1.10E-00	1.00	4.33E-01	0.84	2.13E-03	0.75
1/64	1.67E-01	2.04	5.54E-01	1.00	2.34E-01	0.89	6.47E-04	1.73
$\alpha = -1$								
1/4	1.46E+02	—	1.90E+01	—	7.77E-00	—	2.22E-02	—
1/8	3.85E+01	1.92	7.58E-00	1.32	4.40E-00	0.82	1.91E-02	0.22
1/16	3.37E-00	3.51	2.25E-00	1.75	7.87E-01	2.49	3.63E-03	2.39
1/32	6.99E-01	2.27	1.11E-00	1.03	4.34E-01	0.86	2.14E-03	0.76
1/64	1.77E-01	1.98	5.54E-01	1.00	2.34E-01	0.90	6.51E-04	1.72

## 6. CONCLUDING REMARKS

The results in this paper and [17, 16, 18] have firmly established the feasibility of using nonconforming finite element methods in computational electromagnetics.

We have only treated the source problem in this paper, but the scheme can also be applied to the eigenproblem (1.2). In fact, it follows from the  $L_2$  error estimate in Theorem 4.10 that the classical theory of spectral approximation [40, 7] can be invoked to provide a straightforward convergence analysis for the solution of the eigenproblem by the scheme in this paper. Note that the eigenvalues defined by (1.2) are closely related to Maxwell eigenvalues [25], and by choosing  $\gamma$  large enough the scheme in this paper can be used to compute Maxwell eigenvalues. This will be further investigated in [19].

Since the problem (1.1) resembles a second order elliptic boundary value problem, many of the fast solvers developed for second order elliptic boundary value problems using nonconforming/interior penalty methods can be adopted for the scheme (3.13). The error analysis in this paper provides the foundation for the study of multigrid methods for Maxwell's equations using nonconforming finite elements, which will be carried out in [31].

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