1. Using Taylor series expansion, find the error term (or the leading term in the error term) and order (with respect to \( h \)) of the following approximation for \( f'(x) \):

\[
\frac{4f(x + h) - 3f(x) - f(x - 2h)}{6h}.
\]

2. Using Taylor series expansion, find an approximation for \( f'(x) \) based on the data \( f(x - 2h), f(x - h), f(x), f(x + h) \), that has the highest approximation order (with respect to \( h \)).

3. (Computer problem) Given a smooth function \( f(x) \), and it is known that

\[
F_2(h) = \frac{f(x + h) - 2f(x) + f(x - h)}{h^2}
\]

provides a second order approximation for \( f''(x) \) (with respect to the parameter \( h \)). Apply Richardson extrapolation to the formula. The resulted approximation for \( f''(x) \), denoted as \( F_4(h) \), turns out to be a fourth order approximation instead of a third order one for \( f''(x) \). Demonstrate the performance of the original second order and the new fourth order formula by approximating \( f''(\pi/3) \), where \( f(x) = \sin(x) + xe^{-x} \), with \( h = h_j = 0.1 \times 0.5^j \), with \( j = 1, 2, 3, 4, 5, 6 \).

a.) Tabulate your results and errors, with \( j - th \) row (\( j \geq 2 \)) including the following

\[
h_j, \quad F_2(h_j), \quad e_2(h_j), \quad F_4(h_j), \quad e_4(h_j), \quad \frac{e_2(h_j-1)}{e_2(h_j)}, \quad \frac{e_4(h_j-1)}{e_4(h_j)}
\]

while the first row includes (\( j = 1 \))

\[
h_j, \quad F_2(h_j), \quad e_2(h_j), \quad - , \quad F_4(h_j), \quad e_4(h_j), \quad -.
\]

Here the errors are \( e_2(h) = |F_2(h) - f''(\pi/3)| \) and \( e_4(h) = |F_4(h) - f''(\pi/3)| \).

It is suggested that \( F_2(h_j), F_4(h_j) \) are shown with sufficiently many digits after decimal points, so you can see the change when \( h \) decreases.

b.) Plot \( h_j \) versus \( F_2(h_j) \), \( j = 1, \cdots, 6 \) in loglog scale; on the same figure, plot \( h_j \) versus \( F_4(h_j), j = 1, \cdots, 6 \) in loglog scale;

c.) Discussion: How do your results from a.) b.) confirm /support /contradict the claim that \( F_2(h) \) is a second order approximation for \( f''(x) \), where \( F_4(h) \) is a fourth order approximation for \( f''(x) \)?

4. Apply the composite Trapezoid rule with \( m = 1, 2 \) and 4 panels to approximate the following integrals. Compute the error by comparing with the exact value from calculus.

\[
a) \int_0^2 x \cos(x) dx, \quad b) \int_0^1 \frac{1}{1 + x^2} dx.
\]

5. Apply the composite midpoint rule with \( m = 1, 2 \) and 4 panels to approximate the following integrals.

\[
a) \int_0^2 \frac{dx}{\sqrt{2-x}}, \quad b) \int_0^{\pi/2} \frac{\cos(x)}{\pi/2 - x} dx
\]

For the integral in a), also compute the error by comparing with the exact value from calculus.
6. Find the degree of precision of the following degree four Newton-Cotes rule

\[ \int_{x_0}^{x_4} f(x)dx \approx \frac{2h}{45}(7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4). \]

Here \( x_j = x_0 + jh \), \( y_j = f(x_j) \), \( j = 1, 2, 3, 4 \). And \( h \) is a fixed positive parameter.