MODERN DEVELOPMENTS IN MULTIPHASE FLOW & HEAT TRANSFER

“ENGINEERING APPLICATIONS OF FRACTAL AND CHAOS THEORY”

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1. INTRODUCTION

The study of chaotic phenomena in deterministic dynamical systems relies heavily on the concept of fractal geometry and the topological interpretation of phase space trajectories. This emerging science has already produced some dramatic breakthroughs in our understanding of nonlinear dynamics. It is the purpose of this tutorial paper to summarize the essential concepts necessary for understanding how these ideas and analytical techniques may be applied to physical systems. In particular, applications in the area of single and multiphase flow & heat transfer will be stressed.

This paper will begin by reviewing some important concepts concerning fractals and fractal dimensions. Next, some of the essential elements of nonlinear dynamics will be presented. Examples of static and dynamic bifurcations will be presented and discussed, and phase plane interpretations of both linear and nonlinear oscillations will be shown and generalized to higher order systems of equations.

Next, chaotic phenomena will be discussed and the concepts of a “strange attractor” and basin of attraction will be presented. Bifurcation diagrams, Poincaré sections, first return maps, and Lyapunov exponents, as well as other tests to verify a chaotic response, will be discussed.

Finally, these techniques will be applied to problems of density-wave instability in a single-phase natural circulation loop and in a boiling channel. It will be shown that chaotic phenomena may occur for some operating conditions.

2. FRACTALS

There are many good books on the theory of fractals in which fractals are carefully defined [1, 2, 3, 4]. For our purposes here it is sufficient to define a fractal as, a self-similar mathematical object which is produced by simple repetitive mathematical operations. In order to quantify what is meant, let us consider a few sets which exhibit important fractal properties.

As shown schematically in Figure 1, the Koch set is generated by dividing a line segment of length L into three equal segments (L/3), removing the center segment and in its place forming an equilateral triangle having side length L/3. As the process is repeated a rather fuzzy looking continuous curve, which is nowhere differentiable, is formed. Significantly, in the limit this set has infinite length.

The Cantor set is shown in Figure 2. One way to form it is by again dividing a line segment of length L into three equal segments (L/3), except that in this case the center segment is removed and discarded. As the process is repeated, we obtain, in the limit, a set having zero length and an infinite number of points. The Cantor set is a very important one in the theory of chaos, since the phase space trajectories often form a pattern on so-called Poincaré sections (to be discussed later) which have some properties similar to that of a Cantor set.
Other more complicated sets are also possible using simple iterative algorithms. One of the most famous is the Mandelbrot set. This set is generated from the iteration of the following complex nonlinear function:

\[ z_{n+1} = z_n^2 + C \]  \hspace{1cm} (1)

where, \( z = x + iy \), and, \( C = A + iB \).

Equation (1) is equivalent to iteration of the following coupled real nonlinear functions:

\[ x_{n+1} = x_n^2 - y_n^2 + A \]  \hspace{1cm} (2a)

\[ y_{n+1} = 2x_ny_n + B \]  \hspace{1cm} (2b)
As shown in Figure 3, astoundingly complex patterns can be generated with the relatively simple iterative algorithm discussed above. Moreover, this set demonstrates the fractal property that it repeats itself over and over as we zoom into a particular region. This is an important property of fractal sets, and one which we will find important when we discuss chaotic “first-return maps.”

![Figure 3. The Mandelbrot Set [4].](image)

Let us now turn our attention to how to characterize various sets. In particular, the fractal dimension of a set. There are a number of possible fractal dimensions which one can define. Unfortunately, the fact that they are not all equivalent has often led to confusion. In this chapter we will consider only a few of the most important fractal dimensions for practical applications.

The Hausdorff-Besicovitch dimension ($D_{H-B}$) is defined below:

$$D_{H-B} = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)}$$

hence,

$$N(\varepsilon) \propto \varepsilon^{-D_{H-B}}$$

As shown in Figure 4, $\varepsilon$ is the length of a hypercube, and $N(\varepsilon)$ is the smallest number of hypercubes necessary to enclose the set of hyperspace points shown in Figure 4.
We note that when the set is a single point, \( N(\varepsilon) = 1 \), then Eq. (3b) implies the Hausdorff-Besicovitch dimension is, \( D_{H-B} = 0.0 \). In contrast, when the set is a line segment of length \( L \), the number of hypercubes needed is, \( N(\varepsilon) = L/\varepsilon \). Thus, Eq. (3a) implies:

\[
D_{H-B} = \lim_{\varepsilon \to 0} \left( \frac{\ell n(L/\varepsilon)}{\ell n(1/\varepsilon)} \right) = 1.0
\]

Similarly, when the set is a surface of area \( S \), \( N(\varepsilon) = S/\varepsilon^2 \), thus:

\[
D_{H-B} = \lim_{\varepsilon \to 0} \left( \frac{\ell nS - 2/\varepsilon}{-\ell n\varepsilon} \right) = 2.0
\]

For a fractal set, such as the Koch set, which covers more hyperspace than a line but less than a surface, we expect, \( 1.0 < D_{H-B} < 2.0 \). Indeed, for this set we obtain [5]:

\[
D_{H-B} = \frac{\ell n4}{\ell n3} = 1.26
\]

Similarly, for the Cantor set, which covers more hyperspace than a single point but less than a line, we expect, \( 0.0 < D_{H-B} < 1.0 \). Since the Cantor algorithm implies, \( \varepsilon = (1/3)^m \) and \( N(\varepsilon) = 2^m \), we obtain:

\[
D_{H-B} = \lim_{\varepsilon \to 0} \left( \frac{\ell n2^m}{\ell n3^m} \right) = \frac{\ell n2}{\ell n3} = 0.63
\]

Unfortunately, for many practical applications the limiting operation implicit in the definition of the Hausdorff-Besicovitch dimension converges very slowly. Thus it is not useful for many cases of interest.

Hence, let us next consider the so-called correlation dimension, \( D_C \). This dimension is particularly useful for the analysis of data. It consists of centering a hypersphere about point-i in hyperspace and then letting the radius \( (r) \) of the hypersphere grow until all \( n \) points are enclosed. In practice, since the number of points \( (n) \) is finite, many spheres are used (centered about different points) and the results are averaged. That is, the correlation function, \( C(r) \), is given by:
\[ C(r) = \lim_{n \to \infty} \frac{1}{n^2} \sum_{i \neq j}^n H\left(r - |x_i - x_j|\right) \]  

(4)

where, \( H(\xi) \) is a Heaviside step operation defined such that:

\[
H(\xi) = \begin{cases} 
1.0, & \xi \geq 0 \\
0.0, & \xi < 0 
\end{cases}
\]

It has been found [6] that, in the limit as \( r \to 0 \),

\[ C(r) = A_r r^{D_C} \]  

(5)

Thus,

\[ \ell \log C = D_C \ell \log r + A_2 \]  

(6)

As can be seen in Figure 5, if we plot \( C(r) \) versus \( r \) in a log-log plot then the slope of the line will be the correlation dimension, \( D_C \).

If we have an analytic function that is being numerically evaluated it is straightforward to compute the correlation dimension since the dimension of the hyperspace is known (i.e., all the state variables are known). For example, the Cantor set has a correlation dimension of, \( D_C = 0.63 \). Interestingly this is the same as the Hausdorff-Besicovitch dimension. Unfortunately this is not always true. In fact, it has been found [6] that, \( D_C \leq D_{H-B} \).

Figure 5. Correlation Dimension (\( D_C \)).
Let us now consider evaluating the correlation dimension ($D_C$) of data. If we are processing experimental data we normally do not know how many state variables are needed to characterize the process. Fortunately, it has been shown [7] that a pseudo-phase-space can be constructed using time-delayed measurements of only one temporal measurement, and that the basic topology of the attractor being investigated will be unchanged. This was an important discovery since it allows one to calculate the fractal dimension of a wide range of experimental data.

Let us assume we are measuring some variable $\chi$ (e.g., pressure, temperature, etc.). We then have readings $\chi(t), \chi(t+\tau), \chi(t+2\tau), \ldots$, where the time delay $\tau$ is somewhat arbitrary, however it is often taken to be the period of the maximum energy peak in the power spectral density (PSD) function. To determine $D_C$, we must first assume a dimension for the hyperspace. Let us start by assuming a 3-D space, and plot $\chi(t), \chi(t+\tau)$ and $\chi(t+2\tau)$ as shown in Figure 6. We then center a sphere of radius-$r$ about point-$i$ and apply Eq. (4). Next we make a log-log plot, as in Figure 5, and obtain a slope, $D_C(3)$. We then assume that four state variables are needed to describe the process and apply Eq. (4) to the 4-D hyperspace given by, $\chi(t), \chi(t+\tau), \chi(t+2\tau)$ and $\chi(t+3\tau)$. As before we make a log-log plot of the results and get a new (steeper) slope, $D_C(4)$. We continue this process for 5-D space, then 6-D space and so on, until, as shown in Figure 5, the slope ($D_C$) no longer changes (i.e., $D_C(5) = D_C(6) = \ldots$).

In the example shown in Figure 5, it has been found that the minimum number of state variables ($p$) to describe the process being measured is $p = 5$. This is called the imbedding dimension of the process. As shown in Figure 5, the imbedding dimension can be most easily recognized by plotting $D$ versus $p$. The degree of freedom ($p$) at which $D$ stops changing is the imbedding dimension and this value of $D$ is the correlation dimension ($D_C$) of the process. It should be noted that a truly random process (i.e., one which has no underlying structure) has no finite imbedding dimension. That is, as can be seen in Figure 5, the slope never saturates for so-called “white noise.”

When random noise is superimposed on the chaotic (i.e., deterministic) signal, one expects [8] the slope in the “large r” part of the plot to converge to $D_C$, as discussed above. In contrast, for the “smaller r” part, the slope is different and never converges. Indeed, this part of the curve reflects the random noise in the system.

![Figure 6. 3-D Phase Space Representation of Signal $\chi(t)$](image-url)
Another important and interesting fractal dimension has been discussed by Feder [1]. This dimension is sometimes called the Hurst dimension, $D_{H-B}$. Hurst showed that many time varying chaotic processes, $\xi(t)$, of record length $\tau$, can be correlated by:

$$R(\tau)/S(\tau) = (\tau/2)^{D_H}$$  \hspace{1cm} (7)

where,

$$R(\tau) = \Delta_{t \in \tau} \max \chi(t;\tau) - \min \chi(t;\tau)$$  \hspace{1cm} (8a)

$$\chi(t;\tau) = \frac{1}{\tau} \int_{t-t}^{t} [\xi(t') - \bar{\xi}(\tau)] dt'$$  \hspace{1cm} (8b)

$$\bar{\xi}(\tau) = \frac{1}{\tau} \int_{t-t}^{t} \xi(t') dt'$$  \hspace{1cm} (8c)

and,

$$S = \left\{ \frac{1}{\tau} \int_{t-t}^{t} [\xi(t') - \bar{\xi}]^2 dt' \right\}^{1/2}$$  \hspace{1cm} (8d)

Obviously, $\chi(t;\tau)$ is the cumulative temporal variation of $\xi(t)$ about its mean, $\bar{\xi}$; $R(\tau)$ is the difference between the maximum and minimum value of $\chi(t;\tau)$ in interval $\tau$, and $S$ is the standard deviation of $\bar{\xi}(t)$.

It has been found [1] that random processes are characterized by $D_H = 0.5$, while chaotic processes are correlated by, $D_H > 0.5$. Indeed, processes having $D_H > 0.5$, may have some underlying deterministic structure. Thus the Hurst dimension, $D_H$, is a relatively easy way of determining when a random-looking process may have a hidden structure.

While there is much more than can be said about fractals, the ideas which have been presented above should be sufficient to allow one to understand the fractal nature of chaos. Thus let us now turn our attention to the mathematics which underlies chaos theory. First, we will consider the elements of bifurcation theory.

3. BIFURCATION THEORY

In order to understand how the solutions of nonlinear equations may bifurcate, and what it means when a bifurcation takes place, let us consider some simple examples of static and dynamic bifurcations. This tutorial approach follows the work of Dorning [9]. There are two main classifications of bifurcations; static and dynamic bifurcations.
3.1 Static Bifurcations

The most common and important static bifurcations are the turning point (i.e., saddle point) bifurcation, the transcritical bifurcation and the pitchfork bifurcation. It should be noted that the solution of a single nonlinear differential equation, having only one state variable, may exhibit static bifurcations. In contrast, at least two state variables are required for dynamic bifurcations.

Let us consider some nonlinear first order ordinary differential equations which represent the canonical forms that the various static bifurcations can be reduced to. We begin by considering:

\[ \dot{x}(t) = f_1(x(t), \mu) = \mu - x^2(t) \]  
(9)

The fixed (i.e., steady-state) points of this differential equation are given by setting the time derivative in Eq. (9) equal to zero. Hence, \( f_1(x; \mu) = 0 \), and thus,

\[ x^0(\mu) = \mu \]  
(10a)

or,

\[ x^{(1)}_0 = +\sqrt{\mu}, \quad x^{(2)}_0 = -\sqrt{\mu} \]  
(10b)

In order to examine the stability of these fixed points, we linearize Eq. (9) using,

\[ \delta \dot{x} = \left. \frac{\partial f_1}{\partial x} \right|_{x_0} \delta x \]  
(11a)

Thus,

\[ \delta \dot{x} = -2x_0 \delta x \]  
(11b)

where the perturbation of the state variable (x) is given by:

\[ \delta x \equiv x(t) - x_0 \]  
(11c)

There are two possible values of the steady-state solution, \( x_0 \). These are given in Eq. (10b). Thus we have:

\[ \delta \dot{x}^{(1)} = -2x^{(1)}_0 \delta x = -2\sqrt{\mu} \delta x \]  
(12a)

and,

\[ \delta \dot{x}^{(2)} = -2x^{(2)}_0 \delta x = 2\sqrt{\mu} \delta x \]  
(12b)
Assuming that the parameter $\mu$ is a positive real number, the solution of Eq. (12a) is stable, and is given by:

$$\delta x^{(1)}(t) = \delta x^{(1)}(0)e^{-2\sqrt{\mu}t} \quad (13a)$$

while Eq. (12b) is unstable, and is given by:

$$\delta x^{(2)}(t) = \delta x^{(2)}(0)e^{2\sqrt{\mu}t} \quad (13b)$$

The turning point (i.e., saddle point) bifurcation and its solution flow are given in Figure 7. It can be seen that the lower branch, $x_0^{(2)}$, is shown dashed to denote that it is unstable.

Next we turn our attention to a somewhat more complicated equation given by:

$$\dot{x}(t) = f_x(x(t),\mu) = \mu x - x^2 \quad (14)$$

This equation exhibits what is called a transcritical bifurcation. As before the fixed points are given by:

$$\mu x_o - x_o^2 = 0 \quad (15a)$$

thus,

$$x_o^{(1)} = 0 \quad , \quad x_o^{(2)} = \mu \quad (15b)$$

Figure 7. Turning (i.e., saddle) point bifurcation.
The stability of the fixed points is given by the solutions of the linearization of Eq. (14):

$$\delta \dot{x} = (\mu - 2x_0) \delta x$$  \hspace{1cm} (16)

The solution of Eq. (16) is:

$$\delta x^{(i)}(t) = \delta x^{(i)}(0)e^{(\mu - 2x_0)(t)}$$  \hspace{1cm} (17)

For \(i = 1\), Eqs. (15b) and (17) yield,

$$\delta x^{(1)}(t) = \delta x^{(1)}(0)e^{\mu t}$$  \hspace{1cm} (18a)

while for \(i = 2\),

$$\delta x^{(2)}(t) = \delta x^{(2)}(0)e^{-\mu t}$$  \hspace{1cm} (18b)

Equations (18) show that when \(\mu > 0\), the solution given by Eq. (18a) is unstable while the solution in Eq. (18b) is stable. In contrast, when \(\mu < 0\), the solution in Eq. (18a) is stable while that in Eq. (18b) is unstable. This bifurcation and its solution flow are shown in Figure 8.

Let us next consider the so-called pitchfork bifurcation. This interesting and important static bifurcation occurs in the solution of the following nonlinear ordinary differential equation:

$$\dot{x}(t) = f_3(x(t), \mu) = \mu x - x^3$$  \hspace{1cm} (19)

As before, the fixed points are determined by setting the time derivative to zero. Thus,

$$\mu x_0 - x_0^3 = 0$$  \hspace{1cm} (20a)

hence,

$$x_0^{(1)} = 0 \quad ; \quad \mu < 0$$  \hspace{1cm} (20b)

$$x_0^{(1)} = 0 \quad , \quad x_0^{(2)} = +\sqrt{\mu} \quad , \quad x_0^{(3)} = -\sqrt{\mu} \quad ; \quad \mu > 0$$  \hspace{1cm} (20c)

The stability of these fixed points are determined from the solutions of the linearization of Eq. (19):

$$\delta \dot{x} = (\mu - 3x_0^2) \delta x$$  \hspace{1cm} (21)

The solution of Eq. (21) is given by:
\[
\delta x^{(i)}(t) = \delta x^{(i)}(0)e^{\mu t}
\]  \hspace{1cm} (22)

From Eqs. (20b) and (21) we have:

\[
\delta x^{(i)}(t) = \delta x^{(i)}(0)e^{\mu t}
\]  \hspace{1cm} (23a)

Thus, when \(\mu > 0\) the solution is unstable, while for \(\mu < 0\) the solution is stable.

Next, from Eqs. (20c) and (22), we find stable solutions for both branches of the pitchfork:

\[
\delta x^{(2)}(t) = \delta x^{(2)}(0)e^{-2\mu t}
\]  \hspace{1cm} (23b)

\[
\delta x^{(3)}(t) = \delta x^{(3)}(0)e^{-2\mu t}
\]  \hspace{1cm} (23c)

The pitchfork bifurcation and its solution flow are given in Figure 9. It can be seen that this bifurcation looks similar to a saddle point bifurcation, except that both the upper and lower branch of the pitchfork bifurcation have stable fixed points. As a consequence, the solution flow on the lower branch is different.

Finally, we note that Eq. (19) is a special case of the following nonlinear ordinary differential equation:

\[
\dot{x}(t) = f_x(x(t),\mu,\theta) = \theta + \mu x - x^3
\]  \hspace{1cm} (24)
where $\theta$ is often referred to as the imperfection parameter. Obviously when $\theta = 0$, Eq. (24) reduces to Eq. (19). When $\theta \neq 0$, the fixed points of Eq. (24) are given by:

\[ \theta + \mu x_o - x_o^3 = 0 \]  

which has three roots given by [10],

\[ x_0^{(1)} = (S_1 + S_2) \]  

\[ x_0^{(2)} = \frac{1}{2} (S_1 + S_2) + \frac{i\sqrt{3}}{2} (S_1 - S_2) \]  

\[ x_0^{(3)} = -\frac{1}{2} (S_1 + S_2) - \frac{i\sqrt{3}}{2} (S_1 - S_2) \]

Figure 9. Pitchfork bifurcation.

where,

\[ S_1 = \left[ \theta/2 + \left( \theta^2/4 - \mu^3/27 \right)^{1/2} \right]^{1/3} \]  

\[ S_2 = \left[ \theta/2 - \left( \theta^2/4 - \mu^3/27 \right)^{1/2} \right]^{1/3} \]

As before, the stability of these fixed points is given by linearizing Eq. (24),

\[ \delta \dot{x}(t) = (\mu - 3x_o^2) \delta x \]  

which is the same as Eq. (21). The solution of Eq. (28) is given by Eq. (22):

\[ \delta x^{(i)}(t) = \delta x^{(i)}(0) e^{\left[ \mu - 3x_o^2 \right] t} \]
Thus the stability of the fixed points depends on the value of the three roots given in Eqs. (26), \( x_0^{(i)} \).

It is interesting to consider the static bifurcation diagram of this two parameter, \( \mu \) and \( \theta \), one state variable, \( x(t) \), differential equation. We see in Figure-10 that for \( \theta \neq 0 \) the pitchfork bifurcation unfolds. Indeed we have a so-called cusp catastrophe such that when \( |\theta| \) is large enough, hysteresis causes a sudden change from the one branch of fixed points to the other (e.g., from the lower to the upper branch as \( \theta > 0 \) is increased beyond \( \theta^* \)).

The physical significance of the unfolding of a pitchfork bifurcation has been studied previously in connection with the problem of instabilities in natural convection [11]. That is, for the onset of Benard cells in a square pool of fluid heated from below and inclined by an angle \( \theta \). For example, as can be seen in Figure 11, when the Rayleigh number (Ra) is increased for a horizontal pool (i.e., \( \theta = 0^\circ \)), we have a pitchfork bifurcation (at Ra*) giving rise to a roll cell of either positive or negative polarity. It should be noted that the critical Rayleigh number (Ra*) is closely related to the parameter \( \mu \); indeed, \( \mu = Ra – Ra^* \).

In contrast, when the heated cavity is tilted to \( \theta = 2^\circ \), the pitchfork bifurcation unfolds as shown in figure-1, and the roll cell will normally be in the counter-clockwise direction.

![Figure 10. The unfolding of a pitchfork bifurcation.](image-url)
Dynamic Bifurcations

Let us now turn our attention to the analysis of dynamic, of Hopf, bifurcations [12]. As noted previously, at least two state variables are required for a Hopf bifurcation. A simple example of a system having a Hopf bifurcation is given by:

\[
\begin{align*}
\dot{x}_1(t) &= -x_2 + x_1 \left[ \mu \mp \left( x_1^2 + x_2^2 \right) \right] \\
\dot{x}_2(t) &= -x_1 + x_2 \left[ \mu \mp \left( x_1^2 + x_2^2 \right) \right]
\end{align*}
\] (30a)

Fixed points of this system of equations for real \( \mu \) are given by, \( (x_{10}, x_{20}) = (0,0) \). As usual the stability of these fixed points can be determined by linearizing the system of equations (30) about the fixed points:

\[
\begin{align*}
\delta\dot{x}_1(t) &= -\delta x_2 + \mu \delta x_1 \\
\delta\dot{x}_2(t) &= \delta x_1 + \mu \delta x_2
\end{align*}
\] (31a)

Figure 11. Effect of Rayleigh Number on the Bifurcation for \( \theta = 0^\circ \) and \( \theta = 2^\circ \).
or, equivalently, in matrix notation:

\[
\begin{bmatrix}
\delta x_1 \\
\delta x_2
\end{bmatrix} =
\begin{bmatrix}
\mu & -1 \\
1 & \mu
\end{bmatrix}
\begin{bmatrix}
\delta x_1 \\
\delta x_2
\end{bmatrix}
\] (32)

If we assume a modal solution of the form,

\[
\begin{bmatrix}
\delta x_1 \\
\delta x_2
\end{bmatrix} =
\begin{bmatrix}
a_1 \\
a_2
\end{bmatrix}
e^{\lambda t}
\] (33)

we find that the only non-trivial solution is for:

\[
\det\begin{bmatrix}
(\mu - \lambda) & -1 \\
1 & (\mu - \lambda)
\end{bmatrix} = 0
\] (34)

Hence,

\[
\lambda^2 - 2\mu\lambda + (\mu^2 + 1) = 0
\] (35)

and the eigenvalues are:

\[
\lambda = \frac{2\mu \pm \sqrt{4\mu^2 - 4(\mu^2 + 1)}}{2}
\]

Thus,

\[
\lambda = \mu \pm \sqrt{-1} \equiv \mu \pm i
\] (36)

Equations (36) and (33) imply that we will have an oscillatory solution (with an angular frequency of \(\omega = 1.0\) rad/s) which is damped (i.e., stable) for \(\mu < 0\) (a stable focus), and unstable for \(\mu > 0\) (an unstable focus).

It is interesting to note that \((x_1, x_2) = (0,0)\) is a solution to Eqs. (30), and the only one valid for small perturbations. However, Eqs. (30) also have another, finite amplitude, solution given by:

\[
x_1^2 + x_2^2 = \pm \mu
\] (37)

Equation (37) is a circular orbit in \(x_1 - x_2\) phase space. As shown schematically in Figures-12 for \(\mu > 0\), and the plus (+) sign, in Eqs. (30) and (37), we may have a supercritical bifurcation, while for \(\mu < 0\), and the minus (-) sign, we may have a subcritical bifurcation. As can be seen in Figure 12a, for a supercritical bifurcation, all phase plane (i.e., \(x_1 - x_2\) plane) trajectories (i.e., solution flows) converge to a stable limit cycle of finite amplitude for conditions on the unstable side of the linear stability.
boundary (i.e., $\mu = 0$). Moreover, as given by Eq. (37), the amplitude of this limit cycle is $\sqrt{\mu}$. In contrast, Figure 12b shows that for $\mu < 0$ a subcritical bifurcation, an unstable limit cycle (of amplitude $\sqrt{-\mu}$) exists, such that the phase plane trajectories either converge to the negative $\mu$ axis, $(0,0)$, for small perturbations, or diverge exponentially for large enough perturbations. Thus, in this problem there are two so-called basins of attraction which are separated by the unstable limit cycle.

The occurrence of both subcritical and supercritical bifurcations have been predicted in boiling channels [13] but only supercritical bifurcations (i.e., limit cycles) have been measured to date. It is significant to note that subcritical bifurcations are potentially quite dangerous since they imply that divergent instability can occur in the region of linear stability if large enough amplitude perturbations occur in a boiling channel. This may result in a critical heat flux (CHF) and physical damage to the heated surface.
The Hopf theorem is an existence theorem which states that a dynamic (i.e., Hopf) bifurcation occurs when one has a complex conjugate pair of eigenvalues ($\lambda$) cross the imaginary axis with,

\begin{align}
\text{Im}(\lambda) & \neq 0 \\
\frac{d[\text{Re}(\lambda)]}{d\mu} & \neq 0
\end{align}

We note that the eigenvalues given in Eq. (36) satisfy the Hopf criteria.

The actual application of the Hopf analysis normally involves the use of higher order perturbation theory and a Floquet analysis of the stability of the resultant limit cycle. The application of this analytical method to a boiling channel has been discussed in detail by Achard et al [13] and Rizwan-Uddin et al [14], and thus will not be repeated here.

\subsection*{3.3 Self-Similarity and Mixed Bifurcations}

Many bifurcation diagrams exhibit self-similarity. In particular, let us define:

\begin{align}
\delta_n & = \frac{\mu_n - \mu_{n-1}}{\mu_n - \mu_n} \\
\alpha_n & = \frac{A_{n-1}}{A_n}
\end{align}

As shown schematically in Figure-13, $\mu_n$ is the value of the parameter $\mu$ at the onset of the $n^{th}$ bifurcation and $A_n$ is the maximum amplitude of the oscillation at the end of the $n^{th}$ bifurcation (i.e., at the onset of the $(n+1)^{th}$ bifurcation). A cascade of period doubling bifurcations in the phase plane orbits is normally a precursor to the onset of chaos. Indeed, period doubling is often used as an indication of the approach to chaos.

It has been observed [15] that for many nonlinear systems, that,

\begin{center}
\text{Figure 13. A typical Hopf bifurcation followed by a cascade of period doubling bifurcations.}
\end{center}
\[ \lim_{n \to \infty} \delta_n = 4.669202... \tag{40a} \]
\[ \lim_{n \to \infty} \alpha_n = 2.502907... \tag{40b} \]

These results are called the Feigenbaum numbers (or Feigenvalues), and provide a means of assessing numerical results and of estimating where the next, that is, \((n+1)^{th}\), bifurcation may occur.

Not all bifurcations are simple static and dynamic bifurcations of the type previously discussed. Figure-14a shows an example of a mixed subcritical bifurcation which, at large enough amplitudes, exhibits a stable limit cycle response. Such a system has the features of both a subcritical and a supercritical bifurcation. Moreover, significant hysteresis may occur, as shown in Figure-14b.

Figure-15 shows a mixed supercritical bifurcation, however, in this case if the initial excitation is large enough (i.e., larger than the amplitude of the unstable limit cycle), or if \(\mu > \mu^*\), the solution will diverge with time. Thus, this system of equations has the features of both a supercritical and a subcritical bifurcation.

It is also possible to have both static and dynamic bifurcations in a particular system of equations. Figure-16 shows a pitchfork bifurcation \((0 < \mu < \mu^*)\) which subsequently experiences a supercritical Hopf bifurcation at \(\mu = \mu^*\).

Figure 14a. Mixed Supercritical Hopf Bifurcation

Figure 14b. Hysteresis in Mixed Supercritical Hopf Bifurcation

Physically, this may correspond to the case of a natural circulation loop which begins to circulate at \(\mu = 0\) (a pitchfork bifurcation) and as \(\mu\) is increased the loop becomes unstable (at \(\mu^*\)) due to a Hopf bifurcation. Naturally, more complicated situations are
also possible and depend on the nonlinearities in the mathematical system of equations being investigated.

Now that we have reviewed the elements of the mathematics associated with bifurcations theory, let us consider the application of these concepts to the study of chaos in some engineering systems.

4. CHAOS THEORY

Let us begin with a review of a simple linear oscillator. In particular, a one-dimensional damped spring/mass system which is being excited with a forcing function, \( F(t) \). Such a system can be written as:

\[
m\ddot{x} + \beta \dot{x} + kx = F(t)
\]  \hspace{1cm} (41a)

Dividing through by the mass (m), we obtain:

\[
\dot{x} + \beta \dot{x} + kx = f(t)
\]  \hspace{1cm} (41b)

\[
\frac{dx}{dt}, \beta = \frac{B}{m}, k = \frac{K}{m}, \text{and}, f(t) = \frac{F(t)}{m}.
\]

Equation (41b) can be written as a system of coupled first order coupled differential equations as:

\[
\dot{x} = y
\]  \hspace{1cm} (42a)
\[ \dot{y} = -\beta y - kx + f \]  

(42b)

In matrix form, Eqs. (42) are:

\[
\begin{bmatrix}
0 & 1 \\
-k & -\beta
\end{bmatrix} \begin{bmatrix} \zeta \\ f \end{bmatrix} = M \zeta + f
\]  

(43a)

\[ \zeta = (x \ y)^T \]  

(43b)

Now an autonomous (i.e., unforced) oscillator has \( f = 0 \), and is thus given by:

\[ \dot{\zeta} = M\zeta \]  

(44)

Let us assume a modal solution of the form,

\[ \zeta = a \ e^{\lambda t} \]  

(45)

Combining Eqs. (44) and (45) we obtain:

\[ a \ e^{\lambda t} \ [M - \lambda I] = 0 \]  

(46)

where \( I \) is a matrix having elements given by the Kronecker delta function, \( \delta_{ij} \). The only non-trivial solution to Eq. (46) is:

\[ \det[M - \lambda I] = 0 \]  

(47)
That is,

\[ \det \begin{bmatrix} -\lambda & 1 \\ -k & -(\beta + \lambda) \end{bmatrix} = 0 \]  

(48)

Obviously \( \lambda \) are the eigenvalues of the system matrix, \( M \). Expanding out Eq. (48) we obtain the so-called Characteristic Equation:

\[ \lambda^2 + \beta \lambda + k = 0 \]  

(49)

Thus,

\[ \lambda = \left( -\beta \pm \sqrt{\beta^2 - 4k} \right) / 2 \]  

(50)

We see that \( \lambda \) will be complex if, \( \beta^2 - 4k < 0 \).

Recalling that,

\[ \text{Re}[e^{(\alpha + i\omega)k}] = e^{\alpha} \cos(\omega t) \]  

(51)

We see in Figures-17 that there are three possible solutions. Namely, a stable elliptic attractor, an unstable elliptic repellor (Fig. 17a), and a stable point attractor (Fig. 17b). It is also possible to have an unstable saddle (i.e., turning) point (Fig. 17b), however, this solution is not found in a damped linear spring-mass system. It should be noted that these figures present the locus of the time varying trajectories in the so-called phase plane. This is, a plane defined by the state variables, \( y(t) = \dot{x}(t) \) and \( x(t) \).

Moreover, for the two stable cases, the origin \( (y = x = 0) \) is the stable fixed point of the solution.

The discussion given above for a linear damped spring-mass system may seem overly simple. However, it presents some of the concepts which are necessary to understand more complicated situations. Indeed, the stability analysis of a nonlinear system of differential equations is just a generalization of what has been discussed. This generalization depends on the Center Manifold Theorem, which states in essence that the linear stability analysis of an \( N \)-dimensional nonlinear system can be reduced to the study of an equivalent linear one-dimensional problem on the so-called center manifold [16].

To understand how the stability of a system of equations may be analyzed, let us consider the following \( N \)-dimensional non-linear system:

\[ \dot{x} = F(x; \mu) \]  

(52)

where the underbar denotes a matrix vector and \( \mu \) is a parameter of the system.
As before, the stability of the fixed points, $x_o(\mu)$, of this system can be investigated by linearizing the system. Thus, using a Taylor series expansion:

$$
\delta \dot{x} = \left( \frac{\partial F}{\partial x} \right)_{x_o} \delta x + O(\delta x^2) \tag{53}
$$

Note that the fixed points ($x_o$) are defined by,

$$
F(x_o, \mu) = 0 \tag{54}
$$

Thus neglecting higher order terms in the Taylor series expansion, we can rewrite Eq. (53) as:

$$
\dot{x}(t) = J_{x_o}(\mu) [x(t) - x_o(\mu)] \tag{55}
$$

where, the so-called Jacobian ($J_{x_o}$) matrix of the system is given by:

$$
J_{x_o}(\mu) = \frac{\partial F}{\partial x} |_{x_o(\mu)} \tag{56}
$$

![Figure 17a. Phase Plane Trajectories for Spring/Mass System ($\beta^2 - 4k < 0$).](image1)

![Figure 17b. Phase Plane Trajectories for Spring/Mass System ($\beta^2 - 4k > 0$).](image2)

We note that since $x_o(\mu)$ is a constant we can rewrite Eq. (55) as,

$$
\delta \dot{x} = J_{x_o}(\mu) \delta x \tag{57}
$$

where, the perturbations are defined as:
As before, let us assume a modal solution of the form,

$$\delta \mathbf{x} = g e^{\lambda t}$$  \hspace{1cm} (59)

Equations (57) and (59) yield,

$$a e^{\lambda t} [\lambda I - J_{\infty} (\mu)] = 0$$  \hspace{1cm} (60)

The only non-trivial solution is when,

$$\det[\lambda I - J_{\infty} (\mu)] = 0$$  \hspace{1cm} (61)

Obviously the $\lambda$ are the eigenvalues of the Jacobian, $J_{\infty} (\mu)$.

Thus, from Eq. (59) we note that the N-dimensional system, is linearly stable if all the eigenvalues, $\lambda_i$, of $J_{\infty} (\mu)$ have negative real parts, $\text{Re}(\lambda_i) < 0$. In contrast, the system is linearly unstable when any eigenvalue has a positive real part and is said to be marginally stable when the real part of any eigenvalue is zero. For the latter case, the fixed point is often referred to as a singular point.

It is also interesting to note that the Laplace transform of Eq. (57) yields,

$$[s I - J_{\infty} (\mu)] \delta \mathbf{x}(s) = \delta \mathbf{x}(0)$$  \hspace{1cm} (62)

or,

$$\delta \mathbf{x}(s) = [s I - J_{\infty} (\mu)]^{-1} \delta \mathbf{x}(0)$$  \hspace{1cm} (63)

Thus, recalling the definition of the inverse of a matrix and a transfer function, we find that the Characteristic Equation is given by,

$$\det[s I - J_{\infty} (\mu)] = 0$$  \hspace{1cm} (64)

Comparing Eqs. (61) and (64) we see that the roots, $s$, of the Characteristic Equation are just the eigenvalues, $\lambda$, of the Jacobian, $J_{\infty} (\mu)$. Hence, we confirm that the linear stability of the system of differential equations is determined by the sign of the real part of the most limiting root(s). That is, analysis of the stability of the Nth order system gives the same result as if a one-dimensional stability analysis was performed on the center manifold for the most limiting eigenvalue(s).

Let us now extend some of the ideas that have been discussed for a linear oscillator to a non-linear oscillator. Historically, one of the most important nonlinear oscillators is the Van der Pol oscillator. This autonomous oscillator can be written in the form,
Comparing Eqs. (65) and (41b), we find that the most significant difference is that for the Van der Pol oscillator we have a nonlinear damping term, $\beta \equiv \mu(1-x^2)$, which can change sign depending on the value of the state variable, $x(t)$.

The response of the Van der Pol oscillator depends on the magnitude and sign of the parameter $\mu$. As shown in Figure-19a for $\mu < 0$, one has a supercritical Hopf bifurcation. That is for this negative damping case, $\beta < 0$, when $|x(t)| < 1$, the system behaves as an elliptic attractor. In general, all phase plane trajectories converge to a stable limit cycle for the case shown in Figure-18a.

In Figure-18b we see the case in which $\mu > 0$. This case is called a subcritical bifurcation, and is characterized by the response of an elliptic attractor for $|x(t)| < 1$, and an elliptic repellor for $|x(t)| > 1$. The limit cycle shown is clearly unstable and will not persist. The mechanical analogy for this case is a ball oscillating in a bowl. For small amplitude perturbations, the ball will oscillate and come to rest at the center of the bowl. For large enough perturbations, the ball will jump over the side of the bowl and will fall out. If the perturbation is such that the ball is perched on the rim of the bowl (the unstable limit cycle) it will not remain there since any minor disturbance will cause it either to fall out of, or into, the bowl. As noted previously, subcritical bifurcations are potentially very dangerous occurrences since the system is linearly stable and yet finite amplitude perturbations may cause it to become exponentially divergent.

Let us now turn our attention to the analysis of chaotic, or “strange,” attractors. In order to understand chaos, let us again consider an autonomous second order oscillator of the form:

$$\ddot{x} + \mu(1-x^2)\dot{x} + kx = 0$$ (65)
\[ \dot{x} - ax + x = 0 \]  
\[(66)\]

This equation can also be rewritten in system form as:

\[ \dot{x} = -y \]  
\[(67a)\]

\[ \dot{y} = x + ay \]  
\[(67b)\]

If the damping parameter is positive (i.e., \(a > 0\)), we find that Eqs. (66) and (67) will have negative damping and are thus unstable. Indeed, we have an elliptic repellor in the phase plane, \((y,x)\).

In order to limit the amplitude of \(x(t)\), we can modify the system in Eqs. (67) by introducing a new state variable \((z)\). The resultant system is:

\[ \dot{x} = -y - z \]  
\[(68a)\]

\[ \dot{y} = x + ay \]  
\[(68b)\]

\[ \dot{z} = b + z(x - c) \]  
\[(68c)\]

We see that if \(a, b\) and \(c\) are all positive when \(x(t)\) becomes greater than the parameter \(c\), then \(\dot{z} > 0\) and thus \(z(t)\) will increase, causing \(\dot{x}\) to decrease. This can produce a limit cycle or even a strange attractor. Note that the new state variable, \(z\), represents a nonlinear controller.

Equations (68) are a form of the equations which yield the Rossler band attractor [17]. It is well known that the occurrence of strange attractors require at least three state variables, thus the Rossler band attractor is a simple example of a far more general result.

It is instructive to analyze Eqs. (68) using a methodology which will apply to the most general case. The first thing that one must do is to find the fixed points (i.e., the steady-state solution) of the system. Thus, if we set the time derivatives in Eqs. (68) to zero we obtain:

\[ y_0 = -z_0 \]  
\[(69a)\]

\[ x_0 = -ay_0 \]  
\[(69b)\]

\[ z_{0,2} = \left( c \pm \sqrt{c^2 - 4ab} \right)/2a \]  
\[(69c)\]

where subscript-0 denotes the steady-state and subscripts 1,2 denote the positive and negative root, respectively, of Eq. (69c). Obviously, we have two fixed points in the three-dimensional \((x,y,z)\) phase space.

The next step is to linearize Eqs. (68):
\[ \dot{\delta x} = -\delta y - \delta z \]  
\[ \dot{\delta y} = \delta x + a\delta y \]  
\[ \dot{\delta z} = (x_o - c)\delta z + z_{o_{1,2}} \delta x \]  

These equations can be written in matrix form as:

\[ \dot{\delta \zeta} = \mathbf{M}\delta \zeta \]  

where,

\[ \delta \zeta = (\delta x \ \delta y \ \delta z)^T \]  

and,

\[ M = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z_{o_{1,2}} & 0 & (x_o - c) \end{bmatrix} \]  

As before, if we assume a modal solution of the form,

\[ \zeta = \zeta(0)e^{\lambda t} \]  

Equations (71) and (73) yield a non-trivial solution for:

\[ \det \begin{bmatrix} -\lambda & -1 & -1 \\ 1 & (a - \lambda) & 0 \\ z_{o_{1,2}} & 0 & (x_o - c - \lambda) \end{bmatrix} = 0 \]  

or,

\[ a\lambda^3 - a(x_o - c)\lambda^2 - (z_{o_{1,2}} + 1)\lambda + az_{o_{1,2}} + (x_o - c) = 0 \]  

There are three roots (\( \lambda \)) to this equation for \( z_{o_{1}} \), and three for \( z_{o_{2}} \). These roots are of the form:

\[ \lambda_{1,2}^{(i)} = \sigma^{(i)} \pm i\omega^{(i)} \]  
\[ \lambda_3^{(i)} = -\sigma^{(i)} \]
and,

\[ \lambda^{(2)}_{1,2} = -\sigma^{(2)} \pm i\omega^{(2)} \]  \hspace{1cm} (76a)

\[ \lambda^{(2)}_3 = \epsilon^{(2)} \]  \hspace{1cm} (76b)

We see in Figure-19 that the roots in Eqs. (75) yield the elliptic repellor and point attractor shown on the left of the figure, and the roots in Eqs. (76) yield the elliptic attractor and point repellor shown on the right of the figure. As we shall see shortly, the interaction between these two fixed points produce a phase space vortex which imply stretching and folding of the orbits in the phase plane, and a resultant inability to predict the future response of the deterministic system. This property will lead to what is known as sensitivity to initial conditions (SIC). That is, small differences in initial conditions will be exponentially magnified as the process continues.

Let us now consider the numerical evaluation of the nonlinear system in Eqs. (68). If we fix the value of two of the parameters to be \( b = 2.0 \) and \( c = 4.0 \), and then vary the value of parameter-\( a \), we obtain the phase plane response shown in Figure-20. It is significant to note the cascade of even bifurcations (i.e., period doubling) as the parameter-\( a \) is increased from \( a = 0.3 \) to \( a = 0.3909 \). When \( a = 0.398 \) we have a chaotic response in which the stretched and folded orbits never repeat. This strange attractor is called the Rossler band attractor [17]. It is also interesting to note that as the parameter-\( a \) is further increased that we have a reverse cascade of odd bifurcations.

Many strange attractors have been found in physical systems. Probably the most famous is the butterfly-shaped strange attractor of Lorenz [18]. This attractor resulted from a study by Lorenz of weather prediction using a simplified three state variable model of natural convection:

![Figure 19. Fixed Points of Rossler’s Band Attractor.](image)
Figure 20. Rossler’s Band Attractor (b = 2.0, c = 4.0) [17]

\[
\begin{align*}
\dot{x} &= \sigma(y - x) \quad (77a) \\
\dot{y} &= \rho x - y - xz \quad (77b) \\
\dot{z} &= \beta z + xy \quad (77c)
\end{align*}
\]

where, \(\sigma\), \(\rho\) and \(\beta\) are parameters.

The three fixed points of this system of equations are shown in Figure-21, and the strange attractor in Figure-22. It can be seen that the strange attractor is basically comprised of two foci rotating in the opposite direction. Physically this 3-D phase space motion implies that the motion of the Bernard roll cells is changing chaotically in time.

One of the most important challenges for the analyst is to determine when a chaotic response (i.e., strange attractor) exists and when the response is something else, such as, random noise or the superposition of sinusoids. There are various tests for chaos which can, and should, be used.

When one is working with data, then the various fractal dimensions (\(D_C\) and \(D_H\)) should be computed to determine whether there is some underlying structure in the data. In addition, the power spectral density (PSD) function should be computed. The spectrum
will be broad band for chaotic data as well as white noise. In contrast, if the signal is comprised of superimposed sinusoids, it will have spikes at the frequency of each sinusoid.

It is also useful to plot the data in a 3-D phase plane, [i.e., $\chi(t), \chi(t+\tau), \chi(t+2\tau)$], to appraise the attractor. Other, more quantitative techniques involve the use of a Poincare section. This important technique will be described shortly.

When one is numerically evaluating an analytical model, the approach to chaos is often characterized by a cascade of bifurcations. Such bifurcations are relatively easy to detect since they yield period doubling in the phase place orbits. In addition, a chaotic attractor will always exhibit sensitivity to the initial conditions (SIC). That is, two nearby points in phase space diverge as the orbits progress. Indeed, the points are known to diverge exponentially, and this divergence is characterized by a Lypunov exponent, $\Lambda$. This exponent may be defined in terms base-e, or base-2. That is, if we imagine an initial condition filled hypersphere in phase space of diameter $d_0$, then as the points in this sphere move through phase space the hypersphere will distort into a hyperellipsoid of major axis, $d(t)$. This distortion can be quantified by the Lypunov exponent:

$$
\Lambda = \ln \frac{d(t)}{d_0}
$$

or, alternatively,

$$
\hat{d}(t) = d_0 2^{\hat{\Lambda}t}
$$

Figure 21. Fixed Points of Lorenz Attractor [6]

Figure 22. Lorenz Attractor [6]
In either case a positive Lyapunov exponent ($\Lambda$ or $\hat{\Lambda}$) implies that the orbits become far apart after sufficiently long time. Indeed, even computer roundoff error will lead to nonpredictability due to SIC. Thus, to verify a chaotic response, one should always check the divergence of the orbits to make sure Eqs. (78) are satisfied.

Another excellent way of assessing whether there is a strange attractor or not is to analyze how the phase space orbits pass through a Poincaré section. Thus, let us next discuss this valuable technique.

A typical Poincaré section in a phase space of order three is shown in Figure 23. The penetration of the Poincaré plane by the phase space orbits (i.e., the solution flow) creates a pattern in the plane which is the fractal signature of the attractor under investigation.

Figure 23. Poincaré Section [6]

Figure-24a shows the Rossler band attractor in 3-D state space. Figure-24b shows a Poincaré section after the trajectories have been stretched and folded after completing one orbit in phase space. It can be seen that the original line segment is now a U-shaped segment. Next, as can be seen in Figure-24c, after a second orbit a double-U-shaped pattern emerges. Figure-24d indicates the third orbit produces a quadruple U pattern. Finally, after many orbits we achieve the Poincaré map shown in Figure-24e. This map has a fractal pattern which is similar to a Cantor set. The appearance of a fractal pattern in the Poincaré section is a good indication of a strange attractor.

Other interesting information can be obtained from a Poincaré section. Indeed, if we keep track of the solution flow and plot where it pierces the Poincaré plane on successive orbits we can construct a first-return map, in which we plot state variable $\chi_n$, where the penetration of the plane is now, versus $\chi_{n+1}$, where the corresponding penetration will be at the end of the next orbit. Figure-25 gives the first return map for a Rossler band attractor. We note the non-uniqueness implied by the parabolic shape of the first-return map. That is, two different $\chi_n$ can give the same $\chi_{n+1}$. This loss of uniqueness implies nonpredictability.
In order to appreciate the importance of a first-return map, we note in Figure-25 that one can iterate the first-return map by taking an initial point (say $\chi_{n+1}=A$) and note that on the next iteration this point will become $\chi_n=A$, thus $\chi_{n+1} = B$, and so on. It is more convenient, however, to just draw a $45^\circ$ line and move on a horizontal line from each point, to the $45^\circ$ line then vertically to the first return map. This will give the next iterant ($\chi_{n+1}$). It can be seen that, as expected, the Rossler band attractor is unstable and aperiodic. Such motion in a first return map is the signature of a strange attractor.
Figure 24a-e. The Stretching and Folding Process for a Rossler’s Band Attractor [27]

It is also possible to interpret the first-return map in the time domain [9]. Figures 24 show how a record of analog data might appear in a first-return map for a periodic signal of period $T$ which is sampled on interval $T/4$.

In order to more easily appreciate the iterative procedure in a first-return map, and the stability criterion that controls the motion in this plane, let us consider a simpler problem. In particular, let us consider the so-called Logistic Map of population dynamics. The iterative equation for this problem is given by:

$$
\chi_{n+1} = f(\chi_n) = \mu \chi_n (1 - \chi_n)
$$

where the parameter $\mu$ quantifies the birth rate.

The fixed point $(\chi_0)$ of this problem is where the $45^\circ$ line intersects Eq. (79). That is,

$$x_0 = \mu x_0 (1-x_0)$$
thus,

\[ x_0 = 1 - 1/\mu \]  \hspace{1cm} (80)

The condition for stability is known [17] and given by,

\[ |f'(x_0)| < 1.0 \]  \hspace{1cm} (81)

Now, from Eq. (79),

\[ f'(x) = \mu(1 - 2x) \]  \hspace{1cm} (82)

Combining Eqs. (80) and (82),

\[ f'(x_0) = 2 - \mu \]  \hspace{1cm} (83)
We note that for $\mu = 2.5$, the system is stable, and, as can be seen in Figure-27a, the iteration converges to the fixed point ($x_0$). In contrast, for $\mu = 3.1$ the system is unstable, and results in the “limit cycle” response shown in Figure-27b. Interesting, when $\mu = 3.8$ the system becomes chaotic (i.e., aperiodic), as shown in Figure-27c. This implies that if the birth rate ($\mu$) becomes too large the population dynamics can become completely unpredictable. A frightening prospect at best!
While there is much more than can be said about the analysis of chaos, we conclude with the observation that it is a very exciting and rapidly developing field.

Figure 27a.  

Figure 27b.  

Figure 27c

Figure 27. Behavior of iterated maps: (a) logistic map, showing a transient settling to an attracting equilibrium for $\mu = 2.5$, (b) attracting limiting cycle for $\mu = 3.1$, and (c) aperiodic behavior for $\mu = 3.8$.

Let us now turn our attention to the application of these analytical techniques to problems of interest in thermal-hydraulics. In particular, let us next consider the prediction of nonlinear density-wave instability phenomena in single-phase natural circulation loops and in boiling channels.
5. THE ANALYSIS OF CHAOS IN SINGLE-PHASE NATURAL CONVECTION LOOPS

A thermosyphon is an excellent example with which to demonstrate some analytical procedures that can be used in the analysis of chaos. Thus, let us consider the analysis of nonlinear density-wave instabilities which may occur in a single-phase natural circulation loop. Such phenomena have been previously considered by Bau et al [19]. The discussion below is based on this interesting piece of work.

A quite general thermosyphon loop is shown in Figure-28. It can be seen that the bottom half of the loop is uniformly heated and the upper half is uniformly cooled. For the case of nonsymmetric heating ($\phi \neq 0$), the heated section would be rotated.

Let us begin by deriving the conservation equations which describe fluid motion during natural circulation conditions in the loop shown in Figure-28. Adopting Boussinesq’s approximation, Newton’s second law implies:

$$\rho \frac{d\vec{u}}{dt} = -\frac{1}{R} \frac{\partial p}{\partial \theta} - g \rho \left[ 1 - \beta (T_r - T_{ref}) \right] \cos(\theta + \phi) - \frac{4\tau_w}{D_H}$$  \hspace{1cm} (84)

where,

$$\tau_w = \frac{1}{8} \int p \, \vec{u} \cdot |\vec{u}|$$

and,

$$\vec{u} = \frac{1}{2\pi R} \int_{0}^{\pi} u(r) 2\pi r \, dr$$

The corresponding loop momentum equations comes from integrating Eq. (84) around the loop (i.e., from $\theta=0$ to $2\pi$) and noting that $p(0) = p(2\pi)$. The result of this integration is:

$$\rho \frac{d\vec{u}}{dt} = -\frac{g \rho \beta}{2\pi} \int_{0}^{2\pi} (T_r - T_{ref}) \cos(\theta + \phi) d\theta - \frac{4\tau_w}{D_H}$$  \hspace{1cm} (85)
The corresponding energy equation comes from applying the first law of thermodynamics to a one-dimensional differential control volume which spans the loop:

$$\rho_c \frac{\partial T}{\partial t} + \rho u \frac{1}{R} \frac{\partial T}{\partial \theta} - \frac{k}{R^2} \frac{\partial^2 T}{\partial \theta^2} = \begin{cases} \frac{4H}{D_H^2} (T_h - T_w), & \theta \in T_\text{w} (\theta) \text{ specified} \\ \frac{4q^*}{D_H}, & \theta \in q^* (\theta) \text{ specified} \end{cases}$$  \hspace{1cm} (86)

In accordance with the Boussinesq approximation the liquid in the loop is assumed to be incompressible but the density head term of the loop’s momentum equation varies with temperature. As a consequence the continuity equation is automatically satisfied.

It is convenient to nondimensionalize Eqs. (85) and (86). To this end we define the loop’s time scale as,

$$\tau = \rho_c c_f \frac{D_H}{4H}$$  \hspace{1cm} (87a)

the loop’s Prandtl number as,

$$P_L = 8 \text{Pr}/\text{Nu} = 32 \nu \tau/D_H^2$$  \hspace{1cm} (87b)

and the loop’s Rayleigh number as,

$$\dot{\text{Ra}} = g\beta \Delta T_c \tau^2/(2R P_L) = \frac{g\beta \Delta T_c \rho_c c_f D_h^2/4H}{64R}$$  \hspace{1cm} (87c)

where, $\Delta T_c$ is the instantaneous temperature difference in the liquid from one side of the loop to the other (actually, any two points around the loop will suffice). Similarly, the Biot number of the loop is defined as,

$$B = \frac{1}{\text{Nu} (D_H/2R)^2}$$  \hspace{1cm} (87d)

We can now define the following nondimensional quantities:
\[ \bar{u}^* = \bar{u}_i/(R/\tau) = \bar{u}_i/u_o \]

\[ t^* \equiv t/\tau \]

\[ \bar{T} \equiv (T_i - T_{ref})/\Delta T_i \]

\[ \tau_w^* \equiv \tau_w/\rho_c R^2/\tau^2 \]

Thus, Eq. (85) becomes:

\[ \frac{d\bar{u}^*}{dt^*} = \bar{u}_i \frac{1}{\pi} \bar{R}_c P_L \int_0^{2\pi} \bar{T}^* \cos(\theta + \phi) d\theta - \frac{4\tau_w^*}{(D_h/R)} \] (88)

It is interesting to note that for laminar flow:

\[ \tau_w = \frac{1}{8} \left( \frac{64}{(\bar{u},D_h)/\nu} \right) \rho_c \bar{u}_i^2 = \frac{9\nu \rho_c \bar{u}_i}{D_h} \]

hence,

\[ \frac{4\tau_w^*}{(D_h/R)} = P_L \bar{u}^* \] (89)

Let us next derive the nondimensional form of the loop’s energy equation. Recalling that,

\[ u_o \equiv R/\tau = 4HR \rho_c D_h c_p \]

\[ q_0^* = H\Delta T_i \]

Equation (86) can be rewritten as,

\[ \frac{\partial \bar{T}^*}{\partial t^*} + \bar{u} \frac{\partial \bar{T}^*}{\partial \theta} - B \frac{\partial^{\gamma} \bar{T}^*}{\partial \theta^\gamma} = \begin{cases} \frac{(\bar{T}^* - \bar{T}_w^*)}{(\bar{T}_w^*)}, & \text{if } \bar{T}_w(\theta) \text{ specified} \\ q^*(\theta), & \text{if } q^*(\theta) \text{ specified} \end{cases} \] (90)

Equations (88), (89) and (90) comprise the conservation equations needed for the analysis of a single-phase thermosyphon undergoing laminar natural circulation flow.

In order to use the mathematical machinery which has been developed for the analysis of nonlinear deterministic systems, we shall convert the partial differential equation, given in Eq. (90), into an equivalent ordinary differential equation. Because of the nature of
this particular problem it is convenient to use spectral methods. In particular, to
decompose the various temperature fields into Fourier series.

Thus we assume:

\[ T_w^*(\theta, t) = w_o(t) + \sum_{n=1}^{\infty} w_n(t) \sin(n\theta) \]  

(91)

\[ \bar{T}^*(\theta, t) = \sum_{n=0}^{\infty} \left[ S_n(t) \sin(n\theta) + C_n(t) \cos(n\theta) \right] \]  

(92)

where we have used the fact that the normalized wall temperature \( T_w^* \) is an odd function of \( \theta \).

Introducing Eq. (92) into Eq. (88), and, for simplicity, assuming the validity of Eq. (89), we obtain:

\[ \dot{\bar{u}}^* = \frac{\operatorname{Ra}P_l}{\pi} \sum_{n=0}^{\infty} \left[ S_n(t) \sin(n\theta) + C_n(t) \cos(n\theta) \right] \cos(\theta + \phi) \, d\theta - P_l \bar{u}^* \]  

(93)

Recalling the following orthogonality relations,

\[ \int_0^{2\pi} \sin(n\theta) \cos(m\theta) \, d\theta = \begin{cases} 0, & n = m \\ 0, & n \neq m \end{cases} \]

\[ \int_0^{2\pi} \sin(n\theta) \sin(m\theta) \, d\theta = \begin{cases} \pi, & n = m \\ 0, & n \neq m \end{cases} \]

\[ \int_0^{2\pi} \cos(n\theta) \cos(m\theta) \, d\theta = \begin{cases} \pi, & n = m \\ 0, & n \neq m \end{cases} \]

Equation (93) yields,

\[ \dot{\bar{u}} = \frac{1}{\pi} \frac{\operatorname{Ra}P_l}{\pi} \left[ C_i(t) \cos(\phi) - S_i(t) \sin(\phi) \right] - P_l \bar{u}^* \]  

(94)

Next, we introduce Eqs. (91) and (92) into Eq. (90):
\[
\sum_{n=0}^{\infty} \left[ S_n(t) \sin(n\theta) + C_n(t) \cos(n\theta) \right] + \bar{u} \sum_{n=0}^{\infty} \left[ nS_n(t) \cos(n\theta) - nC_n \sin(n\theta) \right] \\
+ B \sum_{n=0}^{\infty} \left[ n^2S_n(t) \sin(n\theta) + n^2C_n(t) \cos(n\theta) \right]
\]

\[
= \begin{cases} 
q''(\theta), \theta \in q'' \text{ specified} \\
\int_0^{2\pi} q''(\theta) \cos(\theta) d\theta, \theta \in q'' \text{ specified} \\
\int_0^{2\pi} q''(\theta) \sin(\theta) d\theta, \theta \in q'' \text{ specified} \\
\end{cases}
\]

We now multiply Eq. (95) through by \(\cos(\theta)\) and integrate to obtain:

\[
\dot{C}_1 + BC_1 + \bar{u} \ast S_x = \begin{cases} 
\int_0^{2\pi} q''(\theta) \cos(\theta) d\theta, \theta \in q'' \text{ specified} \\
-C_1, \theta \in T_w \text{ specified} 
\end{cases}
\]  (96)

Similarly, if we multiply Eq. (95) through by \(\sin(\theta)\) and integrate we obtain:

\[
\dot{S}_1 + BS_1 + \bar{u} \ast C_x = \begin{cases} 
\int_0^{2\pi} q''(\theta) \sin(\theta) d\theta, \theta \in q'' \text{specified} \\
(w_i - S_i), \theta \in T_w \text{ specified} 
\end{cases}
\]  (97)

For simplicity let us now consider the special case where the wall temperature \((T_w)\) is specified for all \(\theta\). For this case, Eqs. (94), (96) and (97) reduce to:

\[
\dot{\bar{u}}^* = \hat{Ra} P_L \left[ C_1 \cos\phi - S_1 \sin\phi \right] - P_L \bar{u}^* 
\]  (98)

\[
\dot{C}_1 + (1 + B)C_1 + \bar{u} \ast S_x = 0
\]  (99)

\[
\dot{S}_1 + (1 + B)S_1 - \bar{u} \ast C_x = w_i
\]  (100)

It is convenient to make a change of variables as follows:

\[
Ra \equiv \hat{Ra}(1 + B^2)
\]  (101a)

\[
P \equiv P_L / (1 + B)
\]  (101b)

\[
C \equiv \hat{Ra} C_1 / (1 + B)
\]  (101c)

\[
S \equiv \hat{Ra} S_1 / (1 + B)
\]  (101d)
Making these transformations, Eqs. (98)-(100) become:

\[ \dot{u} = \left[ C \cos \phi - S \sin \phi \right] P - Pu \]  
\[ \ddot{C} + C + uS = 0 \]  
\[ \dot{S} + S - uC = Ra \omega_1(t) \]

For the special case of a constant wall temperature, we may let:

\[ A = -w_1 \]

thus, Eq. (104) becomes,

\[ \dot{S} + S - uC + RaA = 0 \]

In order to reduce Eqs. (102), (103) and (106) to a more recognizable form we define,

\[ x \equiv u \]
\[ y \equiv C \]
\[ z \equiv S + R \]

where, \( R \equiv ARa. \)

Hence Eqs. (102), (103) and (106) can be rewritten as,

\[ \dot{x} = \left( y \cos(\phi) - [z - R] \sin(\phi) \right) P - Px \]  
\[ \dot{y} = -x[z - R] - y \]  
\[ \dot{z} = -z + xy \]

We note that for the special case of symmetric heating (i.e., \( \phi = 0 \)) that Eq. (107a) reduces to:

\[ \dot{x} = P[y - x] \]
The system of nonlinear ordinary differential equations given by Eqs. (107b), (107c) and (107d) is a special case (i.e., $\sigma = P$, $\rho = R$, $\beta = 1.0$) of the well-known Lorenz System [18] given in Eqs. (77).

Let us now investigate the system of equations given by Eqs. (107a), (107b) and (107c). First of all, the fixed points of the system are given by setting the time derivations to zero (i.e., $\dot{x} = \dot{y} = \dot{z} = 0$). This yields,

\[ \begin{align*}
x_0 &= y_0 \cos(\phi) - [z_0 - R] \sin(\phi) \\
y_0 &= Rx_0 - x_0 z_0 \\
z_0 &= x_0 y_0
\end{align*} \]

(108a) (108b) (108c)

Combining Eqs. (108) we obtain,

\[ x_0^3 + \left[ 1 - R \cos(\phi) \right] x_0 - R \sin(\phi) = 0 \]

(109)

It is important to note that for the nonsymmetric heating case there are no non-motion (i.e., $x_0 \equiv u_0 = 0$) solutions. Indeed, the formula for a cubic equation implies that there is only one (motion) solution for $x_0 \equiv u_0$ when $R < R_1$, where $R_1$ comes from the solution of:

\[ \left[ 1 - R \cos(\phi) \right]^3 + \frac{27}{4} R_1^2 \sin^2(\phi) = 0 \]

(110)

In contrast, when $R > R_1$ there are three (motion) solutions for $x_0$, however, only two of these solutions are stable. A bifurcation diagram for the case in which $\phi > 0$ is shown in Figure-29. This type bifurcation represents an unfolded pitchfork bifurcation, in which the upper branch is for counterclockwise (CCW) loop flow and the lower branch is for clockwise (CW) loop flow. Clockwise motion implies downflow against the effect of buoyancy (due to nonsymmetric heating). Such steady flows to not occur unless special provisions are made to establish them. Moreover, above a certain generalized Rayleigh number ($R_2$) the loop flow becomes unstable and may reverse direction in going from the lower (CW) branch to the upper (CCW) branch. Such a transition is known as a “catastrophe.”

It should also be noted in Figure-29 that the upper (CCW) branch will become chaotic as the generalized Rayleigh number is increased to $R_3$. Moreover, due to the hysteresis inherent in a subcritical Hopf bifurcation, a chaotic response may occur in the region $R_2 < R < R_3$.

Let us next consider the case of symmetric heating ($\phi = 0$). For this important case, Eqs. (108) imply that the fixed points are given by,

\[ x_0 = y_0 \]

(111a)
Thus we find that for symmetric heating we may have either non-motion solutions (i.e., $x_0 = y_0 = z_0 = 0$) or motion solutions given by,

\begin{align}
    x_0 &= y_0 = \pm \sqrt{R - 1} \\
    z_0 &= R - 1 = x_0^2
\end{align}

Equation (112a) yields the pitchfork bifurcation diagram shown in Figure-30. As noted previously, the upper branch implies counterclockwise (CCW) loop flow, while the lower branch is for clockwise (CW) loop flow. Naturally for the case of symmetric heating, either flow is possible. Also, it should be noted that for both flow directions a subcritical bifurcation and a chaotic response may occur for sufficiently high generalized Rayleigh number (i.e., as $R_b < R \leq R_c = \frac{P(P + 4)}{(P - 2)}$). Moreover for $R > R_c$ a chaotic response with so-called “windows of periodicity” is found. A typical time trace for $R > R_c$ is shown in Figure-31a. It can be seen that deterministic chaos is quite complicated, involving occasional reversals in the direction of the flow. Also, Figure-31b shows that there are no distinctive frequencies in the corresponding Power Spectral Density function. This implies that the time signal shown in Figure-31a is aperiodic.

Figure 29. A Bifurcation Diagram for an Asymmetrically Heated Loop (Dashed Lines Represent Unstable Conditions).
In order to gain more insight into the properties of Eqs. (107d), (107b) and (107c), let us perform a linear stability analysis. If we linearize these equations we obtain:

\[ \dot{x} = P \delta y - \delta x \quad (113a) \]

\[ \dot{y} = -[z_o - R] \delta x - x_o \delta z - \delta y \quad (113b) \]

\[ \dot{z} = -\delta z + x_o \delta y + y_o \delta x \quad (113c) \]

Equations (113) can be written in matrix form as,

\[ \dot{\psi} = J \delta \psi \quad (114) \]

where, the state variable vector is given by,

\[ \delta \psi = [\delta x \delta y \delta z]^T \quad (115) \]

and the so-called Jacobian matrix is,

\[ J = \begin{bmatrix} -P & P & 0 \\ (R - z_o) & -1 & -x_o \\ y_o & x_o & -1 \end{bmatrix} \quad (116) \]
Let us now assume a modal solution of the form,

\[ \delta \psi = \psi_0 e^{\lambda t} \]  

(117)

Combining Eqs. (117) and (114) we find that the only nontrivial solution is when,

\[
\begin{vmatrix}
- (P + \lambda) & P & 0 \\
(R - z_o) & -(1 + \lambda) & -x_o \\
y_o & x_o & -(1 + \lambda)
\end{vmatrix} = 0
\]  

(118)

Figure 31a. The Velocity in the Thermosyphon Loop as a Function of Time (for \( R \geq R_c \)) [19]

Figure 31b. The Power Spectral Density (PSD) Function of the Time Series Shown in Fig. 31a [19]
For the nonmotion solutions (i.e., \( x_0 = y_0 = z_0 = 0 \)), Eq. (118) implies,

\[
\lambda_{1,2} = \frac{-(1+P) \pm \sqrt{(P-1)^2 + 4PR}}{2}
\]

(119a)

\[
\lambda_3 = -1
\]

(119b)

For convenience, we refer to this set of roots as those of fixed point \( C_0 \). Comparing Eqs. (117) and (119) we see that the system is stable (i.e., \( \text{Re}(\lambda_i) < 0 \)) for \( R < 1.0 \). For \( R = 1.0 \) we have neutral stability (i.e., \( \text{Re}(\lambda_i) = 0.0 \)), while for \( R > 1.0 \) the system is unstable (i.e., \( \text{Re}(\lambda_i) > 0.0 \)).

Using the steady loop motion solutions given by Eqs. (112), Eq. (118) implies:

\[
\lambda^3 + (2 + P)\lambda^2 + [(1 + 2P)^2 - P(R - z_0)]\lambda + P[1 + x_0^2 - (R - z_0) + x_0 y_0] = 0
\]

(120)

This cubic equation gives two sets of three eigenvalues depending on which value (i.e., \( \pm \)) is chosen for the \( x_0 = y_0 \) in Eq. (112a). Because these roots are algebraically complicated we shall not write out the set of roots for the fixed points \( C^- \) and \( C^+ \).

Figure-32 shows the fixed points, \( C_0, C^- \) and \( C^+ \) for the Lorenz attractor for \( R > 1.0 \). It can be seen that \( C_0 \) behaves as a point attractor (stable manifold) and a point repellor (unstable manifold). In contrast, \( C^- \) and \( C^+ \) behave as elliptic repellors.

Figure 32. The fixed points of the Lorenz attractor for \( R > 1 \) in phase space. Also shown schematically are the eigenvectors associated with the linearized stability problem and the associated stable and unstable manifolds [19].
(unstable manifold) and point attractors (stable manifold). A number of phase space trajectories are possible. One which starts at the unstable manifold of a given fixed point and returns to the stable manifold of the same fixed point is called a homoclinic orbit. In contrast, orbits which connect different fixed points are called heteroclinic orbits.

Figures-33 show the time response of the Lorenz system for \( R_a < R < R_c \) (see Figure-30). It can be seen that since the system can no longer be at rest (i.e., with higher density fluid over lower density fluid) an oscillatory response from the unstable manifold of \( C_0 \) occurs. This solution may converge to steady flow in the CW direction (\( C^+ \) stable manifold) or in the CCW direction (\( C^- \) unstable manifold). In contrast, at \( R = R_c \) a homoclinic explosion occurs. The resultant strange attractor is shown in Figures-34, where the famous butterfly-shaped Lorenz (strange) attractor can be noted.

It should be noted that hysteresis occurs when going into and out of chaos. For example, when the generalized Rayleigh number (\( R \)) is increased we have the onset of chaos when \( R = R_c \) (see Figure-30). In contrast, when the generalized Rayleigh number is reduced, chaos will persist until we have reduced it to \( R_b \) [5].

Figure-35 shows a Poincare section of the Lorenz attractor. It can be seen to have a very distinctive fractal signature (and as an aside, has a correlation dimension of \( D_c = 2.06 \)). If we zoom in on one of the “line segments” shown in Figure 35, we would find that it consists of a vast number of closely packed sheets, each of which also consist of a large number of sheets. Moreover, the well-known sensitivity to initial conditions (SIC) of the Lorenz attractor (actually for any strange attractor) is clearly shown in Figure-36.

Bau et al [19] have also shown that for a forced time-periodic wall temperature case rather than the time-independent wall temperature case just discussed, that chaos does not suddenly occur after a homoclinic explosion but rather at the end of a cascade of Hopf bifurcations (i.e., period doubling bifurcation). As discussed in this chapter, this property is also characteristic of the onset of chaos in autonomous boiling natural circulation loops. A typical bifurcation diagram for a single-phase thermosyphon which is being forced by a periodic wall temperature (with period \( T \)) is given in Figure-37. It can be seen that there are bands of deterministic chaos for several different ranges of the generalized Rayleigh number (\( R \)). Interestingly, the onset of chaos for the forced case is at a slightly larger generalized Rayleigh number than for the unforced case. That is, for heating from below, forcing the wall temperature has a stabilizing effect.

![Figure 33a](image-url)  

**Figure 33a.** Time series exhibiting the approach to the steady-state solution (\( C^+ \)) for \( R = 7 \) and initial conditions on the RHS unstable manifold of \( C_0 \) [19].
Figure 33b. Time series exhibiting the approach to the steady-state solution (C---) for R = 8 with initial conditions similar to those in Fig. 33a. Note that the change in the Rayleigh number (R) causes trajectories with similar initial data to end up at different fixed points.

Figure 34a. Lorenz attractor in phase space for R = 20 and P = 4. Sufficient time has been allowed for the initial transient to die out [19].

Figure 34b. Projection of the Lorenz Attractor in x-y Phase Plane [19]
Figure 34c. Projection of the Lorenz Attractor in the x-z Phase Plane [19]

Figure 35. Poincare Section through the plane z = R-1 [19]

Figure 36. Sensitivity to initial conditions (SIC). Two solutions with slightly different initial conditions (black and gray lines) exhibit vastly different behavior after sufficient time [19]
Figure 37. The upper half of a bifurcation diagram for a single-phase thermosyphon with a modulated wall temperature. The x values were stroboscopically sampled every period T [19]

6. PRACTICAL APPLICATIONS OF CHAOS THEORY – THE ANALYSIS OF NONLINEAR DENSITY-WAVE INSTABILITIES IN BOILING CHANNELS

The phenomenon of density-wave instabilities in boiling channels is well-known [20]. These oscillations may be found for certain operating conditions of boiling systems which become unstable due to lags in the phasing of pressure-drop feedback mechanisms. The most common manifestation of density-wave instabilities are self-excited oscillations of the flow variables.

The analytical tool which is often used to study the problem of density-wave instabilities is linear frequency-domain stability analysis. Presently rather accurate and reliable models are available for the linear stability analysis of complicated systems such as boiling water nuclear reactors (BWRs).

The study of the non-linear behavior of density-wave instabilities has attracted considerable interest recently. In particular, Hopf bifurcation techniques have been used to study the amplitude and frequency of the oscillations [21,22]. A numerical analysis of the nonlinear dynamics of a steam generator has been performed by LeCoq [23]. Similarly, a numerical analysis was also performed by Rizwan-Uddin & Dorning [22], where a chaotic attractor was indicated for periodically forced flows.

Let us now consider a non-linear analysis of autonomous density-wave instabilities using a lumped parameter model. The model is based on a Galerkin nodal approximation of the conservation equations for a boiling channel.
We start the boiling channel model description by considering the following assumptions made concerning the flow:

- the flow is homogeneous (i.e., no phasic slip)
- the system pressure is constant
- the heat flux is uniform
- both phases are incompressible
- the two phases are in thermodynamic equilibrium
- viscous dissipation, kinetic energy, potential energy and flow work are neglected in the energy equation
- the channel inlet temperature is constant

For these assumptions, the one-dimensional conservation equations can be written as [24]:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z} (\rho u) &= 0 \quad (121a) \\
\frac{\partial}{\partial t} (p h) + \frac{\partial}{\partial z} (p hu) &= \frac{q^* P_i}{A_{t-s}} \quad (121b) \\
\frac{\partial}{\partial t} (\rho u) + \frac{\partial}{\partial z} (\rho u^2) &= \left[ \frac{f}{D_i} + \sum_{i=1}^{N} K_i \delta(z - z_i) \right] \frac{\rho u^2}{2} - pg - \frac{\partial p}{\partial z} \quad (121c)
\end{align*}
\]

and the corresponding equations of state are:

\[
\rho = \rho_f \quad , \text{for } h \leq h_f \quad (122a)
\]

\[
\rho = \left[ v_f + \frac{v_{fg}}{h_{fg}} (h - h_f) \right]^{-1} \quad , \text{for } h > h_f \quad (122b)
\]

The single-phase region of the heated channel extends from the channel inlet to the boiling boundary (i.e., the location where bulk boiling begins). As can be seen in Figure-38, this region has been subdivided into \(N_s\) nodes, having variable length. The partition of the single-phase volume was found necessary to properly describe the propagation of enthalpy waves. These waves are of importance in determining the dynamics of the boiling boundary.

The enthalpy increase from the inlet, \(h_i\), to saturation, \(h_f\), is divided into \(N_s\) equal intervals, \((h_f - h_i)/N_s\). Therefore, the boundary, \(L_n\), between subcooled node-\(n\) and node-\((n+1)\) is defined as the point where the fluid enthalpy is:

\[
h_n = h_i + \frac{n}{N_s} (h_f - h_i) \quad (123)
\]
It should be noted that this enthalpy \( h_n \) is a constant, while its spatial location is a function of time.

\[
\frac{dL_n}{dt} = 2u_i - 2N_s \frac{q^*p_H}{A_{x-s} \rho_f(h_f - h_i)} (L_n - L_{n-1}) - \frac{dL_{n-1}}{dt}
\]

(125)

Integrating the energy equation, Eq. (121b), between \( L_{n-1} \) and \( L_n \), we have, using Liebnitz’s rule:

\[
\int_{L_{n-1}}^{L_n} \left( \frac{d}{dt} \left( \rho \cdot L \right) \right) \, dt = \left( \frac{h_n - h_{n-1}}{L_n - L_{n-1}} \right) (z - L_{n-1}) + h_{n-1}
\]

(124)

The differential equations governing the evolution of the node boundaries, \( L_n \), can be derived using a Galerkin technique by assuming a linear shape function (i.e., enthalpy profile) inside each node. That is,

\[
\frac{dM_{ch}}{dt} = (\rho_i u_i - \rho_{s} u_e) A_{x-s}
\]

(126)

The two-phase mixture’s exit velocity, \( u_e \), can be calculated by first combining Eqs. (121a), (121b) and (122b), which yields:

\[
\frac{\partial u}{\partial z} = \frac{q^*p_H v_{fg}}{A_{x-s} h_{fg}} \equiv \Omega
\]

(127)

Then, integrating Eq. (127) between the boiling boundary \( (\lambda = L_{N_{fg}}) \) and some location, \( z \), in the two-phase part of the heater gives:
\[ u = u_i + \Omega (z - \lambda) \quad (128) \]

In particular, at the exit of the heated channel:

\[ u_e = u_i + \Omega (L - z) \quad (129) \]

The exit density, \( \rho_e \), can be expressed in terms of the total heated channel’s mass, \( M_{ch} \), by also assuming a linear enthalpy profile inside the two-phase region. Combining Eqs. (121a) and (122b), and integrating between the boiling boundary and the channel exit, yields:

\[ M_{2e} = A_{x,\text{t}} (L - \lambda) \rho_t \left[ \frac{\ell(n(\rho_t/\rho_e))}{\rho_t/\rho_e - 1} \right] \quad (130) \]

and the total mass of the heated channel, \( M_{ch} \), is given by:

\[ M_{ch} = \rho_t A_{x,\text{t}} \lambda + M_{2e} \quad (131) \]

An adiabatic riser was also included in the model. It was found that for low flow conditions the presence of the riser can effect the dynamic characteristics of the system.

The riser was divided into \( N_R \) fixed axial nodes of equal length, as can be seen in Fig. 38. Integrating the continuity equation over node-r, gives:

\[ \frac{dM_r}{dt} = A_{x,\text{r}} u_e (\rho_{r-1} - \rho_r) \quad (132) \]

The riser’s node-r mass, \( M_r \), can be expressed in terms of \( \rho_{r-1} \) and \( \rho_r \) using a procedure analogous to that used in the derivation of Eq. (130). This yields:

\[ M_r = \frac{A_{x,\text{r}} L_R}{N_R} \left[ \frac{\ell(n(\rho_{r-1}/\rho_r))}{1 - \frac{1}{\rho_t} - \frac{1}{\rho_{r-1}}} \right] \quad (133) \]

At this point we have \( N_s + N_R + 1 \) equations (i.e., \( N_s = N_{1\phi} + N_{2\phi} \), Eqs. (125), \( N_R \) Eqs. (132) and one Eq. (126)), and \( N_s + N_R + 2 \) unknowns (i.e., \( L_n, M_{ch}, M_r \) and \( \Delta p \)). The model is closed by imposing the external pressure drop (\( \Delta p_{ext} \)) boundary condition on the boiling channel and riser. The momentum equation, Eq. (121c), can be integrated using the assumption of linear enthalpy profiles inside the various nodes. The result is:

\[ \Delta p_{ext} = \Delta p_t + \Delta p_s + \Delta p_f + \Delta p_R + \Delta p_a \quad (134) \]

where:
\[ \Delta p_1 = \int_0^L \frac{\partial}{\partial t} (\rho u) \, dz = \frac{d}{dt} \int_0^L \rho u \, dz \]

thus,

\[ \Delta p_1 = \frac{d}{dt} \left[ \frac{M_{ch}}{A_{k-s}} u + \frac{(u_c - u_i)(L_{\rho_f} - M_{ch}/A_{k-s})}{(\rho_f/\rho_e - 1)} \right] \quad (135) \]

Similarly,

\[ \Delta p_g = \int_0^L \rho g \, dz = \frac{g M_{ch}}{A_{k-s}} - \frac{g \beta \rho_f}{c_p A_{k-s}} \left( \sum_{n=1}^{N_s} E_n - h_r L_{N_s} A_{k-s} \right) \quad (136) \]

where,

\[ E_n = A_{k-s} \int_{L_{n-1}}^{L_n} h \, dz = A_{k-s} (L_n - L_{n-1}) (h_n + h_{n+1}) / 2 \quad (137) \]

The irreversible hydraulic losses are given by:

\[ \Delta p_f = \int_0^L \left[ \frac{f}{D_l} + K_a \delta(z - z_a) \right] \frac{pu^2}{2} \, dz \quad (138) \]

thus, considering only inlet and exit losses,

\[ \Delta p_f = A \left\{ \frac{M_{ch}}{L A_{k-s}} u_i^2 + 2 \frac{(\rho_f - \frac{M_{ch}}{L A_{k-s}})(u_i u_c - u_i^2)}{(\rho_f/\rho_e - 1)} + \frac{(u_c - u_i)^2 \rho_f}{(\rho_f/\rho_e - 1)^2} \left( \frac{L - L_{N_s}}{2L} + \frac{M_{g_e}}{\rho_f L A_{k-s}} \right) \right\} + K_a \frac{u_i^2}{2} + K_e \frac{\rho_e u_e^2}{2} \quad (139) \]

Next, the spatial acceleration term is given by:

\[ \Delta p_a = \int_0^L \frac{\partial^2 p u^2}{\partial z^2} \, dz = \rho_e u_e^2 - \rho_f u_i^2 \quad (140) \]

Finally, the riser pressure drop is given by:

\[ \Delta p_R = \int_L^R \frac{\partial p}{\partial z} \, dz = \frac{d}{dt} \left( u_e M_R / A_R \right) + g M_R / A_R + A_R u_e^2 \frac{M_R}{L_R A_R} + (\rho N_R - \rho_e) u_e^2 \quad (141) \]
We now have derived the nodal model. Let us next consider the results of its evaluation.

The system of differential equations was numerically integrated by means of the Runge Kutta subroutine.

For certain operating conditions of boiling channels, it was found that the system evolves to limit cycles near the linear stability boundary. Indeed, this nodal model allows the simulation of self-sustained oscillations in excellent agreement with a more detailed distributed parameter HEM model [14].

A particularly interesting behavior was found for low Froude (Fr) numbers. Physically, this means operating the boiling channel at low inlet flows. A number of runs were made for the parameters given in Table-I. In these runs only the phase change, or Zuber, number \( N_{pch} \equiv N_{zu} \), was varied.

Figure-39 shows a projection of the limit cycle in the \( \delta u_i^* - \delta \lambda^* \) plane, where \( u_i^* = \frac{u_i}{(L/u)} \), is the normalized inlet velocity. For this condition the fluid velocity tends to drop as the boiling boundary approaches the end of the heated length, due to an increase of the density head in the channel and thus a decrease in the net driving head.

As can be seen in Figures-39, by reducing the channel power a sequence of period doubling bifurcations occurs. On further reduction of the channel power a cascade of bifurcations takes place which leads to a chaotic response. This interesting behavior has been encountered for a large variety of non-linear differential equations [6]. The most common manifestations are so-called strange attractors, which are asymptotic orbits of the system (i.e., the solution flow) describing the trajectories in hyper-phase-space. One important property of strange attractors is the inability to predict future events due to the exponential magnification of any uncertainties. This feature, often known as sensitivity to initial conditions (SIC), may be a source of concern if an accurate knowledge of the system evolution is required, as in nuclear reactor safety problems, for example.

A projection of the strange attractor which was found is shown in Figures-40. This attractor has a correlation dimension [6] of \( D_c = 1.8 \) and an imbedding dimension of six (6). This figure also shows the corresponding temporal evolution of the inlet velocity, the first return map and PDF spectrum. It can be seen that these aperiodic nonlinear oscillations are extremely irregular. Significantly, rather similar chaotic oscillations have been reported for experiments in natural circulating boiling loops having a riser [25,26].
Figure 39. A cascade of bifurcations: (a) limit cycle; (b) period two; (c) period four; (d) T-2 torus
Figure 40. Strange attractor (N_{zu} = 105.5873).

| Table I |
| Parameters Used in the Analysis of Chaos |

| \( N_{SUB} = 100 \) | \( F_r = 0.0016 \) | \( B = 0.0002 \) | \( L_r^* = 3 \) |
| \( K_{IN} = 38 \) | \( K_{EXIT} = 0 \) | \( \Lambda = 0 \) | \( A_r^* = 4 \) |
| \( \Lambda_R = 0 \) | \( K_{RISER} = 0 \) | \( K_{\phi} = 30 \) |
| \( z^*_D = 0 \) | \( z^*_R = 30 \) | \( A_R^* = 4 \) |
| \( N_{1\phi} = 4 \) | \( N_{2\phi} = 1 \) | \( N_R = 3 \) |

7. CLOSURE

It is hopefully clear to the reader that single-phase and boiling natural circulation systems may exhibit a chaotic response. Moreover, the occurrence of such nonlinear instabilities may be very detrimental to the operation of power production or process equipment (e.g., CHF may occur).

Hopefully this brief introduction of the theory of fractals and chaos will be sufficient to allow some of the readers to begin to do work in this exciting field of scientific analysis.
Nomenclature

D  Channel diameter
D_c  Correlation dimension
E  Energy
f  Friction coefficient
g  Gravity
h  Specific enthalpy
h_{fg}  Latent heat of vaporization
h_f  Liquid specific heat
K  Loss coefficient
L  Channel length
M  Mass
M_2  Two-phase mass in the heated channel
N_s  Number of nodes in the subcooled region
N_R  Number of nodes in the riser
P_{HL}  Heated perimeter
p  Pressure
Δp  Pressure drop
q''  Heat flux
q  Total power
t  Time
u  Velocity
v_f  Specific volume of the liquid
v_{fg}  Liquid to vapor specific volume difference
w  Mass flow rate
z  Space variable

Greek
β  Liquid thermal expansion coefficient, $-\frac{1}{\rho} \frac{\partial \rho}{\partial T}$
δχ  Perturbation, χ(t) - χ_o
ρ  Density
ρ_f  Liquid density
Ω  Characteristic frequency
λ  Boiling boundary
Δp  Channel pressure drop

Subscript
a  Acceleration head term
ch  Channel
D  Downcomer
e  Channel exit
ext  External
f  Friction head term
i  Channel inlet
I  Inertial head term
g  Gravity head term
n  n^{th} subcooled node
r  r^{th} riser node
ref  Reference value
R  Riser
2φ  Two-phase
o  Steady-state
Dimensionless Groups

\[ N_{pch} = \frac{q_u}{w_0 h_f g_f v_f} \]  
\[ \text{Phase change number} \]

\[ N_{sub} = \frac{(h_f - h_l) v_f}{h_f v_f} \]  
\[ \text{Subcooling number} \]

\[ \text{Fr} = \frac{u_{iq}^2}{g L_H} \]  
\[ \text{Froude number} \]

\[ \nu = \frac{N_{sub} L}{N_{pch} u_{iq}} = \frac{\nu_f}{u_{iq}} \]

\[ \Lambda = \frac{fL}{2D} \]  
\[ \text{Friction number} \]

\[ b = \frac{\beta h_f v_f}{c_p v_f} \]  
\[ \text{Thermal expansion number} \]

\[ t^* = \frac{t}{\nu} \]

\[ M^* = \frac{M}{(\rho f L \lambda_{sc})} \]

\[ Z^* = \frac{z}{L} \]

\[ A^*_r = \frac{A_r}{A_{sc}} \]

REFERENCES