This homework has 135 points plus 15 bonus points available but, as always, homeworks are graded out of 100 points. Full credit will generally be awarded for a solution only if it is both correctly and efficiently presented using the techniques covered in the lecture and readings, and if the reasoning is properly explained. If you used software or simulations in solving a problem, be sure to include your code, simulation results, and/or worksheets documenting your work. If you score more than 100 points, the extra points do count toward your homework total. You are now welcome to use computational tools to automate calculations of solutions provided that:

- You explain the logic of what you are doing
- You provide the worksheet and/or code showing how you programmed the calculations
- You write out the solution in comprehensible form. You will lose considerable credit if you submit an excessively complicated expression for the solution which a software package produced

The fundamental principle is that your solution should show you understand how to solve the problem.
1 Transcendental Perturbation (15 points)

Obtain an approximate solution to the following problem for small positive \( \epsilon \) which consists of a main term, a nontrivial correction, and an estimate of the error:

\[
\sqrt{2} \sin \left(x + \frac{\pi}{4}\right) - 1 - x + \frac{1}{2}x^2 = -\frac{1}{6}\epsilon
\]

2 Equations with Positive Coefficient of First Order Derivative (25 points)

Consider boundary value problems of the form:

\[
\epsilon \frac{d^2y}{dx^2} + a(x) \frac{dy}{dx} + b(x)y = f(x) \quad \text{for} \quad 0 < x < 1,
\]

\[
\frac{dy}{dx}(0) = \gamma,
\]

\[
y(1) = \beta.
\]

For full credit, show how to obtain a general uniform approximation to the solution of this boundary value problem for \( 0 < \epsilon \ll 1 \) when \( a(x) \) and \( b(x) \) are smooth and \( a(x) > 0 \) for \( x \in [0,1] \). For reduced credit (15 points), choose particular non-constant functions \( a(x) \) and \( b(x) \) satisfying these conditions and solve the particular boundary value problem you chose. In either case, your solution should be expressed in terms of a main term, a nontrivial correction, and an estimate of the error in the approximation.

3 Integral Equation (30 points plus 15 bonus points)

Develop a uniform approximation for the solution to integral problems of the form

\[
\epsilon y(x) = -q(x) \int_0^x [y(s) - f(s)]s \, ds \quad \text{for} \quad 0 \leq x \leq 1
\]

for \( 0 < \epsilon \ll 1 \). Assume \( f(x) \) is smooth, \( q(x) > 0 \) for \( x \in [0,1] \) and that \( q(x) \) is continuously differentiable but not necessarily twice continuously differentiable. Express your solution as a main term, a nontrivial correction, and an estimate of the error in the approximation.

For bonus credit, develop your approximation assuming only \( q \) is continuous, but not necessarily differentiable.
4  Small First and Second Derivative (25 points)

Develop a uniform approximation for the solution to the boundary value problem:

\[ \epsilon \frac{d^2 y}{dx^2} + \epsilon (x + 1)^2 \frac{dy}{dx} - y = x - 1 \text{ for } 0 < x < 1, \]
\[ y(0) = 0, \]
\[ y(1) = -1. \]

Express your solution as a main term, a nontrivial correction, and an estimate of the error in the approximation.

5  A Tale of Two Turning Points (40 points)

Develop a uniform approximation for the solution to the boundary value problem:

\[ \epsilon \frac{d^2 y}{dx^2} + \left( x - \frac{1}{4} \right) \left( x - \frac{3}{4} \right) \frac{dy}{dx} + x(y - 1) = 0 \text{ for } 0 < x < 1, \]
\[ y(0) = -2, \]
\[ y(1) = 2. \]

Here it will suffice to obtain a uniform leading order approximation.