Financial Mathematics and Simulation  
MATH 6740–1 – Spring 2011 
Homework 4

Submission Window: Monday, May 9 – Friday, May 13 at 5:00 PM  
Absolute late deadline: Wednesday, May 18 at 5:00 PM

This homework has 325 points plus 15 bonus points available but, as always, 
homeworks are graded out of 100 points. Full credit will generally be awarded for 
a solution only if it is both correctly and efficiently presented using the techniques 
covered in the lecture and readings, and if the reasoning is properly explained. If 
you used software or simulations in solving a problem, be sure to include your code, 
simulation results, and/or worksheets documenting your work. If you score more 
than 100 points, the extra points do count toward your homework total.

1 Thermal Forcing with Nonlinear Confining Potential (15 points)

Consider a microscale particle moving through a fluid subject to three forces: an 
external applied confining force, friction, and random forces due to interactions (such 
as collisions) with the fluid molecules. A standard way of describing the dynamics 
of such a particle (along a given direction) is through the Itô stochastic differential 
system:

\[
\begin{align*}
\frac{dX}{dt} &= V, \\
\frac{dV}{dt} &= \left[ -\frac{dU}{dx}(X) - \gamma V \right] dt + \sigma(X, V) dW(t),
\end{align*}
\]

with initial data

\[
X(t = 0) = X_0, \quad V(t = 0) = V_0.
\]
Here $X(t)$ denotes the position of the particle as a function of time $t$ (with deterministic initial value $X_0$), $V(t)$ denotes the velocity of the particle (with deterministic initial value $V_0$), $U(x)$ is a smooth, confining potential ($\lim \inf_{|x|\to \infty} |U(x)|/|x| = \infty$), $\gamma$ denotes a friction coefficient, and $\sigma(x, v)$ describes the strength of the thermal forcing (but does not have dimensions of force!).

The laws of equilibrium statistical physics imply that, once a system has settled down to a (statistically stationary) thermal equilibrium state, the probability density for the position and momentum should obey the Gibbs-Boltzmann distribution:

$$p_{X,V}(x,v) = \exp\left( -\frac{1}{2}mv^2 + U(x) \right) / k_BT,$$

where $k_B = 1.38 \times 10^{-23} J/K$ is Boltzmann’s constant and $T$ is the absolute temperature. What functions $\sigma(x, v)$ would make the dynamics described by (1) consistent with this law? Provide a precise mathematical argument.

## 2 Can’t Game the Itô Formula (40 points plus 15 bonus points)

a. (5 points) Suppose Itô’s formula (in integrated form) were to be applied naively to express $(W(T))_+ = \max(W(T), 0)$ in terms of a stochastic integral and ordinary integral over the time interval $[0,T]$. For the moment, ignore the fact that the function $(x)_+$ is not smooth at the origin; it is otherwise piecewise smooth so that one can make sense out of Itô’s formula. Unfortunately, the answer you get in this way is wrong. Show in a precise and concrete way how one can see that $(W(t))_+$ can’t be equal to the integrated expression obtained from a naive application of Itô’s formula.

b. (15 points) The reason for this discrepancy is that Itô’s formula does not work properly for functions $f$ of a stochastic process $X(t)$ when $f$ fails to be continuously differentiable, even at isolated points. One can show how to extend Itô’s formula to functions such as $f(x) = (x)_+$ by following the steps in Exercise 4.20 of Shreve. Develop a numerical simulation which shows (in a suitably approximate way) that this corrected Itô’s formula for $(W(T))_+$ is quantitatively correct.

c. (10 points) Proceed now to examine the stop-loss start-gain paradox in Exercise 4.21 of Shreve. Explain how stochastic calculus contradicts the naive claim that the value of one’s portfolio at the expiry time is the non-negative function $(S(T) - K)_+$. 

2
d. **(10 points plus 15 bonus points)** That may seem like we finessed the problem. To really understand why this stop-loss start-gain strategy doesn’t work, look at an appropriate discrete-time approximation and explain clearly why it doesn’t generate an expected gain. The bonus points are for making your explanation quantitatively precise. The local time correction to the Itô formula is simply encoding this discrete-time intuition into the continuous-time limit where it is more obscure.

3  **40 Rock (40 points)**

The K. E. Parcell Family Bank doesn’t believe in the new-fangled Black-Scholes-Merton theory and relies instead on a version of the older Markowitz theory for option pricing. After all, Markowitz also won a Nobel Prize. In considering the price of a financial derivative, such as a European call or European put, the fair price is set as the expected payoff of the option, i.e., the amount of money which the seller of the financial option has to expect to pay the holder of the financial option at the time the financial option can be exercised. Technically, this pricing model assumes both the buyer and seller have linear utility functions, meaning they are entirely neutral to risk. On the other hand, J. Donaghy & Associates uses the Black-Scholes-Merton option pricing formula to determine the fair price of a financial option.

a. **(10 points)** Develop an explicit deterministic formula describing the K. E. Parcell Family Bank’s fair price of a European call option with strike price $K$, expiry time $T$, money market interest rate $r$, and initial underlying asset price $x$, assuming the standard geometric Brownian motion model for the asset price.

b. **(10 points)** Your answer should be different than the Black-Scholes-Merton option-pricing formula that J. Donaghy & Associates would use. Show how they differ by choosing numerically a set of parameters and plotting the prices each bank would set on a European call option as the strike price or asset volatility coefficient varies. (It’s probably best to make the plots separately with these parameters varying one at a time, but if you can make an intelligible surface plot over both parameters, that’s fine.) The reason it’s interesting to vary these parameters is that one can choose at what strike price to write an option, and the asset volatility might not be entirely certain. (But for the purposes of the problem, we assume both banks agree on the geometric Brownian motion model of the underlying asset, together with the value of its parameters.)

c. **(20 points)** The above development means the two banks disagree on the fair price of the same European call option, which means they will willingly enter into transactions with each other, where each bank thinks it is getting
the better end of the deal. (The sign of the price discrepancy will indicate which bank wants to write (sell) the option for the other bank.) You’d think that J. Donaghy & Associates is the one who is really ripping off K. E. Parcell Family Bank, since they’re using a more modern theory. Explain in detail, with mathematical analysis, whether this is really true, and if so, how it is that J. Donaghy & Associates would (insert favorite financial verb) K. E. Parcell Family Bank. If not, explain how both banks are benefiting from the mutually agreed transaction.

4 Profiting from Others’ Ignorance (85 points)

One can employ two perspectives on the financial markets: 1) the markets are very efficient and it is difficult to game them, so one’s strategy should simply be to build portfolios that hedge against risk and provide a reasonable return, or 2) the markets are full of idiots from whom you can swindle lots of money (the Wall Street term for this is too impolite to put in writing). The former standpoint seems to be the official face of large investment banks and financial advisors. The latter is characteristic of the kind of Wall Street people you read about in *The Big Short*, and in fact several of my day trading graduate student colleagues from the 1990s. We’ll consider the second perspective in this problem.

The classical Black-Scholes-Merton pricing formulas for European calls and puts assumes the underlying asset price $S(t)$ is governed by a geometric Brownian motion model:

$$dS = \alpha S \, dt + \sigma S \, dW(t),$$

$$S(t = 0) = S_0,$$

with constant parameters $\alpha$ and $\sigma > 0$, where $dW(t)$ represents completely unpredictable price fluctuations. The goal is to set a fair price based on establishing a perfect hedge by combining the put/call with a financial investment, the combination of which is perfectly riskless, and must therefore have the same return as the safe money market. A transaction of the European call or put at any other price would allow one party to lock in a profit and the other party should expect to lose money out of the deal. This argument is premised on the asset price actually being governed by the geometric Brownian motion model (2).

As noted in Exercise 4.11 in Shreve, if the true asset price is governed by geometric Brownian motion with a different value of the volatility parameter $\sigma$, then a trader who knows this can set up an arbitrage, guaranteeing a riskless profit, by dealing with another party who is pricing their financial derivatives using the wrong value of $\sigma$. You are asked here to consider the consequences of other misspecifications of the
market model. In each case, suppose a bank is offering to sell you a European call option (with strike price $K$ and exercise time $T$) based on the Black-Scholes-Merton option pricing formula based on the geometric Brownian motion model (2), but you think you know better. Your answers to the following should be expressed as explicit deterministic expressions.

a. **(10 points)** Suppose you think that the asset price is being governed by a geometric Brownian motion model with a different value of $\alpha$. If you’re right, how much money would you expect to earn at time $T$ by buying this call option from the bank? (If you get a negative answer, you can expect to make money by trying to sell (“short”) the bank the call option at the price they are citing, or even somewhat less so they think they are ripping you off and you think you are ripping them off. (Again, forgoing the typical Wall Street verb.) Can you set up an arbitrage that would guarantee yourself a profit from a long or short position on the call option transacted at the bank’s price?

b. **(20 points)** Suppose you think the asset price is in fact governed by a Langevin equation

$$dS = \alpha S \, dt + \beta \, dW(t),$$

(3)

$$S(t = 0) = S_0,$$

(4)

where $\alpha$ is the same parameter as the bank’s asset price model, and $\beta = \sigma S_0$. That is, you don’t think the volatility actually changes with the asset price. (This model has the problem that the asset price can become negative, but for positive $\alpha$ and $\beta$ not too large, this will only rarely happen.) If you’re right, how much money would you expect to earn at time $T$ by buying this call option from the bank? Can you set up an arbitrage that would guarantee yourself a profit from a long or short position on the call option transacted at the bank’s price?

c. **(25 points)** Suppose the bank’s model for the underlying asset price is correct, but you have insider information. We’ll characterize your insider information as knowing, at time 0, what $W(t_i)$ will be for some $0 < t_i < T$. That is, you know something about how the asset price will deviate in the near future, in a general sense, from the growth rate expected by the general market. As explained in Shreve Sec. 4.7, the Wiener process characterizing the market noise can be decomposed over the time interval over which you have useful insider information as:

$$W(t) = \frac{t}{T_i} W(T_i) + Y(t)$$

(5)

where $Y(t)$ is a Brownian bridge from 0 to 0 on the time interval $[0, T_i]$. For you, the first term is a random variable which is measurable with respect to the
σ-algebra of your information at time 0, whereas the second term is independent of the σ-algebra of your information at time 0, and is adapted to the filtration characterizing your information. (For the honest traders, all terms in (5) are independent of the σ-algebra of their information at time 0, and \( W(t) \) (but not \( Y(t) \)) is adapted the filtration characterizing their information.) Describe a strategy that makes use of your insider information to give you an expected profit from transactions involving the European call option and the underlying asset at the fair market price. (Recall that the option-pricing formula was determined so that an honest trader could not expect to make such a profit, given the standard modeling assumptions.) Can you set up an arbitrage that would guarantee you a profit?

To answer this question, you might also want to work with an equivalent formulation of the Brownian bridge process, discussed in Shreve Sec. 4.7.3, which implies that:

\[
dY(t) = -\frac{Y(t)}{T - t} \, dt + d\tilde{W}(t)
\]

where \( \tilde{W}(t) \) is a Wiener process adapted to your (but not the fair traders’) filtration of information. (It is neither the same as the Wiener process \( W(t) \) appearing in the market model (2), nor independent of it.)

d. **(30 points)** Use Monte Carlo numerical simulations for the market model and your arbitrage strategies to verify the analytical results you derived.

### 5 Accumulation of Reward with Growth (75 points)

Consider a function \( V(s, t) \) describing the return on an investment portfolio accumulated beginning at time \( s \) and ending at time \( t \) which obeys the following dynamics in each realization:

\[
\frac{\partial V(s, t)}{\partial t} = \lambda(X(t))V(s, t) + r(X(t), t),
\]

\[
V(s, t = s) = 0.
\]

Here \( X(t) \) is a solution of an Itô stochastic differential equation

\[
dX(t) = a(X, t) \, dt + b(X, t) \, dW(t),
\]

with deterministic functions \( a \) and \( b \), which is supposed to model some sort of market state. \( r(x, t) \) describes the rate of return of the portfolio at time \( t \) when the market is in state \( x \), and \( \lambda(x) \) denotes a growth rate at which the already accumulated reward appreciates or depreciates in value (it depends only on the market state and not explicitly on time). One can think of \( \lambda(x) \) as depending on the reinvestment strategy of the returns (dividends, etc.). The functions \( r(x, t) \) and \( \lambda(x) \) are deterministic.
a. (20 points) Derive a deterministic partial differential equation or system of partial differential equations which will produce the expected value of the accumulated reward by time \( t \), starting from a time \( s \) at which the state of the market is known to be in a given state \( x \). Techniques similar to those used in class will work here, but the result is a little more complicated than those presented in class. Also, certain approaches will run into serious obstacles, so if you find yourself stuck with a mess, consider trying another one of the approaches we developed in class.

b. (15 points) Solve your equation explicitly for the case in which \( a, b, \) and \( \lambda \) are each constant.

c. (15 points) Compare your exact formula against an average over Monte Carlo simulations of the trajectories.

d. (15 points) Examine a model in which \( a, b, \) and/or \( \lambda \) have nontrivial dependence on \( x \), and solve your equations from part (a) either analytically or numerically.

e. (10 points) Compare your result from part (d) against an average over Monte Carlo simulations of the trajectories.

6 Buy and Hold...or Give Up (70 points)

Hedging various financial derivatives (such as so-called American options) involve strategies in which the trader waits for a time at which the price of an asset hits a certain value. We'll consider a simplified version where we simply have an asset whose price \( S(t) \) evolves according to geometric Brownian motion:

\[
\begin{align*}
\frac{dS}{S} &= \alpha dt + \sigma dW(t), \\
S(t=0) &= X_0,
\end{align*}
\]

with constant parameters \( \alpha \) and \( \sigma > 0 \), and we have a strategy which depends on the time at which the value of stock reaches a certain strike price \( K \). The current value of the stock \( X_0 < K \).

a. (15 points) Suppose you have basically bet all your money on this one asset, so if the stock price falls to a certain level \( m < X_0 \), you go bust, give up on your financial fantasies, and join the Peace Corps or some other worthwhile occupation. Compute, as a function of the parameters in the model, the probability that the stock price hits the strike price \( K \) (in which case your investment will pay off big time) before you become disillusioned with finance. You should do this by setting up and solving a deterministic problem.
b. **(15 points)** Set up and solve a deterministic problem to compute the expected amount of time you must wait until your fate with this investment is determined.

c. **(15 points)** Now suppose you are a hearty soul, or perhaps have no soul, and will never give up on your surefire financial strategy. Even if you die, your ghost will come back to haunt Wall Street until your investment pays off. Compute the probability that the asset price will indeed eventually reach the strike price $K$. You should use the results from the previous parts, together with a careful argument based on the properties of geometric Brownian motion discussed in class. Think about what must be happening if the stock price in fact never reaches the strike price $K$.

d. **(10 points)** For some ranges of parameters, you should have found that the asset price is guaranteed to eventually reach the strike price $K$. For this range of parameters, set up and solve a deterministic problem for the expected amount of time you must wait until the asset price reaches the strike price $K$.

e. **(15 points)** Compare your above analytical results with numerical results obtained from an ensemble of Monte Carlo simulations.