Homework 3 due Friday, November 15 at 5 PM.

Last time, in the process of proving the Maximum Modulus Principle, we showed that if a function $f$ is analytic on and inside a circle $C(z_0, \rho)$ and if $|f(z_0)| \geq \sup_{z \in C(z_0, \rho)} |f(z)|$ then $|f(z)| = |f(z_0)|$ for all $z \in C(z_0, \rho)$.

By simply applying this result for all radii $\rho$ for which $f(z)$ remains analytic on and inside $C(z_0, \rho)$ we obtain the following result:

Define $B(z_0, R) = \{ z \in \mathbb{C} : |z - z_0| < R \}$. Then if $f$ is analytic on $B(z_0, R) = \{ z \in C : |z - z_0| \leq R \}$ and $|f(z_0)| \geq |f(z)|$ for all $z \in B(z_0, R)$.

What about having a maximum of $|f(z)|$ over an arbitrary connected domain $D$?

With this argument, one has to verify that a finite number of discs will suffice to connect any two points in the domain. This can be done through the following observations:

- If the domain $D$ is connected, can draw a path in $D$ from $z_0$ to any other point in the domain $D$.
- This path has finite length.
- The path has a finite distance from the boundary of $D$. Therefore, for any point on the path, one can always take a disk of analyticity equal to half this distance from the boundary.
- Therefore a finite number of disks will suffice, since you have to cover a finite distance with overlapping disks with a lower bound on their radius.
This establishes the maximum modulus principle.

An alternative slicker proof is to use the definition of "connected" to show that a connected set has no proper subset that is both relatively open and relatively closed within the connected set.

Therefore to prove a property of a connected set, show that it is true for a subset that is both relatively open and closed w.r.t. the connected set.

- **Openness** by the above argument (just center disc around any point about which one wants to argue constancy in a neighborhood)
- **Closedness**: the subset of constancy of $|f(z)|$ contains all its limit points. This is true by continuity.

Either way, we have established that under the hypotheses of the Maximum Modulus principle, that $|f(z)| = |f(z_0)|$ for all $z \in D$. But we want to show that in fact $f(z) = f(z_0)$ for all $z \in D$. This follows from Cauchy-Riemann equations:

$$f(z) = u(x, y) + i v(x, y) \quad z = x + iy$$

$$|f| \text{ constant} \Rightarrow |f|^2 \text{ constant}$$

$$\left( \Rightarrow \right) u^2 + v^2 \text{ constant}$$

$$0 \leq \frac{2}{\partial_x^2} \left(\frac{u^2 + v^2}{x^2} \right) = u u_x + v v_x$$

$$0 \leq \frac{2}{\partial_y^2} \left(\frac{u^2 + v^2}{y^2} \right) = u u_y + v v_y$$
Either by viewing it as a linear system for \((u,v)\): The determinant is zero or the solution must be zero. Or view as dot product and cross product of two vectors \((u,v)\) and \((u_x,v_x)\) being zero so at least one vector must be zero.

Similarly one shows that for each \((x,y)\in D\), we have either \((u=0,v=0)\) or \((u_y=0,v_y=0)\). Combined with the analyticity of the function in \(D\) (continuously differentiable), this shows that both partial derivatives of \(u,v\) must be zero over the whole domain, which implies that \(f(z)\) must be constant over \(D\).

**Corollaries of the Maximum Modulus Principle:**

**Minimum Modulus Principle:** If \(f(z)\) is a non-constant analytic function on a open domain \(D\), and does not vanish on \(D\) \((f(z) \neq 0 \text{ for } z \in D)\), then \(|f(z)|\) does not take a global minimum value at any \(z \in D\). Also similar extension to maximum modulus principle for regions \(R\) including the boundary (that is, under the conditions stated above over the interior of a region \(R\), with continuity at the boundary, any minimum of \(|f(z)|\) must take place on the boundary of the region \(R\)).

**Proof:** Apply maximum modulus principle to \(1/f(z)\) which satisfies the properties of the maximum modulus principle because \(f(z)\) does not vanish in \(D\).

**Maximum/Minimum Principle for Harmonic Functions (restricted sense):** The real and imaginary parts of an analytic function take their maximum and minimum values over a closed bounded region \(R\) on the boundary of \(R\). In fact, this maximum/minimum principle can be shown to be true for any harmonic functions on simply connected domains.

**Proof:** Consider \(w(z) = \exp(f(z))\). This is another analytic function on the region. Apply maximum modulus principle to \(w(z)\). This implies that \(|\exp(f(z))|\) takes its max/min on the boundary. But
And exponential is a monotonic function.

To show this is true for the imaginary part, apply MMP to $\exp(if(z))$.

As for harmonic functions more generally, one can show by multivariable calculus techniques and Cauchy-Riemann equations that if one is given a harmonic function on a simply connected domain, then one can construct a single-valued analytic functions whose real part is that harmonic function. However, one can show by using techniques outside of complex analysis (just mean value property of solutions to Laplace’s equation) that harmonic functions on arbitrary connected domains take their max/min on the boundary.

A few other useful results for controlling the behavior of analytic functions.

**Subharmonicity of Modulus of Analytic functions:** If $f(z)$ is analytic inside $B(z_0, R)$ and continuous on the closure of this disc, then:

$$|f(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + R e^{i\theta})| \, d\theta$$

This follows directly from the Gauss mean-value theorem and elementary triangle integral inequality. This means that $|f(z)|$ viewed as a function of $z = x + iy$ is a subharmonic function:

$$\Delta |f(x+iy)| \geq 0.$$  

**Schwarz lemma:** If $f(z)$ is analytic on $B(0, R)$, continuous on its closure, and satisfies $f(0) = 0$, then:

$$|f(z)| \leq \frac{M|z|}{R} \quad \text{for } z \in B(0, R) \text{ where } M = \sup_{z \in \partial B(0, R)} |f(z)|$$

Of course these results can be translated to a disc centered anywhere.
Note the stronger control on the size of the function near the center of the disc.

**Proof:** Take \( D = B(0, R) \setminus \{0\} \), punctured disc.

Consider \( g(z) = \frac{f(z)}{z} \). This function is analytic on the domain \( D \), and is continuous on the closure of \( D \) if we define \( g(0) = f'(0) \). (Use definition of derivative \( f'(0) \) and the fact that \( f \) is analytic at \( z = 0 \)). Apply max modulus principle to \( g \) over the closure of \( D \):

\[
\sup_{z \in \overline{D}} |g(z)| \leq \sup_{z \in \overline{D}} |g(z)| \leq \sup_{\{|z|=R, z \neq 0\}} |g(z)|
\]

\[
\leq \max \left( \frac{M}{R}, |g(0)| \right) = \max \left( \frac{M}{R}, \left| f'(0) \right| \right)
\]

For \( |z| = R \):

\[
|g(z)| = \left| \frac{f(z)}{z} \right| = \frac{|f(z)|}{|z|} \leq \frac{M}{R}
\]

Cauchy's inequality implies: \( |f'(0)| \leq M \frac{1!}{R} = \frac{M}{R} \)

\[
\therefore \sup_{z \in \overline{D}} |g(z)| \leq \frac{M}{R}
\]

\[
\sup_{z \in \overline{D}} \left| \frac{f(z)}{z} \right| \leq \frac{M}{R}
\]

\[
\therefore |f(z)| \leq \frac{M}{R} |z| \quad \text{for} \quad |z| \leq R
\]

After these "soft" analytical results concerning analytical functions, we now turn to applying Cauchy's Integral Formula to do concrete calculations. The key vehicle for making this connection is **power series**.

First let's quickly review the notion of **convergence of series of functions**.
We can start with the standard notions of convergence of series of numbers:

\[ \sum_{n=1}^{\infty} c_n \]

Useful ways to prove convergence of such a series is:

- **Ratio test** (look at ratios of successive terms)
- **Root test** (look at \( \limsup_{n \to \infty} |c_n|^{1/n} \). Converge if \(< 1\), diverge if \(> 1\), don't know if \(= 1\).
- Integral test, or more broadly comparison of the given series with other series/functions known to converge/diverge.

Here's an example where the root test shows convergence but ratio test fails to reach a conclusion:

\[ c_n = \begin{cases} \left( -\frac{1}{2} \right)^n & n \text{ odd} \\ \left( \frac{1}{3} \right)^n & n \text{ even} \end{cases} \]

Now consider series of functions: \( \sum_{n=1}^{\infty} f_n(z) \)

We say that such a series **converges (pointwise)** to a function \( g(z) \) if the series converges to \( g(z) \) for each value of \( z \).

That is, for any \( \epsilon > 0 \) and any \( z \) in the domain of convergence, there exists an \( N(\epsilon, z) \) such that for all \( m \geq N(\epsilon, z) \),

\[ \left| g(z) - \sum_{n=1}^{m} f_n(z) \right| < \epsilon \]

We say that such a series **converges uniformly** to a function \( g(z) \) over a set \( S \) if:

For any \( \epsilon > 0 \), there exists an \( N(\epsilon) \) such that for all \( m \geq N(\epsilon) \)
For any \( g(z) \), there exists an \( \varepsilon \) such that for all \( \varepsilon > 0 \),

\[
|g(z) - \sum_{n=1}^{m} f_n(z)| < \varepsilon \quad \text{for all } z \in S.
\]

We'll see that uniform convergence plays nicely with integrals. Now in trying to show convergence of series of functions, one can use the same tests as for series of numbers, with the root test being the most useful. There is one further tool that is useful in proving uniform convergence.

To this end, we introduce the **Weierstrass M-test**:

Suppose that we can find a sequence of positive real numbers \( \{M_n\}_{n=1}^{\infty} \) such that:

- \(|f_n(z)| \leq M_n \) for \( z \in S \)
- \( \sum_{n=1}^{\infty} M_n < \infty \)

Then the series \( \sum_{n=1}^{\infty} f_n(z) \) converges **absolutely and uniformly** on the set \( S \).

(*Absolute convergence* means that the series still converges when the summands are replaced by their modulus, i.e., \( \sum_{n=1}^{\infty} |f_n(z)| \) converges.)