Cauchy Integral Formula Consequences

Monday, October 28, 2013
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Last time we derived Cauchy's Integral Formula, which we will present in somewhat
generalized form:

Cauchy's Integral Formula (plus)

Suppose that \( f(z) \) is analytic on a domain \( D \) which contains a contour \( C \) and its interior.
Then:

\[
\frac{1}{2\pi i} \oint_C \frac{f(s)\,ds}{s-z} = \begin{cases} 
0 & \text{if } z \text{ outside } C \\
f(z) & \text{if } z \text{ inside } C \\
\frac{1}{2}F(z) & \text{if } z \in C \text{ and on smooth part of } C
\end{cases}
\]

Where the contour integral is taken with positive orientation (counterclockwise).

Moreover

\[
\frac{1}{2\pi i} \text{PV} \oint_C \frac{f(s)\,ds}{s-z} = \begin{cases} 
0 & \text{if } z \text{ outside } C \\
f(z) & \text{if } z \text{ inside } C \\
\frac{1}{2}F(z) & \text{if } z \in C \text{ and on smooth part of } C
\end{cases}
\]

Note that the function \( F(z) \) defined by this integral is analytic inside and
outside the contour but discontinuous on the contour.

Cauchy's Integral Theorem is very powerful tool for a number of reasons, among which:

- It allows the function \( f(z) \) to be represented as a contour integral of relatively nice functions and one can apply many mathematical tools to estimate and analyze integral quantities.
- You can think of Cauchy's Integral formula as giving a representation of analytic functions analogous to Fourier and Laplace transforms, which represent functions as integrals over simple functions (exponentials). Here the "simple function" that \( f(z) \) is being represented as an integral over is \( (\zeta - z)^{-1} \). Why is this a nice function? It's analytic everywhere except \( \zeta = z \) and has the property that any simple contour integral around \( z \) will return the constant value \( 2\pi i \). Moreover the fact that \( f \) itself (rather than some transform of it) appears as the weighting function in the Cauchy
integral will allow some bootstrapping arguments to be made to derive strong properties of the analytic function \( f \).

Let's see how this works. First of all notice that from our discussion of contour integrals with respect to a parameter, the Cauchy integral will give an analytic function provided that \( z \) does not fall on the contour, and the derivative will be given by just differentiating under the integral sign. (We can verify that the integrand is analytic w.r.t. \( z \), and after differentiated with respect to \( z \), is continuous w.r.t. \( \zeta \), as required for the conditions in that proposition justifying differentiating under the integral.)

\[
F(z) = \oint_C g(z, \zeta) \, d\zeta
\]

\[
g(z, \zeta) = \frac{1}{2\pi i} \frac{f(\zeta)}{\zeta - z} \quad \text{analytic w.r.t.} \quad \zeta \quad \text{for} \quad z \notin C
\]

\[
\frac{\partial g(z, \zeta)}{\partial z} = \frac{1}{2\pi i} \frac{f(\zeta)}{(\zeta - z)^2}
\]

Continuous \( f \) of \( \zeta \epsilon C \)

Provided \( z \notin C \).

\[
\frac{dF}{dz} = \oint_C \frac{\partial g(z, \zeta)}{\partial z} \, d\zeta = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta
\]

Notice that

\[
g(z, \zeta) = \frac{1}{2\pi i} \frac{f(\zeta)}{(\zeta - z)^2}
\]

Again satisfies the condition of that proposition (analytic in \( z \epsilon D, z \notin C \))

And is continuous w.r.t \( \zeta \epsilon C \) after \( g \) is differentiated with respect to \( z \).

\[
\frac{\partial^2 g(z, \zeta)}{\partial z^2} = \frac{2}{2\pi i} \frac{f(\zeta)}{(\zeta - z)^3}
\]
What this means is, by an induction argument, we can differentiate the Cauchy integral as often as we please, and the derivatives are simply the derivatives taken under the integral sign.

This implies:

**Proposition:** If \( f(z) \) is analytic on a domain \( D \), then it has in fact infinitely many derivatives. Moreover, those derivatives can be expressed as:

\[
\frac{d^n}{dz^n} f(z) = \frac{\eta!}{2\pi i} \int \frac{f(s)}{(s-z)^{n+1}} ds \quad \text{for } n \in \mathbb{N}
\]

Where \( C \) is any simple closed contour lying within \( D \) and containing \( z \) in its interior.

This relation in fact can be used to evaluate many contour integrals, but we'll save that discussion and subsume it into the residue calculus approach.

**Morera's theorem:** Suppose \( f(z) \) is continuous in a domain \( D \), and has the property that for any closed contour \( C \) lying in \( D \), \( \int_C f(z) \, dz = 0 \). Then \( f \) is analytic on \( D \).

This is a converse to the Cauchy-Goursat theorem.

**Proof** of Morera's theorem: The assumption of the theorem, together with standard multivariable calculus arguments, imply that \( f(z) \) has a single-valued antiderivative \( F(z) \) in \( D \):

\[
F(z) = \int_{z_0}^z f(s) \, ds
\]

And this is analytic. So \( F \) is analytic, and therefore it is, in particular, by the previous proposition, twice differentiable, meaning that \( F'(z) = f(z) \) is complex differentiable, i.e., analytic. \( \square \)
Let’s exploit Cauchy’s integral formula, together with integral inequalities to derive inequalities for derivatives of analytic functions.

**Cauchy’s inequalities:** If $f$ is analytic inside and on a simple circular contour $C(z_0, R)$ of radius $R$ centered at $z_0$, then:

$$\left| \frac{d^n f}{dz^n}(z_0) \right| \leq \frac{n! M}{R^n}$$

where $M = \sup_{z \in C(z_0, R)} |f(z)|$.

The intuition behind this result is that the smoothness of an analytic function can be controlled by the distance to any singularity or disruption of analyticity. The farther $z_0$ is from singularities or boundaries, the more control one has on the derivatives. This is actually a typical result in the theory of elliptic partial differential equations, and the linkage to this is that the real and imaginary parts of complex analytic functions are harmonic (solve Laplace’s equation).

**Proof:** Take our above result for the formula for the derivatives of a function in terms of Cauchy integral and apply the standard contour integral inequality.

$$\left| \frac{d^n f}{dz^n}(z_0) \right| \leq \frac{n! L M}{2 \pi R}$$

where $L = \ell(C(z_0, R)) = 2 \pi R$.

Therefore,

$$M = \sup_{z \in C(z_0, R)} \left| \frac{f(z)}{(z-z_0)^{n+1}} \right| \leq \frac{n! L M}{2 \pi R} \Rightarrow \frac{M}{n!} \leq \frac{L}{2 \pi R}.$$
Now a string of consequences from Cauchy's inequality.

\[ |f(z_0)| = R \]
\[ = \sup_{z \in C(z_0, R)} \frac{|f(z)|}{R^{n+1}} = \frac{M}{R^{n+1}} \]
\[ \left| \frac{d^n f(z)}{dz^n} \right| \leq \frac{n!}{2^\pi R^n} \left( \frac{M}{R^{n+1}} \right) = \frac{n!M}{R^n} \]

\[ \square \]

Now a string of consequences from Cauchy's inequality.

**Liouville Theorem:** If \( f(z) \) is analytic on the whole (non-extended) complex plane and is bounded on the (non-extended) complex plane, then it must be constant.

In other words, every entire function \( f(z) \) (a function analytic on the whole, non-extended complex plane) is unbounded or constant.

**Proof:** Let's argue that the derivative is zero by bounding its magnitude by arbitrarily small numbers.

\[ M = \sup_{z \in \mathbb{C}} |f(z)| < \infty \quad \text{by boundedness} \]

Consider a circle \( C(z_0, R) \) about any point \( z_0 \in \mathbb{C} \) and any radius \( R \):

Because \( f \) will be analytic inside and on the circle:

\[ \left| \frac{d^n f(z_0)}{dz^n} \right| \leq \frac{n!M}{R^n} \rightarrow 0 \quad \text{as} \quad R \rightarrow \infty \]

So by taking arbitrarily large circles about arbitrary points \( z_0 \) we see that we must have:

\[ \left| \frac{d^n f(z_0)}{dz^n} \right| = 0 \quad \text{for all} \quad z_0 \in \mathbb{C} \]

\[ = \Rightarrow f'(z_0) = 0 \quad \text{for all} \quad z_0 \in \mathbb{C} \]

\[ = \Rightarrow f(z) \text{ constant} \]

\[ f(z) = \left( \int_C f(z) \, dz \right) \]

\[ f(z) = \int_C f(z) \, dz \]

\[ f(z) = \int_C f(z) \, dz + f(z_0) \]
Fundamental Theorem of Algebra: Any complex-valued polynomial $P(z) = \sum_{j=0}^{n} a_j z^j$ with $a_j \in \mathbb{C}$ has $n$ complex roots (allowing for repetition).

Proof: If we can show that at least one root $c$ exists, then by factoring out $(z-c)$, we get a degree $n-1$ polynomial, and can show by induction that $n$ roots exist.

Suppose on the contrary that $P(z)$ had no roots on the complex plane. Then consider $1/P(z)$. Because $P(z)$ is analytic everywhere, $1/P(z)$ will be analytic wherever $P(z) \neq 0$, which is nowhere, so $1/P(z)$ is an entire function. Moreover, we can show it's bounded. Clearly, by standard results of continuous functions on compact sets, $1/P(z)$ will be bounded over any finite subset of $\mathbb{C}$. So the only way to have boundedness fail is for the function to become unbounded "at $\infty$". Let's see what happens at $\infty$.

$$\lim_{z \to \infty} \frac{1}{P(z)} = \lim_{w \to 0} \frac{1}{P(\frac{1}{w})} = \lim_{w \to 0} \frac{w^n}{\sum_{j=0}^{n} a_j w^{-j}} = 0 \text{ for } n \geq 1$$

because $\lim_{w \to 0} \frac{w^n}{\sum_{j=0}^{n} a_j w^{-j}} = a_0$

$$- \lim_{w \to 0} w^n = 0$$

What if $a_0 = 0$? Then $z = 0$ clearly a root.

So we see that since $1/P(z)$ has a finite limit at infinity, it must be bounded in a neighborhood of infinity.

$$\frac{1}{|P(z)|} < \epsilon$$

for $\frac{1}{2} |z| < \delta$
Therefore $1/P(z)$ is a bounded entire function. By Liouville's theorem, this function must be constant, but it isn't for $n \geq 1$ so by contradiction a root must exist for $n \geq 1$.

**Gauss's Mean Value Theorem:** If $f(z)$ is analytic on and inside a simple circular contour $C(z_0, R)$, then:

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + R e^{it}) \, dt$$

This theorem emphasizes the rigidity of analytic functions. The value at the center of a circle is determined by the values on the circle.

**Proof:** This is just applying Cauchy's integral formula with the parameterization $\zeta = z_0 + R e^{it}$, $0 \leq t < 2\pi$

$$f(z_0) = \frac{1}{2\pi i} \oint_{C(z_0,R)} \frac{f(\zeta)}{\zeta - z_0} \, d\zeta = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{z_0 + Re^{it} - z_0} \, R e^{it} \, dt$$
Applying Gauss's mean value theorem to the real and imaginary parts of $f(z)$ tells us that real-valued harmonic functions also have this same mean value property.

**Maximum Modulus Principle:** If $f(z)$ is analytic and nonconstant over some open connected domain $D$, then $|f(z)|$ has no maximum value in $D$.

**Corollary:** If $f(z)$ is analytic and non-constant over the interior of a closed bounded region $R$, and continuous on the closed bounded region $R$, then $|f(z)|$ must take its maximum on the boundary of $R$.

**Proof of Maximum Modulus Principle:** Suppose to the contrary that $f(z)$ is analytic and nonconstant on an open domain $D$ and there exists a $z_0 \in D$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in D$. Now apply Gauss's mean value theorem to a circle $C(z_0, \rho)$, which, along with its interior, belongs to $D$.

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{it}) \, dt$$

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{it})| \, dt$$

$$\frac{1}{2\pi} \int_0^{2\pi} |f(z)| \, dt \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{it})| \, dt$$

$$0 \leq \frac{1}{2\pi} \int_0^{2\pi} \left| f(z_0 + \rho e^{it}) \right| \, dt - |f(z_0)| \, dt$$

$$|f(z_0 + \rho e^{it})| \leq |f(z_0)|$$
An integral of a nonpositive function is nonnegative. This can only happen if the integrand is zero almost everywhere, but since the functions are continuous, integrand must be zero everywhere. This implies that:

\[ \left| f(x) \right| \leq \left| f(z) \right| \quad \text{for all } \quad z \in \mathbb{D} \]

Proof to be continued...