Renormalized Resonance Quartets in Dispersive Wave Turbulence

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Using the (1+1)-D Majda-McLaughlin-Tabak (MMT) model as an example, we present an extension of the Wave Turbulence (WT) theory to systems with strong nonlinearities. We demonstrate that nonlinear wave interactions renormalize the dynamics, leading to (i) a possible destruction of scaling structures in the bare wave systems and a drastic deformation of the resonant manifold even at weak nonlinearities, and (ii) creation of nonlinear resonance quartets in wave systems for which there would be no resonances as predicted by the linear dispersion relation. Finally, we derive an effective WT kinetic equation and show that our prediction of the renormalized Rayleigh-Jeans distribution is in excellent agreement with the simulation of the full wave system in equilibrium.

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For many wave phenomena, due to their inherent complexity and turbulent nature, statistical ensembles rather than individual wave trajectories render the natural observations. In numerous branches of physics, including surface waves, capillary waves, internal waves, waves on liquid hydrogen, Alfven and Langmuir waves in plasmas, and turbulence in nonlinear optics, wave turbulence (WT) [2–4] arises through interactions of weakly nonlinear resonant waves in a dispersive medium. In contrast to strong turbulence in incompressible fluids, the weak nonlinearity in wave interactions potentially allows for a systematic treatment of WT utilizing near-identity transformations, statistical averaging and Hamiltonian structure [4]. The resulting kinetic equation in WT theory captures the time evolution of wave action [4]. In addition to the equilibrium Rayleigh-Jeans (RJ) distribution, there are Zakharov-Kolmogorov stationary solutions [4] to the kinetic equation for homogeneous, scale-invariant wave systems, which capture the direct and inverse cascades of wave excitations. These were believed to be universal (i.e., independent of the details of driving and damping) nonequilibrium spectra in an inertial range where neither driving nor damping exists.

Invoking random phase approximation (RPA), near Gaussianity in wave statistics (so no coherent structures) and resonant wave-wave interactions, WT theory was formally developed for describing the long-time statistical behavior of waves. Yet a major question remains, namely, how well it can describe real wave systems. Many studies attempted to verify the results of WT theory using direct numerical simulations of the underlying wave equations, but careful examination of the validity conditions of WT theory is further needed, in particular, on questions of what happens if any of the assumptions leading to WT theory are violated. This requires careful analysis of the related wave and (integro-differential) kinetic equations, with accurate simulations for precise statistical convergence. For (1+1)-D dispersive waves, this was carried out by introducing the MMT model [5], whose Hamiltonian in the Fourier space is given by \( \mathcal{H} = \mathcal{H}_2 + \mathcal{H}_4 \),

\[
\begin{align*}
\mathcal{H}_2 &= \int \omega_k |a_k|^2 \, dk, \\
\mathcal{H}_4 &= \frac{1}{2} \int T_{1234} a_k \alpha_k \beta_k \beta_k^* \beta_k^* \delta(k_1 + k_2 - k_3 - k_4) \, dk_{1234},
\end{align*}
\]

with \( \omega_k = |k|^\alpha \), \( T_{1234} = |k_1 k_2 k_3 k_4|^{\beta/4} \), parameters \( \alpha > 0 \) and \( \beta \), \( dk_{1234} = dk_1 dk_2 dk_3 dk_4 \) and \( \delta(\cdot) \) denoting the Dirac delta function.

In this Letter, using the MMT model as a prototypical example, we show how nonlinearity strongly modifies the dynamics of resonance structure and discuss its consequences for the long-time dynamics of WT. Our study reveals (i) the scaling-structures in the bare Hamiltonian (1) can be destroyed and resonant manifolds are qualitatively modified by the renormalized dispersion relation, which is a generalization of weak turbulence results [14] to strong nonlinearities; and (ii) nonlinear interactions can create resonances in wave systems whose bare dynamics has no resonance according to the linear dispersion relation. Finally, we extend the WT theory to include renormalized resonance dynamics.

The MMT model (1) is a prototypical example of a homogeneous, scale-invariant system that allows for four-wave resonances in 1-D in case of a concave dispersion law, i.e., \( \alpha < 1 \) (for which, there are no three-wave resonances). Its canonical equation of motion is

\[
\frac{\partial a_k}{\partial t} = \frac{\delta \mathcal{H}}{\delta \alpha_k^*},
\]

If \( \alpha = 2 \) and \( \beta = 0 \), it corresponds to the nonlinear Schrödinger equation (NLS) while the case of \( \alpha = 1/2 \), \( \beta = 3 \) mimics the scalings present in water waves. Numerical studies [5, 6] of system (1) reveal self-similar, often coexistent, spectra, including one apparently inconsistent with WT theory, as well as coherent structures, such as solitons, quasisolitons, and collapses, which greatly complicate the WT picture of the MMT system. Studies of this model thus show that, even in the weakly nonlinear limit, WT theory may not be able to capture
fully the rich behavior of the nonlinear wave system [6, 7].

Since the long-time statistical behavior of the nonlinear system is often controlled by resonances, WT theory focuses on the resonant wave interactions [4] determined by the linear dispersion relation \( \omega_k = \omega(k) \):

\[
\begin{align*}
\Delta_{k,k_k,k_k} &\equiv k_1 + k_2 - k_3 - k_4 = 0, \\
\Delta_{k,k_k,k_k} &\equiv \omega_k + \omega_{k_2} - \omega_{k_3} - \omega_{k_4} = 0.
\end{align*}
\]

(3a) (3b)

WT theory assumes that waves interact weakly, and thus, in equilibrium, give rise to the RJ distribution—a stationary solution of the kinetic equation, independent of the details of the nonlinearity [4]. However, nonlinear wave interactions tend to renormalize dispersion relations [8], which may have a strong impact on wave-wave interactions and resonant structures. Using the MMT system as a WT model, we investigate the consequences of dispersion renormalization for resonant wave interactions in both weakly and strongly nonlinear limits.

We first show that, in equilibrium, the Zwanzig-Mori (ZM) theory [9] can successfully describe how the dispersion relation is renormalized for long waves. This theory yields a generalized Langevin equation governing effective dynamics of slow observables. For a single dynamical variable \( a_k(t) \), this exact Langevin equation is given by \( \partial a_k(t)/\partial t = -\Omega_k a_k(t) - \int_0^t K(t-s) a_k(s) \, ds + F_k(t) \), where \( F_k \) is the random force related to the memory kernel \( K \) by the fluctuation-dissipation theorem [9]. Using the equipartition theorem \( \theta = a_k^2 \partial \mathcal{H}/a_k \), where \( \theta \) is the temperature of the MMT system, and \( \langle \cdot \rangle \) denotes the average over the Gibbs measure \( e^{-\mathcal{H}/\theta} \), we can show that the effective dispersion relation is

\[
\Omega_k = \frac{\theta}{\langle a_k^2 \rangle} = |k| + |k| \int |k_1 k_2 k_3| \frac{\langle a_k a_k a_k a_k \rangle}{\langle a_k^2 \rangle} \beta(\Delta_{k,k,k}) \, dk_{123}.
\]

The ZM projection formalism usually results in a linear, non-Markovian process. However, for a slow dynamical variable such as a long-wave mode, Markovian behavior results, and the renormalized dispersion relation \( \Omega_k \) characterizes the temporal frequency of \( a_k(t) \) [10]. For short waves, it is not clear that there is a time-scale separation among the linear dispersion, memory kernel and random forcing, therefore it would be difficult to interpret \( \Omega_k \) as the oscillation frequency of \( a_k \) for high \( k \)'s. However, it will be seen below that, surprisingly, \( \Omega_k \) accurately describes the oscillations of \( a_k \) for all \( k \)'s.

The renormalized ZM dispersion (4) further reduces to

\[
\Omega_k \equiv |k| + \left( 2 \int |k|^2 \langle a_k a_k \rangle \, dk \right) |k|^2
\]

(5)

by RPA. Via \( \Omega_k = \theta/\langle a_k a_k \rangle \), Eq. (5) becomes \( \Omega_k = |k|^\alpha + \theta \left( 2 \int |k|^\beta/2 \Omega_k^{-1} \, dk \right) |k|^\beta/2 \), from which \( \Omega_k \) and \( \theta \) can be determined after invoking the conservation of wave action, \( \int |a_k|^2 \, dk = N \), i.e., \( \int \theta/\Omega_k \, dk = \langle N \rangle \), where \( N \) is set by the initial condition. The connection between this renormalized dispersion and wave interactions can be seen by considering the collective effect of the trivial resonances, i.e., \( k_1 = k_3 \) or \( k_1 = k_4 \) in Conditions (3).

More precisely, the trivial resonant terms in \( \mathcal{H}_S \), \( \mathcal{H}_S \equiv 2 \int |k|^{3/2} |a_k|^2 \, dk \), can be approximated by \( \mathcal{H}_S = \int \left( 2 \int |k|^{3/2} \langle a_k a_k \rangle \, dk \right) |k|^{\beta/2} |a_k|^2 \, dk \) via a mean-field argument that each \( a_k \) interacts effectively with the thermal background waves \( \langle a_k |^2 \rangle \). The combination of \( \mathcal{H}_S \) and \( \mathcal{H}_R \) yields an effective quadratic interaction \( \mathcal{H}_S \equiv \int |k|^{\alpha} + \left( 2 \int |k|^{3/2} \langle a_k a_k \rangle \, dk \right) |k|^{\beta/2} |a_k|^2 \, dk \rangle. \) Hence, the dispersion relation (5). Therefore, the long-time dynamics can be described by an effective Hamiltonian \( \mathcal{H}_S = \mathcal{H}_S + \mathcal{H}_S - \mathcal{H}_S \) representing the nonlinear interactions. This dispersion renormalization, arising from trivial resonant interactions, effectively weakens the averaged nonlinear interactions. Note that Eq. (5), which is not limited to weak nonlinearities, is a non-perturbative generalization of the perturbatively corrected dispersion relation for weak nonlinearities [14, 15].

We now turn to the examination of our predictions (4) and (5). We numerically solve Eq. (2) [11, 12] to obtain the spatiotemporal spectrum \( \langle a_k a_k \rangle \) in equilibrium, \( \langle a_k a_k \rangle \) is not a simple rescaling of the bare \( \langle a_k a_k \rangle \), which may have a strong impact on wave-wave interactions. This dispersion renormalization, arising from trivial resonant interactions, effectively weakens the averaged nonlinear interactions. Note that Eq. (5), which is not limited to weak nonlinearities, is a non-perturbative generalization of the perturbatively corrected dispersion relation for weak nonlinearities [14, 15].

![Figure 1: Measured \( \omega_k \), the bare \( \omega_k = |k|^\alpha \), Eq. (4), and Eq. (5) with \( \alpha = 1/2, \beta = 6 \), are depicted as solid, dotted, dashed, and dashed-dotted lines, respectively. \( N = 1024 \). Inset: The same for NLS (\( \alpha = 2, \beta = 0 \).)](image-url)
overall concavity of the bare $\omega_k = |k|^{1/2}$ to the convexity of the renormalized curves at high $k$'s. This qualitative change of the dispersion relation takes place whenever $\alpha < 1$ and $\beta/2 > 1$, as seen in Eqs. (4) and (5), which generalizes, to strong nonlinearity, the corresponding results at weak nonlinearity [14]. It is important to note that the renormalization correction is $O(|k|^{3/2})$, which can always dominate over the bare dispersion relation $|k|$ for large $k$'s if $\beta/2 > \alpha$, no matter how small the nonlinearity. Furthermore, except for $\beta = 2\alpha$, the scaling structures in the bare dynamics (1) are destroyed by renormalization even at weak nonlinearities, thus, giving rise to a new resonance manifold not determined by the original scaling symmetry, as discussed below.

The theoretic resonance structure (3) can be visualized by projecting

$$|\Delta^{\omega}_{\omega_1}\omega_2| = |\omega_{k_1} + \omega_{k_2} - \omega_{k_3} - \omega_{k_4}|$$

on $(k_1,k_2)$ with $k_3$ being fixed and $k_4$ from (3a). Figs. 2(a),(b),(c) display surface plots of (6) for the bare $\omega_k = |k|^{\alpha}$, and the renormalized $\omega_k = \Omega_k$ (note that using $\Omega_k$ gives similar results) for $\beta = 4$ (Fig. 2(b)) and $\beta = 8$ (Fig. 2(c)), respectively. In these figures, the resonance manifold determined by $|\Delta^{\omega}_{\omega_1}\omega_2| \approx 0$, as signified by the dark strips, undergoes a deformation as $\beta$ increases and the resonance structures determined by the renormalized $\Omega_k$ are clearly different from those by the bare $\omega_k$. To approximate the MMT model, we use $N$ Fourier modes, and move all to the First Brillouin zone. The resonances within the area in Fig. 2(a) bounded by the two dashed lines are system intrinsic, i.e., not caused by the periodicity of the finite system.

In the traditional WT theory, waves interact through resonances controlled by the bare $\omega_k$. Here, we demonstrate a different picture. Since the resonances control the contribution of terms such as $a_{k_1}a_{k_2}a_{k_3}^*a_{k_4}^*\delta(\Delta_{k_3k_4})$ in the long-time limit, we use the long-time average

$$A_{k_1k_2k_3k_4} \equiv \langle a_{k_1}a_{k_2}a_{k_3}^*a_{k_4}^*\delta(\Delta_{k_3k_4}) \rangle$$

to reveal the resonance structures manifested in the dynamics (1). Here $\delta$ equals 1 if $\Delta_{k_1k_2}$ is a multiple of $N$ and, 0, otherwise. This $\delta$, instead of the Dirac $\delta$, is used to account for the discrete approximation of Eq. (2). For $\beta = 8$, $|A_{k_1k_2k_3k_4}|$ is displayed in Fig. 2(f), whose comparison with Fig. 2(c) reveals an excellent agreement between the locations of the peaks (dark strips) of the long-time average and the loci of the resonances (6) determined by the renormalized $\Omega_k^R \equiv \Omega_k$ or $\Omega_k$ (The WT theory would predict the resonance structures as in Fig. 2(a) for these cases). The physical picture derived from these results is that wave resonances are renormalized and they are governed by the renormalized $\Omega_k^R$. We note in passing that $\Omega_k^R = |k|^{\alpha} + \text{const}$ for NLS, therefore, its renormalized resonance structures should be the same as those predicted by WT theory, as is confirmed in our study.

We stress that both the nonlinearity parameter $\beta$ and the linear frequency exponent $\alpha$ play important roles in $\Omega_k^R$. In particular, if $\alpha > 1$ (for which there is no nontrivial four-wave resonances by the relation $\omega_k = |k|^{\alpha}$), resonances controlled by $\Omega_k^R$ may arise if $0 < \beta/2 < 1$. Shown in Fig. 2(e) is such a result where new resonance structures for $\alpha = 2$, $\beta = 1$, are created. For comparison, the resonance structure (6) for $\omega_k = |k|^2$ is displayed in Fig. 2(d) which does not possess new resonant strips appearing in Fig. 2(e). This result shows that the nonlinearity renormalizes the linear dispersion relation to modify the resonance manifold, thereby creating new resonant interactions even when there would be no bare resonance as dictated by the linear dispersion relation. We note that there is a surprising similarity in the resonance structure between Fig. 2(b) and Fig. 2(e). This similarity arises because both $\Omega_k^R$ have the asymptotic form of $c_1|k|^{1/2} + c_2|k|^2$. The resonance structure of Fig. 2(b) is in a weak turbulence regime while Fig. 2(d) is in a strong nonlinear regime with $\mathcal{H}_4/\mathcal{H}_2 \sim 1$.

The classical kinetic equation of WT theory [4] cannot be used to find the spectra with strong nonlinearities. Here, we develop a WT-like theory for this regime based on the frequency renormalization derived above. Approximating $\langle |a_k(t)|^2 \rangle$ by $n_k(t) \equiv \langle |a_k(t)|^2 \rangle_{\text{eff}}$ aver-
In conclusion, a new dynamical picture of WT emerges: The linear dispersion relation is effectively renormalized, allowing one to treat systems with strong nonlinearity. This renormalization can create new resonances that are not present in the bare resonances, giving rise to WT dynamics which cannot be captured by the classical WT theory. Going beyond the classical perturbative perspective of WT, our work has revealed a non-perturbative nature of WT with the spectrum $n_k$ of WT dynamics determined by an intertwining self-consistent process: The trivial resonant scatterings of waves off of background waves characterized by $n_k$ control the true, renormalized, dispersion relation. This renormalized dispersion relation, in turn, controls nontrivial resonances of the full dynamics, thus giving rise to a self-consistent wave spectrum $n_k$.

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[11] The numerical simulation of Hamiltonian system (2) can be viewed as a representation of microcanonical ensemble for system (2). A symplectic algorithm is used [12].
[13] System (2) is additionally forced by random noise at low $k$ and dissipated at both low and high $k$.