1. The following example shows that in some physical situations, non-uniqueness is natural and obvious, not pathological. Consider a water bucket with a hole in the bottom. If you see an empty bucket with a puddle beneath it, can you figure out when the bucket was full? No, of course not! It could have finished emptying a minute ago, ten minutes ago, or whatever. The solution to the corresponding differential equation must be non-unique when integrated backwards in time.

Here is a crude model of the situation. Let \( h(t) \) be the height of the water remaining in the bucket at time \( t \), \( a \) the area of the hole, \( A \) the cross-sectional area of the bucket (assumed constant), and \( v(t) \) the velocity of the water passing through the hole.

a) Show that \( av(t) = A \dot{h}(t) \). What physical law are you invoking?

b) To derive an additional equation, use conservation of energy. First, find the change in potential energy in the system, assuming that the height of water in the bucket changes by an amount \( \Delta h \) and that the water has density \( \rho \). Then find the kinetic energy transported out of the bucket by the escaping water. Finally, assuming all the potential energy is converted into kinetic energy, derive the equation \( v^2 = 2gh \).

c) Combining (a) and (b), show that \( \dot{h} = -C\sqrt{h} \), where \( C = \sqrt{2g} \left( \frac{a}{A} \right) \).

d) Given \( h(0) = 0 \) (bucket empty at \( t = 0 \), show that the solution for \( h(t) \) is non-unique in backwards time, i.e., for \( t < 0 \).

2. Homogeneous Differential Equation: Show that choosing a new dependent variable \( u = y/x \) transforms the differential equation

\[
y' = f \left( \frac{y}{x} \right)\]

into a separable equation.

(ii) Solve the initial-value problem

\[
y' = \frac{y}{x} - \frac{x^2}{y^2}, \quad y(1) = 1.
\]
In what interval is this solution valid?

3. Solve the initial-value problem

\[ y' = \frac{y \cos x}{1 + 2y^2}, \quad y(0) = 1 \quad (1) \]

as far as you can. Furthermore, conclude that the solution \( \phi(x) \) of (1) can be extended to the whole interval \(-\infty < x < \infty\) by completing the following outline:

(i) By finding the maximum and minimum values of \( y/(1 + 2y^2) \), show that

\[
\left| \frac{y}{1 + 2y^2} \right| \leq \frac{1}{2\sqrt{2}}
\]

for all \( y \), and hence that the absolute value of the right-hand side of the differential equation in (1) is bounded by \( 1/2\sqrt{2} \) for all \( x \) and \( y \).

(ii) Use the result of part (i) to show that

\[
|\phi(x) - 1| \leq \frac{|x|}{2\sqrt{2}}
\]

for all \( x \). Conclude that \( \phi(x) \) can indeed be extended to the whole interval \(-\infty < x < \infty\).

HINT: First formally integrate (1) as it stands to derive an integral equation for \( \phi(x) \). Also, it is clear that \( \phi(x) \) can be extended to \(-\infty < x < \infty\) precisely when it never blows up.

4. Consider the initial-value problem

\[ y' = f(x)g(y), \quad y(x_0) = y_0, \quad (2) \]

where the functions \( f(x) \) and \( g(y) \) are continuous near \( x_0 \) and \( y_0 \), respectively. Show that

(i) If \( g(y_0) \neq 0 \), then there exists a unique solution of (2). Which one?

(ii) If \( g(y_0) = 0 \), and \( g(y) \neq 0 \) for all \( y \) close enough to \( y_0 \), then there exists a unique solution of (2) if the integral

\[
\int_{y_0}^{y} \frac{1}{g(\eta)} \, d\eta
\]

diverges. What is this solution?

5. Solve the equation

\[ x(2 + x)y' + 2(1 + x)y = 1 + 3x^2. \]
Sketch some representative integral curves, and investigate the behavior of solutions as $x \to \pm \infty$. Then find the particular solution that satisfies the initial condition $y(-1) = 1$, and determine the maximal interval to which it can be extended.

6. (i) Consider Bernoulli’s equation

$$y' + p(x)y = q(x)y^n.$$ 

Show that if $n \neq 0, 1$, then the substitution $u = y^{1-n}$ reduces Bernoulli’s equation to a linear equation.

(ii) What equations does Bernoulli’s equation reduce to for $n = 0$ and $n = 1$?

(iii) Bernoulli’s equation

$$\frac{dy}{dt} = (\Gamma \cos t + T) y - y^3,$$

where $\Gamma$ and $T$ are constants, occurs in the study of the stability of fluid flow. Solve this equation as far as possible.

7. Integrating Factors: It is always possible (why?) to convert a differential equation written in the form

$$M(x, y)dx + N(x, y)dy = 0$$

into an exact equation if we multiply it by some appropriate function $\mu(x, y)$, which we call the integrating factor. However, this procedure is usually even less practical than solving equation (3) directly, since the function $\mu(x, y)$ must satisfy a partial differential equation.

(i) What partial differential equation must the function $\mu(x, y)$ satisfy?

(ii) In some special cases, looking for an integrating factor is advantageous. In particular, this is the case if we suspect that $\mu(x, y)$ is really just a function of $x$, of $y$, or of some specific combination of these variables. Thus, find the conditions for the functions $M(x, y)$ and $N(x, y)$ such that $\mu = \mu(x)$, $\mu = \mu(y)$, or $\mu = \mu(x^2 + y^2)$.

(iii) Find an integrating factor for the equation

$$(2x^2 + 2xy^2 + 1)y \, dx + (3y^2 + x) \, dy = 0,$$

and integrate it.

(iv) Compute an integrating factor of the form $\mu = \mu(x)$ for the first-order linear equation

$$y' = f(x)y + g(x).$$

Use this factor to re-derive the general solution of this equation.
8. The potential of two long parallel wires a distance 2a apart with charge densities ±q, respectively, is given by

\[ U(x, y) = \frac{q}{2\pi} \ln \frac{(x + a)^2 + y^2}{(x - a)^2 + y^2}. \]

(i) Assuming that the problem is essentially two-dimensional, find the lines of force generated by these two charged wires. Recall that lines of force are curves that are everywhere perpendicular to the curves \( U(x, y) = \text{constant} \).

HINT: Take the differential of the potential \( U(x, y) \) and replace \( dx \rightarrow dy \) and \( dy \rightarrow -dx \). (Explain why!) The resulting equation is exact as it stands. After you have integrated it, use the formula for the tangent of the sum of two angles to simplify a difference of two arctangents. The lines of force you should find are circles.

(ii) Find the limiting expressions for the potential \( U(x, y) \) and the results of (i) if you let \( a \rightarrow 0 \), and simultaneously increase the charge density so that \( qa \) remains constant. The results are the potential and the lines of force for a two-dimensional dipole.

9. The Grönwall Inequality: Let \( \phi, \psi, \) and \( \chi \) be real-valued continuous functions on a real \( t \)-interval \( I: a \leq t \leq b \). Let \( \chi(t) > 0 \) on \( I \), and suppose for \( t \in I \) that

\[ \phi(t) \leq \psi(t) + \int_a^t \chi(s) \phi(s) \, ds. \]

Prove that on \( I \)

\[ \phi(t) \leq \psi(t) + \int_a^t \chi(s) \psi(s) \exp \left( \int_s^t \chi(u) \, du \right) \, ds. \]

HINT: Let \( R(t) = \int_a^t \chi(s) \phi(s) \, ds \) and show that \( \dot{R} - \chi R \leq \chi \psi \).

10. Daniel Bernoulli’s work in 1760 had the goal of appraising the effectiveness of a controversial inoculation program against smallpox, which at the time was a major threat to public health. His model applies equally well to any other disease which, once contracted and survived, confers a lifetime immunity.

Consider the cohort of individuals born in a given year \( t = 0 \) and let \( n(t) \) be the number of these individuals surviving \( t \) years later. Let \( x(t) \) be the number of members of this cohort that have not had smallpox by year \( t \), and who are therefore still susceptible. Let \( \beta \) be the rate at which susceptibles contract smallpox, and let \( \nu \) be the rate at which people who contract smallpox die from the disease. Finally, let \( \mu(t) \) be the death rate from all causes other than smallpox. Then \( \frac{dx}{dt} \), the rate at which the number of susceptibles declines, is given by

\[ \frac{dx}{dt} = -[\beta + \mu(t)] x; \quad (4) \]
the first term on the right side of Eq. (4) is the rate at which susceptibles contract the smallpox, while the second term is the rate at which they die from all other causes. Also,

\[ \frac{dn}{dt} = -\nu \beta x - \mu(t)n, \]  

where \( dn/dt \) is the death rate of the entire cohort, and the two terms on the right side are the death rates due to smallpox and to all other causes, respectively.

(i) Let \( z = x/n \), and show that \( z \) satisfies the initial-value problem

\[ \frac{dz}{dt} = -\beta z(1 - \nu z), \quad z(0) = 1. \]  

Observe that the initial-value problem (6) does not depend on \( \mu(t) \).

(ii) Find \( z(t) \) by solving Eq. (6).

HINT: The calculation will be much easier if you make the substitution \( u = 1/z \).

(iii) Bernoulli estimated that \( \nu = \beta = \frac{1}{8} \). Using these values, determine the proportion of 20-year-olds who have not had smallpox.

NOTE: Based on the model previously described and on the best mortality data available at the time, Bernoulli calculated that if deaths due to smallpox could be eliminated \( (\nu = 0) \), then approximately 3 years could be added to the average life expectancy (in 1760) of 26 years 7 months. He therefore supported the inoculation program.

11. Consider the differential equation

\[ y' = y^{1/3}. \]

(i) Show that, locally, there are infinitely many solutions that pass through every point \( (x_0, 0) \).

(ii) Show that, locally and globally, there is only one solution passing through each point \( (x_0, y_0) \) with \( y_0 \neq 0 \).

Draw some appropriate pictures.

12. Investigate how the number of equilibria and their stability for the equation

\[ \dot{x} = rx - x^3 \]
depend on the parameter \( r \). Show that there are three qualitatively different cases, \( r > 0 \), \( r = 0 \) and \( r < 0 \). Draw the phase portrait in the \( x - \dot{x} \) diagram, and some representative integral curves in the \( x - t \) diagram for each qualitatively different case.

13. Suppose that the smooth function \( \phi(x, \varepsilon) \) is a solution of the differential equation

\[
y' = f(x, y, \varepsilon),
\]

depending on the parameter \( \varepsilon \).

(a) Is \( \phi(x, 0) \) a solution of the equation

\[
y' = f(x, y, 0)\]

Explain your answer, for instance by quoting a known theorem.

(b) Prove that the derivative

\[
\psi(x) = \frac{\partial \phi}{\partial \varepsilon}(x, 0)
\]

satisfies the inhomogeneous linear equation

\[
\psi' = \frac{\partial f}{\partial y}(x, \phi(x, 0), 0) \psi + \frac{\partial f}{\partial \varepsilon}(x, \phi(x, 0), 0).
\]

14. **Model of an Epidemic:** Suppose that, during an outburst of an infectious disease, the population can be divided into three classes: \( x(t) = \) the number of healthy people; \( y(t) = \) the number of sick people; \( z(t) = \) the number of dead people. The model will be based on the following three assumptions:

(i) The total population remains constant in size, except for deaths due to the epidemic. That is, the epidemic evolves so rapidly that we can ignore the slower changes in the populations due to births, emigration, or deaths by other causes.

(ii) Healthy people get sick at the rate proportional to the product of \( x \) and \( y \). This would be true if healthy and sick people encounter each other at a rate proportional to their numbers, and if there were a constant probability that each such encounter would lead to transmission of the disease.

(iii) Sick people die at a constant rate \( \ell \).

Under these assumptions, the model is

\[
\dot{x} = -kxy, \quad \dot{y} = kxy - \ell y, \quad \dot{z} = \ell y.
\]
(a) Which of the above three assumptions are reflected in each of the equations?

(b) Show that $x + y + z = N$, where $N$ is a constant. Which assumption does this constant reflect?

(c) Use the $\dot{x}$ and $\dot{z}$ equations to show that $x(t) = x_0 \exp(-kz(t)/\ell)$, where $x_0 = x(0)$.

(d) Show that $z$ satisfies the first-order equation $\dot{z} = \ell \left[ N - z - x_0 \exp(-kz(t)/\ell) \right]$.

(e) Show that, after an appropriate rescaling of all the variables and parameters, this equation becomes

$$\frac{du}{d\tau} = a - bu - e^{-u}.$$ 

(f) Show that $a \geq 1$ and $b > 0$.

(g) Determine the number of equilibrium points and classify their stability.

(h) Show that the maximum of $\dot{u}(t)$ occurs at the same time as the maximum of both $\dot{z}(t)$ and $y(t)$. (This time is called the peak of the epidemic, denoted $t_{\text{peak}}$. At this time, there are more sick people and a higher daily death rate than at any other time.)

(i) Show that if $b < 1$, then $\dot{u}(t)$ is increasing at $t = 0$ and reaches its maximum at some time $t_{\text{peak}} > 0$. Thus things get worse before they get better. (The term epidemic is reserved for this case.) Show that $\dot{u}(t)$ eventually decreases to 0.

(j) On the other hand, show that $t_{\text{peak}} = 0$ if $b > 1$. (Hence no epidemic occurs if $b > 1$.)

(k) The condition $b = 1$ is the threshold condition for an epidemic to occur. Can you give a biological interpretation of this condition?

15. Flows on the Circle: Let $\theta$ be an angle variable, that is, let it parametrize a circle $x = a \cos \theta$, $y = a \sin \theta$. Let

$$\dot{\theta} = f(\theta),$$

where $f(\theta + 2\pi) = f(\theta)$ for every $\theta$, be a differential equation defining a flow on the circle.

(i) Why does the function $f(\theta)$ have to be periodic in $\theta$ with period $2\pi$?

(ii) Is $\dot{\theta} = \omega$, where $\omega$ is a constant, a differential equation on the circle? If yes, what type of motion on the circle does it represent? If no, why not? Answer the same questions for $\dot{\theta} = \theta$.

(iii) Show that, if $f(\theta)$ has only simple zeros (that is, $f(\bar{\theta}) = 0$, $(df/d\theta)(\bar{\theta}) \neq 0$, so that $f(\theta)$ actually crosses the $\theta$-axis at $\theta = \bar{\theta}$), there must be an even number of them, and so the
equation (7) has an even number of equilibria.

(iv) Show that if (7) has no equilibria, all the solutions on the circle are periodic in time $t$ with period

$$T = \int_0^{2\pi} \frac{d\theta}{f(\theta)}.$$ 

(v) Consider the equation $\dot{\theta} = \omega - \sin \theta$, where $\omega$ is a positive constant. How many qualitatively different phase portraits of this equation exist as $\omega$ is varied? Draw them. (They should look the same as phase portraits on the line, just that you should draw them on the circle.) Find all the equilibria of this equation and determine their stability. Also, when there are no equilibria, find the period of the solution.

HINT: For the very last question, make the substitution $u = \tan(\theta/2)$. Also, integrate from $-\pi$ to $\pi$ instead of from 0 to $2\pi$. Why can you do that?

16. Given scalar functions $f_1(x), \ldots, f_n(x)$, continuous on $a \leq x \leq b$, show that they are linearly independent if and only if det $A \neq 0$, where $A = (a_{ij})$, and

$$a_{ij} = \int_a^b f_i(x)f_j(x) \, dx, \quad i, j = 1, \ldots, n.$$ 

17. Compute the Wronskian of the equation

$$y'' - \frac{1}{x}y' + y = 0.$$ 

What appears to be “wrong” with it? Explain how this apparent contradiction is resolved.

18. (i) Let $y_1(x)$ be a solution of a second-order linear differential equation. If $W(x)$ is the Wronskian corresponding to this equation, show that the second linearly independent solution is

$$y_2(x) = y_1(x) \int \frac{W(x)}{[y_1(x)]^2} \, dx.$$ 

(ii) One solution of the equation

$$y'' - \frac{x + 1}{x}y' - \frac{2(x - 1)}{x}y = 0$$

is $y_1(x) = e^{2x}$. What is the general solution of this equation?
19. **Euler’s Equation** reads

\[ x^2y'' + a_1xy' + a_0y = 0. \]  \hspace{1cm} (8)

(i) Show that if \( y(x) \) is a solution of (8), then so is \( y(-x) \). Deduce that it is enough to consider solutions of (8) for \( x > 0 \).

(ii) Show that assuming \( y = x^r \) in (8) leads to a quadratic equation for \( r \). Find the form of two linearly independent solutions of (8) if the roots of this quadratic equation are real \( r_1 \neq r_2 \).

(iii) Find the form of two linearly independent solutions of (8) if the roots of the quadratic equation for the exponent \( r \) are complex conjugate \( r_1 = \lambda + i\mu, r_2 = \lambda - i\mu \).

(iv) Use problem 18 to find the solution of (8) if the quadratic equation for the exponent \( r \) has two equal real roots.

20. Euler’s equation (8) is a perfect counter-example used to show what all can go wrong with the solutions of a differential equations at points where the existence theorem does not hold.

(i) Where does Euler’s equation have such points?

(ii) How far can the solution of Euler’s equation with the initial condition \( y(x_0) = y_0 \), with \( x_0 > 0 \), be extended.

(iii) Consider the situations in which the exponents \( r_1 \) and \( r_2 \) corresponding to a particular Euler’s equation are: (a) \( r_1 = 1, r_2 = 2 \); (b) \( r_1 = 1, r_2 = -1 \); (c) \( r_1 = r_2 = 1 \); (d) \( r_1 = i, r_2 = -i \); (e) \( r_1 = 1 + i, r_2 = 1 - i \); (f) \( r_1 = -1 + i, r_2 = -1 - i \). Write down the solutions in each case, draw their graphs, and discuss their behavior near the origin. Also, explain what goes wrong with the initial-value problem \( y(0) = y_0 \) in each case.

21. (i) Verify that given \( f(t) \), continuous on \( -k < t < k \), then

\[ z(t) = \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} f(t_n) \, dt_n \]

is the solution of the initial-value problem

\[ z^{(n)} = f(t), \quad , z(0) = \dot{z}(0) = \cdots = z^{(n)}(0) = 0. \]  \hspace{1cm} (9)

(ii) Use the formula

\[ \int_a^b dx \int_a^x F(x, y) \, dy = \int_a^b dy \int_y^b F(x, y) \, dx \]
to show that
\[ z(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s) \, ds. \]  
(10)

(iii) Verify directly that (10) provides the solution to the problem (9).

22. **Floquet Theory:** Let the matrix \( A(t) \) be continuous for all \( t \) and periodic in \( t \) with period \( p \), that is, \( A(t + p) = A(t) \). Show the following statements:

(i) If \( Y(t) \) is a fundamental matrix of the system

\[ \dot{y}(t) = A(t)y(t), \]  
(11)

then so is \( Y(t + kp) \) for any integer \( k \).

(ii) For every fundamental matrix \( Y(t) \) of (11), there exists a nonsingular matrix \( C \) that satisfies the equation

\[ Y(t + kp) = Y(t)C^k. \]

Compute \( C \) explicitly.

(iii) If \( Y(t) \) is a fundamental matrix of (11), and \( Z(t) = Y(t)B \) for some nonsingular matrix \( B \), then \( Z(t + p) = Z(t)B^{-1}CB \).

(iv) For every eigenvalue \( \lambda \) of the matrix \( C \), there exists a solution \( y(t) \) of (11) such that \( y(t + p) = \lambda y(t) \).

23. Find the real general solution of the system

\[ \dot{x} = -3x + 2z, \quad \dot{y} = x - y, \quad \dot{z} = -2x - y. \]

How does this solution behave as \( t \to \infty \)?

24. Find the real general solutions and sketch the phase portraits of the \( 2 \times 2 \) linear systems \( \dot{x} = Ax \), where \( A \) is the matrix

(i) \( \begin{pmatrix} 2 & 4 \\ 5 & 3 \end{pmatrix} \),  
(ii) \( \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \),  
(iii) \( \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \),  
(iv) \( \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \),  
(v) \( \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \).

Determine the number, location, stability, and type of the equilibria in each case.
25. Determine the stable, unstable, and center subspaces of the equilibria for the $2 \times 2$ linear systems $\dot{x} = Ax$, where $A$ is the matrix

\[(i) \begin{pmatrix} 2 & 4 \\ 5 & 3 \end{pmatrix}, \quad (ii) \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}, \quad (iii) \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}, \quad (iv) \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}, \quad (v) \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} .\]

Which of these equilibria are hyperbolic?

26. Compute the real general solutions, sketch the phase portraits (in the eigen-coordinates), and determine the stable, unstable, and center subspaces of all the equilibrium points for the $3 \times 3$ linear systems $\dot{x} = Ax$, where the eigenvalues and the eigenvectors of the matrix $A$ are

\[(i) \lambda_1 = -1, \quad x^{(1)} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}, \quad \lambda_2 = -2, \quad x^{(2)} = \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}, \quad \lambda_3 = 1, \quad x^{(3)} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix},\]

\[(ii) \lambda_1 = -1, \quad x^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \lambda_2, 3 = -2 \pm 5i, \quad x^{(2),(3)} = \begin{pmatrix} 1 \pm 3i \\ -2 \pm i \\ -1 \end{pmatrix},\]

\[(iii) \lambda_1 = 1, \quad x^{(1)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \lambda_2 = -2, \quad x^{(2)} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \lambda_3 = 0, \quad x^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\]

\[(iv) \lambda_1 = -1, \quad x^{(1)} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \quad \lambda_2, 3 = 1 \pm 2i, \quad x^{(2),(3)} = \begin{pmatrix} 2 \mp i \\ -1 \\ 1 \pm 2i \end{pmatrix},\]

\[(v) \lambda_1 = 2, \quad x^{(1)} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \quad \lambda_2, 3 = \pm 2i, \quad x^{(2),(3)} = \begin{pmatrix} \mp i \\ 1 \\ 2 \pm 3i \end{pmatrix}.\]

27. Let $A$ be a constant $n \times n$ matrix, and consider the system

\[x \frac{dy}{dx} = Ay, \quad y \in \mathbb{R}. \quad (12)\]
(i) Show that the substitution \( x = e^t \) transforms (12) into a system with constant coefficients, and determine the coefficient matrix of that system.

(ii) Deduce that Euler’s equation \( x^2 y'' + a_1 x y' + a_0 y = 0 \) can be reduced to a second-order linear equation with constant coefficients via the substitution \( x = e^t \).

(iv) Find the general solution of the system
\[
\frac{dy}{dx} \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix} y.
\]

28. (i) Compute the Wronskian of the matrix \( e^{At} \), where \( A \) is a constant \( n \times n \) matrix. Thus, convince yourself that \( e^{At} C \), where \( C \) is a constant \( n \)-dimensional vector, is indeed the general solution of the system \( \dot{x} = Ax \).

(ii) Let \( \phi_1, \phi_2 \ldots, \phi_n \) be \( n \)-dimensional vectors. It is well known that the volume of the parallelepiped spanned by the vectors \( \phi_1, \phi_2 \ldots, \phi_n \) is equal to the absolute value of the determinant of the matrix \( \Phi \) whose columns are precisely these vectors. Using this fact, compute the volume of the parallelepiped spanned by the vectors \( e^{At} \phi_1, e^{At} \phi_2 \ldots, e^{At} \phi_n \).

(iii) Show that the volumes of all \( n \)-dimensional subsets of \( \mathbb{R}^n \) change by an equal factor during the same time interval. What is this factor?

(iv) When does the flow of the system \( \dot{x} = Ax \) preserve volume?

29. A Hamiltonian system has the form
\[
\dot{p} = -\frac{\partial H(p, q)}{\partial q}, \quad \dot{q} = \frac{\partial H(p, q)}{\partial p},
\]
where \( p = (p_1, \ldots, p_n) \) and \( q = (q_1, \ldots, q_n) \) are \( n \)-dimensional vectors and \( H(p, q) \) is a real-valued function of \( 2n \) variables. Show that every linear Hamiltonian system is of the form
\[
\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = JA \begin{pmatrix} p \\ q \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}
\]
where \( 0 \) and \( I \) are the zero and identity \( n \times n \) matrices, and \( A \) is a symmetric \( 2n \times 2n \) matrix, respectively. What is the function \( H(p, q) \) in this case?

HINT: First show that \( J^2 = -I \), the \( 2n \times 2n \) identity matrix.

30. Find the general solutions of the \( 3 \times 3 \) linear systems \( \dot{x} = Ax \), where \( A \) is the matrix
\[
\begin{align*}
\text{(i)} & \quad \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}, & \text{(ii)} & \quad \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix}, & \text{(iii)} & \quad \begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix}.
\end{align*}
\]
HINT: The eigenvalues of the respective matrices are: (i) $-1$, (ii) $2$, (iii) $1$.

31. Find the general solutions of the equations

(i) $y''' + y'' + y' + y = e^{-x} + 4x$,

(ii) $y'' + 2y' + y = \frac{e^{-x}}{1 + x^2}$,

(iii) $x^2y'' + xy' - y = x \ln x$, for $x > 0$.

32. The motion of a charged particle with a unit charge in the constant electric field $E$ and the constant magnetic field $B$ is given by the equation

$$\dot{\mathbf{v}} = E + \mathbf{v} \times B,$$

where $\mathbf{v} = \dot{\mathbf{r}}$ is the velocity of the particle, $\mathbf{r}$ its position, and $\times$ denotes the cross product of two three-dimensional vectors. Find $\mathbf{r}(t)$ as a function of time $t$, and initial conditions $\mathbf{v}(0) = \mathbf{v}_0$ and $\mathbf{r}(0) = \mathbf{r}_0$.

HINT: Choose your coordinate system so that the $z$-axis is aligned with the magnetic field. Also, guess the particular solution corresponding to $E$ rather than using variation of constants.

33. Find the general solution of the system

$$\dot{\mathbf{x}} = \frac{1}{t} \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x} + te^t \begin{pmatrix} t + 1 \\ t + 3 \end{pmatrix}.$$

34. Consider the nonlinear initial-value problem

$$\dot{\mathbf{x}} = A\mathbf{x} + f(\mathbf{x}), \quad \mathbf{x}(0) = \mathbf{x}_0,$$

where $f(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^2)$. Use variation of constants to show that this problem is equivalent to the integral equation

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 + \int_0^t e^{A(t-s)}f(\mathbf{x}(s)) \, ds.$$

35. (i) Find the solution of the initial-value problem

$$\ddot{u} + 2i\dot{u} + \omega_0^2 u = f \cos \omega t, \quad u(0) = u_0, \quad \dot{u}(0) = \dot{u}_0.$$
if $\beta < \omega_0$.

(ii) Let $u_0 = 0$ and $\dot{u}_0 = 0$ in the result of (i), and show that, for fixed $t$,

$$\lim_{\omega \to \omega_0} \lim_{\beta \to 0} u(t) = \lim_{\beta \to 0} \lim_{\omega \to \omega_0} u(t) = \frac{f}{2\omega_0} t \sin \omega_0 t.$$

36. We have shown in class that the particular solution of the equation $\ddot{u} + 2\beta \dot{u} + \omega_0^2 u = f \cos \omega t$ that retains its form for large $t$ is $u_p(t) = R \cos(\omega t - \delta)$, where

$$R = \frac{f}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta \omega)^2}}, \quad \text{and} \quad \delta = \arctan \frac{2\beta \omega}{\omega_0^2 - \omega^2}.$$

Write the non-dimensional amplitude $A = R\omega_0/f$ and the phase $\delta$ in terms of the non-dimensional variables $x = \omega/\omega_0$, and $\gamma = 2\beta/\omega_0$. Investigate the behavior of the functions $A(x)$ and $\delta(x)$. In particular, find the maximum of the function $A(x)$ and the inflection point of the function $\delta(x)$. What is the limit of this inflection point as $\gamma \to 0$? What is $\delta(1)$? Draw the appropriate sketches of the functions $A(x)$ and $\delta(x)$. How many qualitatively different pairs of sketches are there?

37. **Weakly nonlinear oscillations**: Find the dependence of the period of oscillations of a pendulum described by the equation

$$\ddot{x} + \sin x = 0 \quad (13)$$
on the amplitude $A$, assuming that $A$ is small, by filling in the steps of the following outline:

Consider the solution of the pendulum equation (13) with the initial condition $x(0) = A$, $\dot{x}(0) = 0$ as a function of $A$.

(i) Explain why we can expand $x(t) = A x_1(t) + A^2 x_2(t) + A^3 x_3(t) + \mathcal{O}(A^4)$.

(ii) By inserting the above expansion into the equation (13), and collecting terms with the like powers of the amplitude $A$, compute the equations

$$\ddot{x}_1 + x_1 = 0, \quad \ddot{x}_2 + x_2 = 0, \quad \ddot{x}_3 + x_3 - \frac{1}{6} x_1^3 = 0.$$

Also, derive the initial conditions $x_1(0) = 1$, $x_2(0) = x_3(0) = \dot{x}_1(0) = \dot{x}_2(0) = \dot{x}_3(0) = 0$.

(iii) Show that the solutions of the initial-value problems computed in part (ii) are

$$x_1 = \cos t, \quad x_2 = 0, \quad x_3 = \frac{1}{192} (\cos t - \cos 3t) + \frac{1}{16} t \sin t.$$
(vi) Argue on mathematical or physical grounds that the period $T$ of the oscillations is just at the point at which $x(t)$ has its maximum, and is near $2\pi$ for small $A$. To find this point, use the condition $\dot{x}(T) = 0$. Assume $T = 2\pi + u$, where $u$ is small (why can you do that?), compute that

$$u = 2\pi \frac{A^2}{16} + O(A^3),$$

and deduce the corresponding expansion for $T$.

(v) Argue on mathematical or physical grounds that the period $T$ must be an even function of the amplitude $A$, and thus improve the remainder estimate in the above result for $T$ by one order of magnitude. Thus, show that

$$T = 2\pi \left[ 1 + \frac{A^2}{16} + O(A^4) \right].$$

38. **Geometric ray optics:** According to Fermat’s principle, the path of a light ray moving in an inhomogeneous, two-dimensional medium with velocity $v(x, y)$ between the points $(x_1, y_1)$ and $(x_2, y_2)$ is the curve that minimizes the time

$$T = \int_{x_1}^{x_2} \sqrt{1 + \frac{y'^2}{v(x, y)}} \, dx,$$

during which this curve is traversed. Show that if the optical medium is the upper half-plane, and $v(x, y) = y$, the light rays are segments of circles with centers on the $x$-axis.

39. Show that the extremals of the functional

$$J = \int_{x_0}^{x_1} F(x, y, y', y'', \ldots, y^{(n)}) \, dx$$

among $2n$-times continuously differentiable functions are given by the equation

$$F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} - \ldots + (-1)^n \frac{d^n}{dx^n} F_{y^{(n)}} = 0.$$

**HINT:** In your derivation, use the function $\eta(x) = (x - c + \delta)^{2n}(x - c - \delta)^{2n}$ for $|x - c| < \delta$ and $\eta(x) = 0$ elsewhere.

40. In polar coordinates,

$$x = r \cos \phi \sin \theta, \quad y = r \sin \phi \sin \theta, \quad z = r \cos \theta,$$
a cone is given by the equation $\theta = \text{const}$. Find geodesic curves on the cone, that is, the shortest curves connecting pairs of points on the cone.

HINT: Look for a function $r(\phi)$.

41. Find the extrema of the integral

$$J[y] = \int_0^\pi y'^2(x) \, dx,$$

subject to the conditions

$$\int_0^\pi y^2(x) \, dx = 1, \quad y(0) = y(\pi) = 0.$$

42. Let the upper half of the $xy$-plane be covered with a mass having continuous density $\mu(x, y)$.

(i) Show that from among all curves $y(x)$ of given length $l$ that pass through the points $(x_0, 0)$ and $(x_1, 0)$, the one which together with the interval $[x_0, x_1]$ on the $x$-axis encloses the largest amount of mass satisfies the equation

$$\frac{y''}{\sqrt{1 + y'^2}} = \frac{\mu(x, y)}{\lambda}, \quad (14)$$

where $\lambda$ is a constant. (The right-hand side of this equation is the curvature of the curve $y(x)$ at the point $(x, y(x))$.)

(ii) Show that if $\mu = \mu(y)$ only, equation (14) may be replaced by the equation

$$\frac{\lambda}{\sqrt{1 + y'^2}} + \rho(y) = C,$$

where $C$ is a constant and

$$\rho(y) = \int_0^y \mu(s) \, ds.$$

(iii) Find $y(x)$ when $\mu = 1$.

43. The Lagrangian function for a two-dimensional mass-spring system with mass $m$ and spring constant $k$ both equal to $m = k = 1$ is

$$L = \frac{1}{2} \dot{x}^2 + \frac{1}{2} \dot{y}^2 - \frac{1}{2} x^2 - \frac{1}{2} y^2.$$
(i) Find Lagrange’s equations and compute the motion of this system in the rectangular $xy$-coordinates.

(ii) Find Lagrange’s equations in the polar $r\theta$-coordinates, and find the orbits $r(\theta)$.

44. A body of least resistance: When a body of revolution is moving through the air, the normal component of the pressure at every point of the body is given by the expression $p = 2\rho v^2 \sin^2 \theta$, where $\rho$ is the density of the air, $v$ is the velocity of the air relative to the body, and $\theta$ is the angle between the velocity and its tangential component.

(i) Sketch an appropriate figure of this situation.

(ii) Show that the total force of air resistance on the body is

$$F = 4\pi \rho v^2 \int_0^l \sin^3 \theta y\sqrt{1+y'^2} \, dx,$$

where $l$ is the length of the body.

(iii) Show that

$$\sin \theta = \frac{y'}{\sqrt{1+y'^2}}.$$

(iv) Assume that $y'$ is small, and derive the approximate expression

$$F = 4\pi \rho v^2 \int_0^l y^3 \, dx.$$

(v) Find the shape $y(x)$ of the body with $y(0) = 0$ and $y(l) = R$ such that $F$ is minimal from this approximation.

45. A particle with unit mass moves in the spherically symmetric potential

$$U(q) = \frac{1}{2} |q|^2 \left(1 - \frac{1}{2} |q|^2\right),$$

where $q = (q_1, q_2, q_3)$ are the coordinates of the particle, and $|q| = \sqrt{q_1^2 + q_2^2 + q_3^2}$.

(i) Find the Lagrangian for this particle and derive Lagrange’s equations.

(ii) Show that the vector of momenta $p$ conjugate to the coordinates $q$ is equal to $p = \dot{q}$. Derive the Hamiltonian and Hamilton’s equations for this problem in the $p - q$ phase space.

(iii) Show that $p \times q = L = \text{const.}$, and deduce that the particle’s motion is planar. What is the physical meaning of $L$?
(iv) From (iii), conclude that you can restrict yourself to the case \( q = (q_1, q_2) \). Introduce polar coordinates \( q_1 = q \cos \theta, \ q_2 = q \sin \theta \). Find the Lagrangian and Lagrange’s equations in this polar coordinate system. Which (if any) coordinates are cyclic?

(v) Find the momenta \( p_q \) and \( p_\theta \) conjugate to the polar coordinates \( q \) and \( \theta \), and derive the corresponding Hamiltonian and Hamilton’s equations.

(vi) Do you think that, once you have found that this problem is planar, the polar coordinates are preferable to the rectangular coordinates? Explain your answer.

Do not try to solve any of the Lagrange’s or Hamilton’s equations that you have computed.

46. **Poisson Brackets:** Consider the system of Hamilton’s equations

\[
\dot{q} = \frac{\partial H(p, q)}{\partial p}, \quad \dot{p} = -\frac{\partial H(p, q)}{\partial q}
\]  

for the coordinates \( q = (q_1, \ldots, q_n) \) and momenta \( p = (p_1, \ldots, p_n) \) in the \( p - q \) phase space, and the Hamiltonian \( H(p, q) \).

(i) For any scalar function \( F(p, q) \) of the coordinates \( q \) and momenta \( p \), show that (15) imply

\[
\hat{F} = \frac{\partial F(p, q)}{\partial q} \cdot \frac{\partial H(p, q)}{\partial p} - \frac{\partial F(p, q)}{\partial p} \cdot \frac{\partial H(p, q)}{\partial q} \equiv \{F, H\},
\]

where

\[
\frac{\partial A(x, y)}{\partial x} \cdot \frac{\partial B(x, y)}{\partial y} = \sum_{i=1}^{n} \frac{\partial A(x, y)}{\partial x_i} \frac{\partial B(x, y)}{\partial y_i}.
\]

The quantity \( \{F, H\} \) is called the **Poisson Bracket** of \( F \) and \( H \). Note that we can define \( \{F, G\} \) for any scalar functions \( F(p, q) \) and \( G(p, q) \), without either of them being the Hamiltonian.

HINT: Use the chain rule and (15).

(ii) Show that \( \{F, G\} = -\{G, F\} \), and that \( \{F, \lambda G + \mu H\} = \lambda\{F, G\} + \mu\{F, H\} \). Also, deduce a similar identity for \( \{\lambda F + \mu G, H\} \).

(iii) Show that \( \{p_i, p_j\} = 0, \ \{q_i, q_j\} = 0, \ \text{and} \ \{q_i, p_j\} = \delta_{ij} \) for \( i, j = 1, \ldots, n \), where \( \delta_{ij} \) is the **Kronecker delta**, that is, \( \delta_{ij} = 0 \) if \( i \neq j \) and \( \delta_{ij} = 1 \) if \( i = j \).

(iv) Let \( n = 3 \), that is, \( q = (q_1, q_2, q_3) \) and \( p = (p_1, p_2, p_3) \), and let the angular momentum \( L \) be \( L = (L_1, L_2, L_3) = q \times p \). Show that

\[
\{L_1, L_2\} = L_3, \quad \{L_2, L_3\} = L_1, \quad \{L_3, L_1\} = L_2.
\]
HINT: First show that for any \( F, G, \) and \( H, \)
\[
\{ F, GH \} = G \{ F, H \} + H \{ F, G \}. 
\]
Show the first one of these identities, and deduce the other two by carefully renaming the coordinate axes, that is, making sure that your coordinate system remains right-handed after the renaming. What happens if it becomes left-handed?

(v) Show that \( \{|L|^2, L_i \} = 0 \) for \( i = 1, 2, 3, \) where \( |L|^2 = L_1^2 + L_2^2 + L_3^2. \)

REMARKS: The Poisson bracket as defined in this problem for classical mechanics has an extremely important analog in quantum mechanics, the commutator, which has identical algebraic properties. In particular, the three identities shown in (ii) imply that the Poisson bracket is anti-commutative or skew-symmetric, and bilinear. The Jacoby identity \( \{ F, \{ G, H \} \} + \{ G, \{ H, F \} \} + \{ H, \{ F, G \} \} = 0 \) is another very important property of the Poisson bracket. (The algebra needed to compute it is completely straightforward but excruciating.) These three properties, the anti-commutativity, the bilinearity, and the Jacoby identity ensure that the vector space of all scalar functions defined on the \( p - q \) phase space, equipped with the operation \( \{ F, G \} \) has the structure of a Lie algebra. This structure is very important in classical and even more in quantum mechanics, but is well beyond the scope of this course. The results of (iv) and (v), known in quantum mechanics as the commutation rules for the angular momentum, offer a small glimpse into this vast topic.

47. Lagrangian and Hamiltonian descriptions of a charged particle moving in an electromagnetic field: Recall Maxwell’s equations
\[
\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0, \quad \nabla \cdot \mathbf{D} = 4\pi \rho \\
\n\nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} = \frac{4\pi \mathbf{j}}{c}, \quad \nabla \cdot \mathbf{B} = 0.
\]
Here, \( \mathbf{E} \) and \( \mathbf{H} \) are the electric and magnetic field strengths, \( \mathbf{D} \) and \( \mathbf{B} \) are the electric and magnetic field densities, \( \rho \) is the charge density, \( \mathbf{j} \) is the current density and \( c \) is the speed of light. All these quantities (except, of course, for \( c \)) depend on the position vector \( \mathbf{r} = (x, y, z) \) and possibly the time \( t. \)

(i) Show that from the fourth of these equations, it follows that \( \mathbf{B} = \nabla \times \mathbf{A} \) for some vector potential \( \mathbf{A}. \)

HINT: Quote the appropriate result from Advanced Calculus.

(ii) Deduce from (i) and the first Maxwell’s equation that
\[
\mathbf{E} = -\nabla \phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t},
\]
where $\phi$ is some scalar potential.

HINT: Again quote the appropriate result from Advanced Calculus.

(iii) It is known in physics that the Lorenz force on a charged particle in the electromagnetic field given by $E$ and $B$ is
\[
F = e\left( E + \frac{1}{c}v \times B \right),
\]
where $e$ is the charge and $v = \dot{r}$. Deduce from (i) and (ii) that
\[
F = e\left[ -\nabla\phi - \frac{1}{c} \frac{\partial A}{\partial t} + \frac{1}{c} v \times (\nabla \times A) \right],
\]
and compute that
\[
F = e\left[ -\frac{\partial}{\partial r} (\phi - \frac{1}{c} v \cdot A) - \frac{1}{c} \frac{d}{dt} \left( \frac{\partial}{\partial v} (A \cdot v) \right) \right].
\]

HINT: Show the last identity in components.

(iv) Since Newton’s third law tells us that $F = m\ddot{r} = m\dot{v}$, where $m$ is the mass of the particle, deduce that the particle’s motion is described by Lagrange’s equations corresponding to the Lagrangian $L = T - U$, where
\[
T = \frac{1}{2} m |v|^2 \quad \text{and} \quad U = e\left( \phi - \frac{1}{c} A \cdot v \right),
\]
are the kinetic and potential energies of the particle, respectively.

(v) Show that the momentum vector $p$ conjugate to the position vector $r$ for the Lagrangian system described in (iv) is
\[
p = mv + \frac{e}{c}A,
\]
and the corresponding Hamiltonian is
\[
H = \frac{1}{2m} \left| p - \frac{e}{c} A \right|^2 + e\phi.
\]
Derive Hamilton’s equations for this Hamiltonian.

48. Consider the planar system
\[
\dot{r} = -r(1 - r)(2 - r), \quad \dot{\theta} = r^2,
\]
which is expressed in terms of the polar coordinates $r$ and $\theta$. How many equilibrium and periodic solutions (up to a time shift) does this system possess? What is the stability type
of each of these equilibrium and periodic solutions? Draw the phase portrait of this system in the \(x - y\) plane, \(x = r \cos \theta, y = r \sin \theta\). What are the periods of the periodic solutions?

49. Does the system
\[
\ddot{y} + 2\beta \dot{y} + \omega_0^2 y = f \cos \omega t, \quad 0 < \beta < \omega_0,
\]
possess a periodic solution? If yes, what is its stability type?

HINT: To save time and work, use as much information from the lectures as you can.

50. For each of the following Newtonian systems find its Hamiltonian function, and sketch its phase portrait in the \(x - p\) plane (i.e., \(x - \dot{x}\) plane):

(i) \(\ddot{x} = -x - x^3\), (ii) \(\ddot{x} = x - x^3\), (iii) \(\ddot{x} = -x + x^3\).

51. Investigate the stability of all the solutions of the equation \(\dot{x} = -x - t + 1\) as \(t \to \infty\). Sketch these solutions.

52. Consider the system
\[
\dot{x} = -y - x \left(x^2 + y^2 - 1\right), \quad \dot{y} = x - y \left(x^2 + y^2 - 1\right), \quad \dot{z} = -2z. \tag{16}
\]

(i) By transforming the \(x - y\) piece of this system into polar coordinates, \(x = r \cos \theta, y = r \sin \theta\), show that (16) has an asymptotically stable periodic solution
\[
x = \cos t, \quad y = \sin t. \tag{17}
\]

HINT: First show that
\[
\dot{r} = \frac{x\dot{x} + y\dot{y}}{r}, \quad \dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}.
\]

(ii) Show that the linearization of the system (16) about the solution (17) is
\[
\dot{\xi} = -2(\cos^2 t)\xi - (1 + 2 \cos t \sin t)\eta, \quad \dot{\eta} = (1 - 2 \cos t \sin t)\xi - 2(\sin^2 t)\eta, \quad \dot{\zeta} = -2\zeta.
\]

(iii) Show that the substitution \(\xi = a \cos t - b \sin t, \eta = a \sin t + b \cos t\) transforms the system obtained in (ii) into one with constant coefficients, and solve this system. What does this substitution do geometrically? What is the stability of the zero solution, and why?
Figure 1: $N - 1$ coupled pendula, with $N = 9$.

53. Consider $N - 1$ pendula coupled as shown in Figure 1.

Let $m$ be the mass and $l$ the length of each pendulum, $a$ the length and $k$ the stiffness of each spring, $b$ the height at which the springs are attached to the pendula, and $u_i$ the displacement of the $i$-th pendulum from its equilibrium position. Neglect all the effects of air resistance and friction.

(i) Show that, if $u_i \ll 1$, the equation governing the oscillations of the $i$-th pendulum is

$$m \ddot{u}_i + mg \frac{b}{l} (u_{i+1} + u_{i-1} - 2u_i), \quad i = 1, \ldots, N - 1,$$

where $g$ is the acceleration of gravity. Explain why you must also assume $u_0 = u_N = 0$.

(ii) Verify that in the $n$-th normal mode of oscillation of the system (18), the position of the $i$-th pendulum is given by the expression

$$u_{i,n} = (A_n \cos \omega_n t + B_n \sin \omega_n t) \sin \frac{in\pi}{N}, \quad i, n = 1, \ldots, N - 1.$$

Here

$$\omega_n^2 = \omega^2 + \kappa \left(1 - \cos \frac{n\pi}{N}\right), \quad \omega^2 = \frac{g}{l}, \quad \kappa = \frac{2kb^2}{ml^2},$$

and $A_n$ and $B_n$ are amplitudes obtained from the initial conditions.

(iii) Let $u_n = (u_{1,n}, \ldots, u_{N-1,n})$ be the $n$-th normal mode. Denote by $v_n$ the spatial amplitude of $u_n$, that is $u_n = (A_n \cos \omega_n t + B_n \sin \omega_n t)v_n$, so that $v_{i,n} = \sin(in\pi/N)$. Compute the frequencies $\omega_n$ and sketch the spatial shapes $v_n$ of the normal modes explicitly for $N = 4$.

(iv) Assume that the number of pendula is very large, $N \gg 1$, and that they are close together, $a \ll 1$, so that we can consider them as “continuously distributed.” In this case, you can consider the mass of the pendula to also be continuously distributed, $m = \rho a$, where $\rho$ is the mass density per unit length. If you assume that the stiffness $k$ of each spring increases linearly with decreasing $a$ so that

$$\frac{kb^2}{ml^2} = \frac{c^2}{a^2}, \quad c = \text{const.},$$
and let \( x = ia \), you can rewrite (18) as

\[
\ddot{u}(x, t) + \omega^2 u(x, t) = \frac{c^2}{a^2} \left( u(x + a, t) + u(x - a, t) - 2u(x, t) \right).
\]

Show that for \( a \ll 1 \), in the first approximation, this equation becomes the Klein-Gordon equation

\[
u_{tt} - c^2 u_{xx} + \omega^2 u = 0,
\]

where the subscripts denote partial differentiation. Show that the appropriate boundary conditions for the Klein-Gordon equation in this limit are \( u(0, t) = 0 \) and \( u(L, t) = 0 \), where \( L = Na \). Find the normal modes of this equation, and show that the normal modes of the discrete system of pendula limit smoothly on these continuous normal modes.

54. Let \( (x(t, H), p(t, H)) \) be a one-parameter family of solutions of the Newtonian system \( \dot{x} = p, \dot{p} = f(x) \), parametrized by the energy \( H \). Show that the vector functions \( (\dot{x}(t, H), \dot{p}(t, H)) \) and \( (x_H(t, H), p_H(t, H)) \), both evaluated at \( H = h \), form a fundamental system of solutions for the linearization of the Newtonian system \( \dot{x} = p, \dot{p} = f(x) \) about its solution \( (x(t, h), p(t, h)) \). In particular, by differentiating the energy equation, show that the Wronskian of the above two vector solutions is equal to one.

55. A linear triatomic molecule is simulated by a configuration of masses and ideal springs that looks like the following diagram:

\[
\begin{array}{ccc}
  \text{m} & \text{k} & \text{M} & \text{k} & \text{m} \\
\end{array}
\]

The equilibrium length of the springs is \( b \). Find the eigenfrequencies and normal modes for longitudinal vibration. Describe the physical meaning of the modes.

56. In Problem 45, you have considered a particle with unit mass moving in the circularly symmetric potential

\[
U(q) = \frac{1}{2}|q|^2 \left( 1 - \frac{1}{2}|q|^2 \right),
\]

23
where $q = (q_1, q_2)$ are the coordinates of the particle, and $|q| = \sqrt{q_1^2 + q_2^2}$. You have shown that, in the polar coordinates $q = (q \cos \theta, q \sin \theta)$, the Hamiltonian function of this particle is

$$H(q, p_q, \theta, p_\theta) = \frac{1}{2} \left( p_q^2 + \frac{p_\theta^2}{q^2} \right) + \frac{1}{2} q^2 \left( 1 - \frac{1}{2} q^2 \right),$$

where $p_q$ and $p_\theta$ are the momenta conjugate to the (generalized) coordinates $q$ and $\theta$. You have also shown the corresponding Hamilton’s equations to be

$$\dot{q} = p_q, \quad \dot{p}_q = -q \left( 1 - q^2 \right) + \frac{p_\theta^2}{q^3}, \quad \dot{\theta} = \frac{p_\theta}{q^2}, \quad \dot{p}_\theta = 0.$$

Observe that the system for the variables $q$ and $p_q$ is planar, and can thus be solved without the knowledge of $\theta$, and that $\theta$ can be computed by an integration as soon as the system for $q$ and $p_q$ has been solved.

(i) Sketch the phase portraits of the $q - p_q$ system for zero and nonzero values of $p_\theta$.

(ii) What geometric objects in the full $q - \dot{q}$ phase space do the periodic orbits in the diagram with $p_\theta \neq 0$ in the $q - p_q$ phase plane correspond to?

(iii) (This problem is very short and very hard.) For zero $p_\theta$, the phase portrait in the $q - p_q$ plane and the reconstruction of the trajectories in the $q - \dot{q}$ coordinates from those in the $q - p_q - \theta - p_\theta$ coordinates appear to run into a contradiction. What is this contradiction and how is it resolved?

57. Consider Newton’s equation $\ddot{x} = -U'(x)$, where $U(x)$ is some smooth function representing the potential energy of a particle with unit mass. Consider a periodic orbit with energy $E$, and assume that this orbit intersects the $x$-axis in the $x - \dot{x}$ phase plane only at the points $x_1(E)$ and $x_2(E) > x_1(E)$. Show that the period of this orbit equals

$$T = \int_{x_1(E)}^{x_2(E)} \frac{\sqrt{2} \, dx}{\sqrt{E - U(x)}}.$$

Sketch an appropriate figure.

58. Use the result of problem 57 (even if you didn’t show it) to compute the period of oscillations of the mass-spring system $m\ddot{x} + kx = 0$.

59. Show that the system

$$\dot{x} = -\omega y - \alpha x \left( x^2 + y^2 - a^2 \right), \quad \dot{y} = \omega x - \alpha y \left( x^2 + y^2 - a^2 \right),$$

$$\dot{u} = -\Omega v - \beta u \left( u^2 + v^2 - b^2 \right), \quad \dot{v} = \Omega u - \beta v \left( u^2 + v^2 - b^2 \right),$$

...
possesses an asymptotically stable invariant torus in its phase space. What are its two radii? For what parameter values are the solutions on this torus periodic? Dense?

HINT: Let \( x = r \cos \phi, \ y = r \sin \phi, \ u = \rho \cos \theta, \ v = \rho \sin \theta. \)

60. Find the equilibrium points and their stability, and sketch the phase portrait of the predator-prey system

\[
\dot{x} = x \left(1 - \frac{1}{2} y\right), \quad \dot{y} = y \left(-\frac{3}{4} + \frac{1}{4} x\right).
\]

HINT: Before drawing the trajectories, find an integral of motion.

61. Find the equilibrium points and their stability, and sketch the phase portrait of the competing species system

\[
\dot{x} = x \left(1 - x - y\right), \quad \dot{y} = y \left(\frac{3}{2} - x - y\right).
\]

Show that the straight line \( y = \frac{3}{2}(1 - x) \) is invariant under the flow of this system.

62. Find the equilibrium points and their stability, and sketch the phase portrait of the damped pendulum equation

\[
\ddot{x} + \beta \dot{x} + \sin x = 0.
\]

Interpret your results physically. What happens to the phase portrait at \( \beta = 2 \)?

HINT: First transform this equation into a system.

63. Show that for \( \beta > 0 \), the system

\[
\dot{x} = \left(\alpha + 1 - x^2 - y^2\right) y - \beta x \quad \dot{y} = \left(\alpha - 1 + x^2 + y^2\right) x - \beta y
\]

has no closed orbits.

64. Let \( r = \sqrt{x^2 + y^2}. \)

(a) Show that for the system

\[
\dot{x} = -x - \frac{y}{\ln r}, \quad \dot{y} = -y + \frac{x}{\ln r}
\]
the origin is a spiral sink, but that for its linearization, the origin is a stable star.

(b) Show that for the system

\[ \dot{x} = -y - xr, \quad \dot{y} = x - yr \]

the origin is a spiral sink, but that for its linearization, the origin is center.

(c) Show that for the system

\[ \dot{x} = -y + xr^2 \sin \frac{\pi}{r}, \quad \dot{y} = x + yr^2 \sin \frac{\pi}{r} \]

all the circles \( C_n \) around the origin with radii \( r_n = 1/n, \ n = 1, 2, \ldots \), are periodic orbits, and that all trajectories between two such circles tend towards one of them. Also show that the trajectories outside \( C_1 \) tend to infinity. Show that the corresponding linearization of this system about the origin has the origin as a center.

65. Sketch the bifurcation diagram of the system

\[ \dot{x} = \mu + \frac{1}{2}x - \frac{x}{1+x} \]

Determine what kind of bifurcations occur in this system, and at what values.

66. Sketch the bifurcation diagram of the system

\[ \dot{x} = x (\mu - e^x) \]

Determine what kind of bifurcations occur in this system, and at what values.

67. Sketch the bifurcation diagram of the system

\[ \dot{x} = x + \frac{\mu x}{1 + x^2} \]

Determine what kind of bifurcations occur in this system, and at what values.

68. Consider the biased van der Pol oscillator

\[ \ddot{x} + \mu (x^2 - 1) \dot{x} + x = a. \]

Find the curves in \( \mu - a \) space at which Hopf bifurcations occur.

69. Consider the system

\[ \dot{x} = \mu x - \sin x. \]
(a) For the case $\mu = 0$, find and classify all the equilibrium points, and sketch the phase portrait.

(b) Show that when $\mu > 1$, there is only one equilibrium point. What is its stability type?

(c) As $\mu$ decreases from $\infty$ to 0, classify all the bifurcations that occur.

(d) For $0 < \mu \ll 1$, find an approximate formula for the values of $\mu$ at which bifurcations occur.

(e) Classify all the bifurcations that occur as $\mu$ decreases from 0 to $-\infty$.

(f) Plot the bifurcation diagram for $-\infty < \mu < \infty$, and indicate the stability of the various branches of equilibrium points.