Problems for MATH-6300

Complex Analysis

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This list will change as the semester goes on. Please make sure you always have the newest version of it.

1. Prove the following

Theorem 1 *For every power series* $\sum_{n=0}^{\infty} a_n z^n$ *there exists a number* R*,* $0 \leq R \leq \infty$*, called the radius of convergence, with the following properties:*

(i) *The series converges absolutely for every* z *with* $|z| < R$*. If* $0 \leq \rho < R$ *the convergence is uniform for* $z \leq \rho$*.*

(ii) *If* $|z| > R$ *the terms of the series are unbounded, and the series is consequently divergent.*

(iii) *In* $|z| < R$ *the sum of the series is an analytic function. The derivative can be obtained by termwise differentiation, and the derived series has the same radius of convergence.*

The radius of convergence is given by the formula

$$\frac{1}{R} = \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} .$$

HINTS: (a) To prove convergence and uniform convergence, use an appropriately chosen geometric series as a majorant.

(b) To prove divergence, show that there are infinitely many unbounded terms.

(c) To prove that the derived series $\sum_{n=1}^{\infty} n a_n z^n$ has the same radius of convergence, first prove that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$ by setting $n^{\frac{1}{n}} = \delta_n$, and using the first two terms of the binomial theorem to show that $\delta_n^2 < 2/n$.

(d) Write $f(z) = \sum_{n=0}^{\infty} a_n z^n = s_n(z) + R_n(z)$, where $s_n(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}$, and $R_n(z) = \sum_{k=n}^{\infty} a_k z^k$. Write also $f_1(z) = \sum_{n=1}^{\infty} n a_n z^n = \lim_{n\to\infty} s_n(z)$. Use the identity

$$\frac{f(z) - f(z_0)}{z - z_0} - f_1(z_0) = \left(\frac{s_n(z) - s_n(z_0)}{z - z_0} - s_n'(z_0)\right) + (s_n'(z_0) - f_1(z_0)) + \left(\frac{R_n(z) - R_n(z_0)}{z - z_0}\right),$$

for $z \neq z_0$ and $|z|$, $|z_0| < \rho < R$. Estimate the three terms in this identity, in particular, estimate the last term by the inequality

$$\left|\frac{R_n(z) - R_n(z_0)}{z - z_0}\right| \le \sum_{k=n}^{\infty} k\,|a_k|\,\rho^k.$$

2. We say that an open subset of the complex plane is connected if it cannot be decomposed into two disjoint open sets. Prove

Theorem 2 *A nonempty open set in the plane is connected if and only if any two of its points can be joined by a polygon whose sides are parallel to the coordinate axes.*

3. Show the *Schwartz lemma:*

Theorem 3 *If $f(z)$ is analytic for $|z| < 1$ and satisfies the conditions $|f(z)| \le 1$, $f(0) = 0$, then $|f(z)| \le |z|$ and $|f'(0)| \le 1$. Equality holds if and only if $f(z) = cz$ with a constant c of absolute value 1.*

HINT: Apply the maximum principle to the function $f_1(z)$ which is equal to $f(z)/z$ for $z \neq 0$ and $f'(0)$ for $z = 0$.

4. Let $f(z)$ be analytic in the closed domain bounded by the contour C; $z_1, z_2, \ldots, z_n$ are arbitrary distinct points within C and $\omega_n(z) = (z - z_1)(z - z_2)\ldots(z - z_n)$. Show that the integral

$$P(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\omega_n(\zeta)}\,\frac{\omega_n(\zeta) - \omega_n(z)}{\zeta - z}\,d\zeta$$

is a polynomial of degree $(n-1)$ which is equal to $f(z)$ at the points $z_1, z_2, \ldots, z_n$.

5. The Hermite polynomials $H_n(z)$ are defined by the expansion

$$e^{2tz - t^2} = \sum_{n=0}^{\infty} \frac{H_n(z)}{n!} t^n.$$

Prove the following relations:

1. $H_{n+1}(z) - 2zH_n(z) + 2nH_{n-1}(z) = 0, \qquad (n \geq 1).$
2. $H_n'(z) = 2nH_{n-1}(z), \qquad (n \geq 1).$
3. $H_n''(z) - 2zH_n'(z) + 2nH_n(z) = 0, \qquad (n \geq 0).$
4. $H_n(z) = (-1)^n e^{z^2} \dfrac{\mathrm{d}^n e^{-z^2}}{\mathrm{d}z^n}.$

6. Let $f(z)$ be analytic and $f(z) \neq 0$ in the open, bounded subset Ω of the complex plane, and let $|f(z)|$ be continuous on the closure of Ω. Show that $|f(z)|$ attains its minimum at the boundary of Ω.

7. The Fibonacci numbers are defined by $c_0 = 0$, $c_1 = 1$, and $c_n = c_{n-1} + c_{n-2}$. Show that the c_n are Taylor coefficients of a rational function, and determine a closed expression for c_n.

8. Write down all possible Laurent series about $z = 0$ of the function

$$f(z) = \frac{1}{1-z^2} + \frac{1}{3-z},$$

and determine the regions in which they converge.

9. For all values of z and t, except $t = 0$, we have the Laurent series

$$\exp\left[\frac{1}{2}z\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(z)t^n,$$

where the Laurent coefficient

$$J_n(z) = \frac{1}{2\pi i}\oint_C \frac{1}{v^{n+1}} \exp\left[\frac{1}{2}z\left(v - \frac{1}{v}\right)\right]\, dv$$

is calculated over any positively-oriented contour encircling the origin, and is called the Bessel function of order n.

(a) Write $v = 2t/z$, use this substitution to expand the integrand in a power series in z. Integrate this series to obtain the Taylor series for $J_n(z)$ around the origin for any integer n.

(b) Show that $J_{-n}(z) = (-1)^n J_n(z)$.

(c) Either from the integral formula or the Taylor series in (a) show that $J_n(z)$ satisfies the differential equation

$$z^2\frac{\mathrm{d}^2 J_n(z)}{\mathrm{d}z^2} + z\frac{\mathrm{d}J_n(z)}{\mathrm{d}z} + (z^2 - n^2)J_n(z) = 0.$$

(d) Show that, when n is an integer,

$$J_n(y+z) = \sum_{n=-\infty}^{\infty} J_m(y) J_{n-m}(z).$$

(e) Show that if $r^2 = x^2 + y^2$

$$J_0(r) = J_0(x)J_0(y) + \sum_{n=1}^{\infty} (-1)^n J_{2n}(x) J_{2n}(y).$$

10. (a) Given the function

$$A(z) = \int_z^{\infty} \frac{e^{-1/t}}{t^2}\, dt$$

find a Laurent expansion in powers of z for $|z| > R$, $R > 0$. Why will the same procedure fail if we consider

$$E(z) = \int_z^{\infty} \frac{e^{-t}}{t}\, dt \text{ ?}$$

(b) A formal series for $E(z)$ in part (a) can be obtained by repeated differentiation by parts, that is,

$$\begin{aligned} E(z) &= \frac{e^{-z}}{z} - \int_z^{\infty} \frac{e^{-t}}{t^2}\, dt \\ &= \frac{e^{-z}}{z} - \frac{e^{-z}}{z^2} + \int_z^{\infty} \frac{2e^{-t}}{t^3}\, dt = \cdots \end{aligned}$$

If this procedure is continued, show that the result is

$$E(z) = \frac{e^{-z}}{z}\left(1 - \frac{1}{z} + \ldots + \frac{(-1)^n n!}{z^n}\right) + R_n(z),$$

with

$$R_n(z) = (-1)^{n+1}(n+1)! \int_z^{\infty} \frac{2e^{-t}}{t^{n+2}}\, dt.$$

Explain why the series obtained as $n \to \infty$ does not converge.

(c) In (b), consider $z = x$ real. Show that

$$|R_n(x)| \leq (n+1)! \frac{e^{-x}}{x^2}.$$

Explain how to approximate the integral $E(x)$ for large x, given some n, and why this approximation holds true for $\mathrm{Re}\, z > 0$. Why does the approximation fail as $n \to \infty$?

11. Investigate the singularities of the function

$$f(z) = \log\left(5 + \sqrt{\frac{z+1}{z-1}}\right).$$

12. Let $f(z)$ be analytic and single-valued in the ring-shaped region $r_1 \leq |z| \leq r_3$. Let $|z| = r_2$ be a circle satisfying $r_1 < r_2 < r_3$. Denote the maximun of $f(z)$ on r_j by $M(r_j)$, $j = 1,\, 2,\, 3$. Then prove *Hadamard's three-circles theorem*, which states that $\log M(r)$ is a convex function of $\log r$, in the sense that

$$\log M(r_2) \leq \frac{\log r_3 - \log r_2}{\log r_3 - \log r_1}\log M(r_1) + \frac{\log r_2 - \log r_1}{\log r_3 - \log r_1}\log M(r_3).$$

When would equality occur?

HINT: Consider $w(z) = z^s f(z)$, where s is real. Since $|z|^s$ is single-valued, the maximum-modulus theorem is easily seen to apply. Then choose s appropriately.

13. Draw the Riemann surface of the functions

$$f(z) = \Big[(x-1)(x-2)(x-3)(x-4)(x-5)\Big]^{\frac{1}{2}}.$$

and

$$g(z) = \Big[(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)\Big]^{\frac{1}{2}}.$$

Topologically, is there any difference between the two?

14. Show that the power series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(z-1)^n}{n} \quad \text{and} \quad i\pi - \sum_{n=1}^{\infty}\frac{(z+1)^n}{n}$$

are analytic continuations of one another.

HINT: Both lie on the same branch of $\log z$, so just find an appropriate path.

15. Show that the function

$$f(z) = \sum_{n=0}^{\infty} z^{n!}$$

tends to infinity along every radius whose argument is a rational multiple of π, and thus has the unit circle as a natural boundary.

16. Show that the sum of the series

$$\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})}$$

equals

$$\frac{1}{(1-z)^2}, \qquad \text{when} \quad |z|<1,$$

and

$$\frac{1}{z(1-z)^2}, \qquad \text{when} \quad |z|>1.$$

How is this fact connected with the theory of uniform convergence? Why type of a singularity does the above series have on the unit circle?

17. Comparing the coefficients in the Laurent developments of $\cot \pi z$ and of its expression as a sum of partial fractions, show that

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = 2^{2k-1} \frac{B_{2k}}{(2k)!} \pi^{2k}.$$

Here, B_{2k} are the Bernoulli numbers that we discussed in class.

18. Using the infinite product representation for $\sin z$, show that

$$\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right) = \frac{1}{2}.$$

19. For integer m, let

$$L(w,m) = \frac{1}{w-1} + 1 + w + w^2 + \cdots + w^{m-1}.$$

Prove directly (without quoting the theorem proven in class) the following

Theorem 4 *Let $\{z_k\}$ and $\{a_k\}$ be sequences with z_k distinct, $|z_k| \to \infty$ as $k \to \infty$, and m an integer such that*

$$\sum_{j=1}^{\infty} \frac{|a_j|}{|z_j|^{m+1}} < \infty.$$

Then the most general meromorphic function whose only singularities are simple poles at $z_0 = 0$ and z_k with residues a_k, for $k = 0, 2, \ldots$, is given by the formula

$$f(z) = \frac{a_0}{z - z_0} + \sum_{n=1}^{\infty} \left(\frac{a_n}{z_n}\right) L\left(\frac{z}{z_n}, m\right) + h(z),$$

where $h(z)$ is an entire function.

HINT: Show that

$$L(w, m) = \frac{w^m}{w - 1},$$

and so for $|w| < 1/2$ the estimate $L(w, m) \leq 2|w|^m$ holds. If $|z| < R$ and for all j such that $|z_j| < 2R$, deduce that

$$\frac{a_j}{z_j} L\left(\frac{z}{z_j}, m\right) \leq 2\frac{2R^m|a_j|}{|z_j|^{m+1}}.$$

20. Euler's *gamma function* $\Gamma(z)$ is defined by the integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

(a) Show that $\Gamma(z)$ is analytic at least in the half-plane $z > 0$, that $\Gamma(z + 1) = z\Gamma(z)$, and that $\Gamma(n + 1) = n!$.

(b) Using

$$e^{-t} = \lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n$$

prove

$$\Gamma(z) = \lim_{n\to\infty} n^z \int_0^1 (1 - \tau)^n \tau^{z-1} d\tau.$$

Justify the interchanging of limits. Then integrate by parts to obtain

$$\Gamma(z) = \frac{1}{z} \prod_1^\infty \left[\left(1 + \frac{1}{n}\right)^z \left(1 + \frac{z}{n}\right)^{-1}\right].$$

(c) Show that

$$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right)$$

exists.

(d) Use (b) and (c) to show that

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_1^\infty \left[\left(1 + \frac{z}{n}\right) e^{-z/n}\right].$$

Conclude that $\Gamma(z)$ is meromorphic in the entire plane, with poles at $z = -n$, $n = 0, 1, 2, 3, \ldots$, and is without zeros.

(e) Reason that the formula $\Gamma(z+1) = z\Gamma(z)$ holds for all z except $z = -n$, $n = 0, 1, 2, 3, \ldots$.

(f) Use (a), (d), and the infinite product expansion of $\sin z$ to show that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

Deduce that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

(g) Show that the residue of $\Gamma(z)$ at $z = -n$ is $(-1)^n/n!$.

(h) Show that

$$\frac{d}{dz}\frac{\Gamma'(z)}{\Gamma(z)} = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}.$$

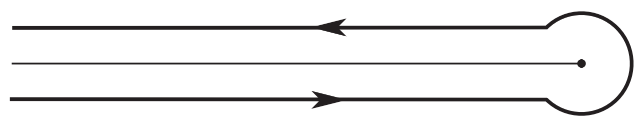
(i) Show that $\Gamma(z)\Gamma(z+\frac{1}{2})$ and $\Gamma(2z)$ have the same poles and use (a), (f), and (h) to deduce that

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{2z-1}\Gamma(2z).$$

(j) Show that

$$\Gamma(z) = \frac{1}{2i \sin \pi z} \int_C t^{z-1} e^t \, dt, \tag{1}$$

where C is the path in the figure encircling the origin, for all z not equal to a positive integer.



HINT: First assume $\operatorname{Re} z > 0$. For large negative real parts of z, estimate the integral along short vertical segments emanating from the negative real axis. Conclude that the integration path C can be deformed into the union of the two half-rays $\{ve^{\pm i\pi} \mid \rho < v < \infty\}$, taken in the appropriate directions, and the circle $\{\rho e^{i\theta} \mid -\pi < \theta < \pi\}$. Show that the integrals on the two rays give you (1), and that the integral along the circle vanishes as $\rho \to 0$. Then argue that the integral on the right-hand side of (1) must exist for all z.

(k) Use (j) and (f) to show that

$$\frac{1}{\Gamma(z)} = \frac{i}{2\pi} \int_C t^{z-1} e^t \, dt.$$

(l) Compute

$$\int_{-\infty}^{\infty} e^{-x^2} \, dx.$$

21. Show that

$$\int_0^\infty \frac{\cosh ax}{\cosh \pi x} dx = \frac{1}{2}\sec\left(\frac{a}{2}\right), \qquad |a| < \pi.$$

HINT: Use a rectangular contour with corners $\pm R$ and $\pm R + i$.

22. (a) Use principal value integrals to show that

$$\int_0^\infty \frac{\cos kx - \cos mx}{x^2} = \frac{\pi}{2}(|m| - |k|).$$

HINT: The function $(e^{ikz} - e^{imz})/z^2$ has a simple pole at the origin.

(b) Let $k = 2$, $m = 0$ to deduce that

$$\int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}.$$

23. Show that

$$\int_0^\infty \frac{x^{-k}}{x^2 + 2x\cos\phi + 1} dx = \frac{\pi}{\sin k\pi} \frac{\sin(k\phi)}{\sin\phi}$$

for $0 < k < 1$, $0 < \phi < \pi$.

24. A sequence $\{\Gamma_n\}$ of nested closed contours is called *regular*, if they all contain the origin, and if $S_n/d_n < C$, where S_n is the length of Γ_n, and d_n the shortest distance between the origin and Γ_n.

(a) Show the following Theorem: Let the meromorphic function $f(z)$ be analytic at $z = 0$, and let all its poles b_ν, $\nu = 1, 2, 3, \cdots$, be simple. Furthermore, let all the b_ν satisfy the inequalities $|b_1| \le |b_2| \le |b_3| \le \cdots$. If there exists a positive constant M such that $|f(z)| \le M$ on a regular sequence of contours $\{\Gamma_n\}$, then

$$f(z) = f(0) + \sum_{\nu=1}^\infty A_\nu \left(\frac{1}{z - b_\nu} + \frac{1}{b_\nu}\right),$$

where A_ν is the residue of $f(z)$ at $z = b_\nu$. This series converges uniformly on every compact subset of the complex plane with the poles of $f(z)$ deleted.

HINT: Use the integrals

$$I_n(z) = \frac{1}{2\pi i} \oint_{\Gamma_n} \frac{f(\zeta)d\zeta}{\zeta(\zeta - z)}.$$

(b) Use (a) to show that

$$\cot z = \frac{1}{z} + 2z \sum_{n=1}^\infty \frac{1}{z^2 - n^2\pi^2}.$$

HINT: Use

$$f(z) = \cot z - \frac{1}{z},$$

and let Γ_n be the square with corners $(n + 1/2)\pi(\pm 1 \pm i)$.

(c) Deduce from (b) that

$$\sin z = z \prod_{n \neq 0} \left(1 - \frac{z}{n\pi}\right) e^{z/n\pi} = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right).$$

(d) Express explicitly the function represented by the product

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{n^2\pi^2}\right).$$

25. The *Riemann zeta function* is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad s = \sigma + i\tau.$$

(a) If $\operatorname{Re} s > 1$, show that

$$\frac{1}{\zeta(s)} = \prod_{n=1}^{\infty} \left(1 - \frac{1}{p_n^s}\right),$$

where $\{p_n\}$ is the ascending sequence of all the prime numbers.

HINT: To show the convergence of the product on the right-hand side, use the appropriate theorem linking this convergence to the convergence of an appropriate series. To show the equality, show that

$$\zeta(s)(1 - 2^{-s})(1 - 3^{-s}) \cdots (1 - p_N^{-s}) = \sum m^{-s},$$

where the sum runs over all the integers that contain none of the prime factors 2, 3, ..., p_N.

(b) In (a), we have tacitly assumed that there are infinitely many primes. If you do not assume this fact, reason as in (a) to prove it.

HINT: If not, show that you would have $\lim_{s \to 1} \zeta(s) < \infty$.

(c) Show that

$$\Gamma(s)\zeta(s) = \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx.$$

HINT: Replace x by nx in the integral

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\, dx$$

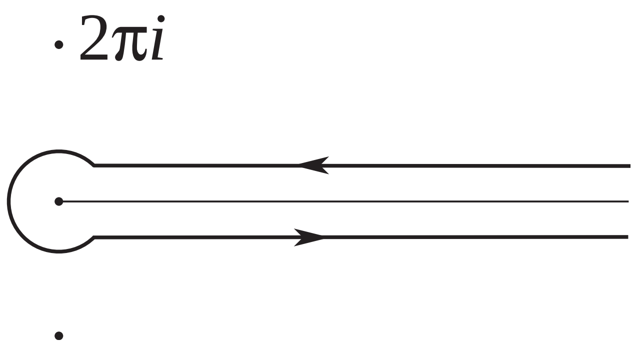
and sum the appropriate series. Justify the

(d) Show that

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i} \oint_C \frac{(-z)^{s-1}}{e^z - 1} dz,$$

where $(-z)^{s-1}$ is defined on the complement of the real axis as $e^{(s-1)\log(-z)}$ with $-\pi < \operatorname{Im} m \log(-z) < \pi$, and C is the path shown in the figure.

HINT: Let the radius of the circle go to 0. Also, somewhere in the calculation, use part (c) and the formula $\Gamma(s)\Gamma(1-s) = \pi/\sin \pi s$.



(e) Conclude that the ζ-function is meromorphic in the whole plane and its only pole is a simple pole at $s = 1$ with residue 1.

HINT: Argue that the zeros and poles of $\Gamma(s)$ and the integral on the right-hand side of the formula in part (d) cancel when s is a positive integer. At $s = 1$, use problem 20 (g) and compute the integral by residues to show that it is 1.

26. Determine explicitly the largest disks about the origin whose respective images under the mappings $w = z^2 + z$ and $w = e^z$ are one-to-one.

27. Use the open mapping theorem to prove the maximum modulus theorem.

28. Use Rouché's theorem to show the fundamental theorem of algebra.

29. (i) Let $w = f(z)$ be analytic near the point $z = z_0$ and let $f(z_0) = w_0$. If $f'(z_0) \neq 0$, show that the inverse function $z = f^{-1}(w)$ is given by

$$z = f^{-1}(w) = z_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \left\{ \frac{d^{n-1}}{dz^{n-1}} \Big[\psi(z)\Big]^n \right\}_{z=z_0} (w - w_0)^n,$$

where

$$\psi(z) = \frac{z - z_0}{f(z) - w_0}.$$

(ii) Show that

$$\frac{1}{\sqrt{1 - 2tw + w^2}} = \sum_{n=0}^{\infty} P_n(t) w^n,$$

where the *Legendre polynomials* $P_n(t)$ are given by the formula

$$P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n} (t^2 - 1)^n.$$

HINT: Let

$$w = f(z) = 2\frac{z - t}{z^2 - 1},$$

and set $z_0 = t$ and $w_0 = 0$. Show that

$$z = f^{-1}(w) = t + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\frac{d^{n-1}}{dz^{n-1}} \left(\frac{z^2 - 1}{2} \right)^n \right]_{z=t} w^n,$$

and also compute directly that

$$z = f^{-1}(w) = \frac{1 - \sqrt{1 - 2tw + w^2}}{w}.$$

Finally, compute $\partial z / \partial t$ from both expressions.

(iii) Show that the solution to Kepler's equation

$$z - \tau = w \sin z$$

is given by

$$z = \tau + \sum_{n=1}^{\infty} \left[\frac{d^{n-1}}{dz^{n-1}} (\sin z)^n \right]_{z=\tau} w^n.$$

30. (i) If $f(z)$ is analytic in the whole plane, real on the real axis, and purely imaginary on the imaginary axis, show that $f(z)$ is odd.

(ii) If $f(z)$ is analytic in a symmetric region Ω, show that it can be written in the form $f_1 + i f_2$, where f_1 and f_2 are analytic in Ω and real on the real axis.

(iii) If $f(z)$ is analytic in $|z| \leq 1$ and $|f| = 1$ on $|z| = 1$, show that $f(z)$ is rational.

31. (i) Show that the most general linear fractional transformation of the unit circle $\{|z| < 1\}$ onto itself is given by

$$f(z) = e^{i\lambda} \frac{z - a}{\bar{a} z - 1},$$

with $\lambda \in \mathbb{R}$ and $|a| < 1$.

HINT: Inverse points a and $1/\bar{a}$ must be mapped into 0 and ∞, respectively.

(ii) Show that the most general linear fractional transformation of the circle $\{|z| < \rho\}$ onto the circle $\{|w| < r\}$ is given by

$$w = \rho r\, e^{i\lambda} \frac{z-a}{\bar{a}z - \rho^2},$$

with $\lambda \in \mathbb{R}$ and $|a| < \rho$.

(iii) Show that the most general linear fractional transformation that maps the upper half-plane $\operatorname{Im} z > 0$ onto the unit circle $\{|w| < 1\}$ is given by the formula

$$f(z) = e^{i\lambda} \frac{z-a}{z-\bar{a}},$$

where $\lambda \in \mathbb{R}$ and $\operatorname{Im} a > 0$.

HINT: Conjugate values of some z must map into 0 and ∞, respectively.

32. (i) Show that a one-to-one conformal mapping of $\{|z| < 1\}$ onto itself that preserves the origin must be a linear function of the form $f(z) = e^{i\alpha} z$, with $\alpha \in \mathbb{R}$.

HINT: Use the Schwartz lemma (Problem 3) and the fact that an analytic function with constant modulus must be a constant.

(ii) Use (i) to show that any one-to-one conformal mapping of $\{|z| < 1\}$ onto itself must be a linear fractional transformation.

33. Prove the following

Theorem 5 *If $f(z)$ is analytic for $|z| < 1$, $\operatorname{Re}[f(z)] > 0$, and $f(0) = a > 0$, then $f'(0) \leq 2a$.*

HINT: Compose f with a linear fractional transformation ϕ so that the function $g = \phi \circ f$ maps $\{|z| < 1\}$ onto itself and $g(0) = 0$. Cauchy's formula on circles with radii $1 - \varepsilon$ shows $|g'(0)| \leq 1$. Invert ϕ to find $f(z)$ in terms of $g(z)$, and relate the corresponding derivatives.

34. Show that the mapping

$$w = -\frac{R}{\alpha} \frac{z-\alpha}{z-\beta}$$

for some real α and β, maps the region contained between the circles $|z-a| = r$ and $|z| = R$, $0 < a < R - r$, conformally onto some annulus $\rho < |w| < 1$. Compute that

$$\alpha = \frac{1}{2a}\left(R^2 + a^2 - r^2 - A\right), \qquad \beta = \frac{R^2}{\alpha},$$

where

$$A = \left(R^2 + a^2 - r^2\right)^2 - 4a^2 R^2,$$

and

$$\rho = \left|\frac{R}{\alpha}\right| \sqrt{\frac{\alpha - a}{\beta - a}}.$$

HINT: The equation $|z - \alpha| = k|z - \beta|$, $k > 0$ is the equation of a circle with respect to which α and β are symmetric to one-another. You can choose $\alpha, \beta \in \mathbb{R}$ (why?) and vary k so that $|z - a| = r$ and $|z| = R$ are part of a family of circles with this property. This will let you compute α and β. Finally, choose $k = -R$.

35. Prove the following

Theorem 6 *The only one-to-one conformal mapping of the extended plane onto itself is the linear fractional transformation.*

HINT: First show that a one-to-one conformal mapping $f(z)$ of the finite plane onto itself must be the linear transformation as follows:

(i) Argue that $f'(z) \neq 0$ in $\mathbb{C}$, since $f(z) - f(z_0)$ would have at least two distinct roots near any z_0 with $f'(z_0) = 0$.

(ii) Use the open mapping theorem to show that the image of $|z| < 1$ contains some disk $|w - w_0| < A$ and deduce from the Weierstrass theorem that ∞ cannot be an essential singularity for $f(z)$.

(iii) The function $f(z)$ must have a pole at ∞, and so must be a polynomial. Since $f'(z) \neq 0$, it must be linear.

(iv) Show that the general case follows by applying the transformation $\zeta = 1/(z - z_0)$, where z_0 is the points that is mapped to ∞.

36. Prove that in any region Ω the family of analytic functions with positive real part is normal. Under what added condition is it locally bounded?

HINT: Consider the functions e^{-f}.

37. Show that the functions z^n, where n is a nonnegative integer, form a normal family in $|z| < 1$ and also in $|z| > 1$, but not in any region that contains a point on the unit circle.

38. If the domain Ω contains the point at infinity and is mapped onto the interior of the unit circle in such a way as to make the points $z = \infty$ and $w = 0$ correspond to each other, show that the mapping function $w = f(z)$ has the expansion

$$f(z) = \sum_{n=1}^{\infty} \frac{a_n}{z^n}, \qquad a_1 \neq 0,$$

which converges in the exterior of a sufficiently large circle.

39. Suppose $f(z)$ is a one-to-one conformal mapping of $|z| < 1$ onto a square with center at 0, and $f(0) = 0$. Prove that $f(iz) = if(z)$. If $f(z) = \sum_{n=0}^{\infty} c_n z^n$, prove that $c_n = 0$ unless $n - 1$ is a multiple of 4. Generalize this to regular n-gons.

40. Let Ω be a simply connected region which is not the whole plane, z_0 a point in Ω, and $f(z)$ the unique, one-to-one, conformal mapping of Ω onto $|w| < 1$ such that $f(z_0) = 0$ and $f'(z_0) > 0$. If z_0 is real and Ω is symmetric with respect to the real axis, prove by the uniqueness that f satisfies the symmetry principle $f(\bar{z}) = \overline{f(z)}$.

41. Show that

$$w = \int_0^z \frac{d\zeta}{\sqrt{\zeta(1-\zeta^2)}}$$

maps the upper half-plane $\operatorname{Im} z > 0$ onto the interior of the square of side length

$$\frac{1}{2\sqrt{2\pi}}\ \Gamma^2\left(\frac{1}{4}\right).$$

Compare this function to the result of Problem 39.

42. Show that

$$w = \int_0^z \frac{d\zeta}{(1-\zeta^n)^{n/2}}$$

maps $|z| < 1$ onto the inside of a regular n-gon whose side length is

$$\frac{1}{n} 2^{1-\frac{4}{n}} \frac{\Gamma^2\left(\frac{1}{2} - \frac{1}{n}\right)}{\Gamma\left(1 - \frac{2}{n}\right)}.$$

Compare this function to the result of Problem 39.

43. Show that

$$w = \int_0^z \frac{(1-\zeta^5)^{2/5}}{(1+\zeta^5)^{4/5}}\, d\zeta$$

maps the unit circle $|z| < 1$ onto the inside of a five-pointed star with the acute internal angle $\pi/5$ and the reflex internal angle $7\pi/5$. Show that the radius R of the circumscribed circle equals

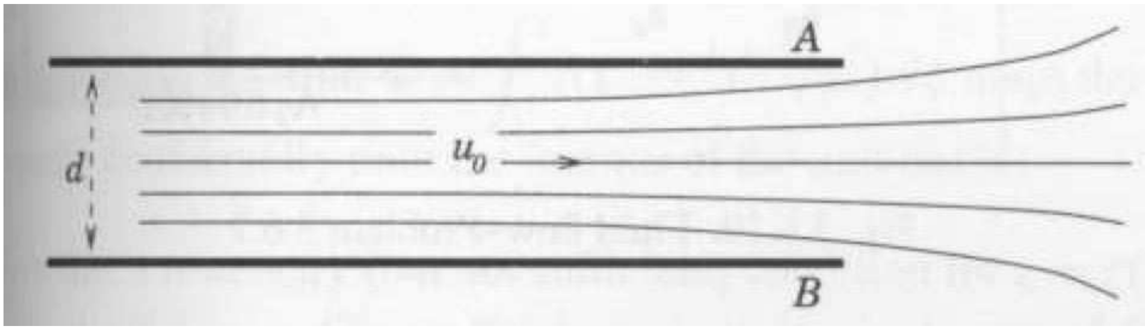
$$R = \int_{-1}^{0} \frac{(1-t^5)^{2/5}}{(1+t^5)^{4/5}}\, dt = \frac{\Gamma\left(\frac{1}{10}\right)\Gamma\left(\frac{1}{5}\right)}{5 \cdot 2^{2/5}\, \Gamma\left(\frac{3}{10}\right)}.$$

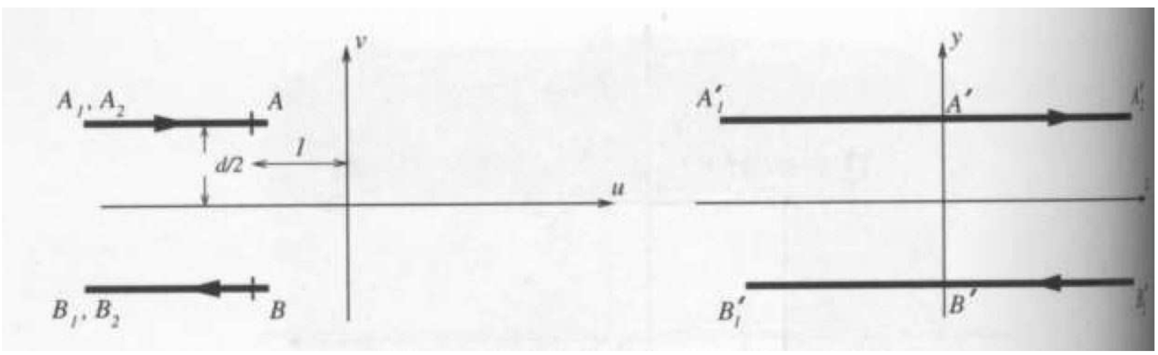
HINT: For the last equality, use the substitution

$$u = \left(\frac{1+t^5}{1-t^5}\right)^2.$$

44. A fluid flows with initial velocity u_0 through a semiinfinite channel of width d and emerges through the opening AB of the channel (see the top figure). Find the speed of the fluid at any point.

HINT: First show that the conformal mapping $w = z + e^{2\pi z/d}$ maps the channel $|y| < d/2$ onto the w-plane excluding slits, as indicated in the bottom figure.





45. Derive the complex potential $\Omega(z)$ for flow past an elliptic cylinder with semiaxes a and b, where the flow has velocity $U + iV$ at infinity, and circulation γ.

HINT: First show that

$$w = \frac{1}{a+b}\left(z + \sqrt{z^2 - c^2}\right), \qquad c^2 = a^2 - b^2,$$

(choose the positive square root) maps the exterior of the ellipse with semiaxes a and b onto the exterior of the circle $|w| = 1$.

ANSWER:

$$\Omega(z) = \frac{\gamma}{2\pi i}\log\left(z + \sqrt{z^2 - c^2}\right) + \frac{U}{a-b}\left(az - b\sqrt{z^2 - c^2}\right) + \frac{iV}{a-b}\left(bz - a\sqrt{z^2 - c^2}\right).$$

46. Find the flow past an infinitely long and infinitely thin vertical wall of height s, if the velocity far away from this wall is u_0 in the horizontal direction.

HINT: Use the Schwartz-Christoffel transformation to show that the upper half-plane with a "slit" between 0 and is is mapped onto the upper half of the z plane by the (inverse of the) function

$$w = s\sqrt{z^2 - 1}.$$

47. Consider Bessel's equation

$$z^2 w'' + z w' + (z^2 - \nu^2) w = 0.$$

(i) Show that one solution is always the Bessel function of order ν,

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu + n + 1)} \left(\frac{z}{2}\right)^{2n+\nu}.$$

Show that, if ν is not an integer of a half-integer, a second linearly independent solution is $J_{-\nu}(z)$.

(ii) For the Bessel function $J_\nu(z)$, show the relations

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z),$$

and

$$\frac{d}{dz}\left[z^{-\nu} J_\nu(z)\right] = -z^{-\nu} J_{\nu+1}(z).$$

Conclude by induction that

$$z^{-\nu-n} J_{\nu+n}(z) = (-1)^n \left(\frac{d}{z dz}\right)^n \left[z^{-\nu} J_\nu(z)\right].$$

(iii) If ν is a non-negative integer, show that a second linearly independent solution is the Neumann function of order ν,

$$Y_\nu(z) = 2J_\nu(z)\left[\log\left(\frac{z}{2}\right) + \gamma\right] - \sum_{n=0}^{\nu-1} \frac{(\nu - n - 1)!}{n!} \left(\frac{z}{2}\right)^{-\nu+2n}$$
$$- \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(\nu+n)!} \left(\sum_{k=1}^{\nu+n} \frac{1}{k} + \sum_{k=1}^{n} \frac{1}{k}\right) \left(\frac{z}{2}\right)^{2n+\nu}.$$

Show also that

$$Y_\nu(z) = \lim_{\mu \to \nu} \frac{J_\mu(z) \cos \pi\mu - J_{-\mu}(z)}{\sin \pi\mu}.$$

(iv) Show that

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z, \qquad J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cos z,$$

then use part (ii) or the series representation to show that, for positive integer n,

$$J_{n+\frac{1}{2}}(z) = \frac{(-1)^n (2z)^{n+\frac{1}{2}}}{\sqrt{\pi}} \left(\frac{d}{dz^2} \right)^n \frac{\sin z}{z}.$$

Show that

$$\frac{(2z)^{n+\frac{1}{2}}}{\sqrt{\pi}} \left(\frac{d}{dz^2} \right)^n \frac{\cos z}{z}$$

also solves the equation for $J_{n+\frac{1}{2}}(z)$, and that, in fact, this function is $J_{-n-\frac{1}{2}}(z)$.