

Homework Problems

Find the general solutions of the following first-order linear differential equations:

1. $y' + y = xe^{-x} + 1.$

2. $xy' + 2y = \sin x.$

3. $y' + 2xy = 2xe^{-x^2}.$

4. $(1 + x^2)y' + 4xy = \frac{1}{(1 + x^2)^2}.$

Find the solutions to the following problems:

5. $y' + \frac{2}{x}y = \frac{\cos x}{x^2}, \quad y(\pi) = 0.$

6. $x^2y' + 3xy = \frac{\sin x}{x}.$

7. $xy' + y = e^x, \quad y(1) = 1.$

8. $y' + y = \frac{1}{1 + x^2}, \quad y(0) = 0.$

9. Find the general solution of the equation

$$y' - \frac{1}{x}y = x.$$

What happens to all the solutions as $x \rightarrow 0$?

Find the solutions of the given differential equations:

10. $y' = \frac{x^2}{y}.$

11. $y' + y^2 \sin x = 0.$

12. $xy' = \sqrt{1 - y^2}.$

Find the **explicit** solutions to the initial value problems:

13. $y' = \frac{x(x^2 + 1)}{4y^3}, \quad y(0) = -\frac{1}{\sqrt{2}}.$

14. $y' = \frac{x^2}{y(1 + x^3)}, \quad y(0) = -1.$

15. Einsteinium-253 decays at a rate proportional to the amount present. Determine the half-life τ , if this material loses one third of its mass in 11.7 days.

16. Suppose that 100 mg of thorium-234 are initially present in a closed container, and that thorium-234 is added to the container at a constant rate of 1 mg/day.

(a) Find the amount $Q(t)$ of thorium-234 in the container at any time, given that its decay rate is $0.02828 \text{ days}^{-1}$.

(b) Find the limiting amount Q_1 of thorium-234 in the container as $t \rightarrow \infty$.

(c) How long a time period must elapse before the amount of thorium-234 in the container drops to within 0.5 mg of the limiting value Q_1 ?

(d) If thorium-234 is added to the container at a rate of k mg/day, find the value of k that is required to maintain a constant level of 100 mg of thorium-234.

17. A tank initially contains 120 liters of pure water. A mixture containing γ g/liter of salt enters the tank at a rate 2 liters/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of γ for the amount of salt in the tank at any time t . Also, find the limiting amount of salt in the tank as $t \rightarrow \infty$.

18. Suppose that a room containing 120 ft^3 of air is originally free of carbon monoxide. Beginning at time $t = 0$ cigarette smoke, containing 4% of carbon monoxide, is introduced to the room at a rate of $0.1 \text{ ft}^3/\text{min}$, and the well-circulated mixture is allowed to leave the room at the same rate.

(a) Find an expression for the concentration $x(t)$ of carbon monoxide in the room at any time $t > 0$.

(b) Extended exposure to a carbon monoxide concentration as low as 0.00012 is harmful to the human body. Find the time τ at which this concentration is reached.

In each of the following two problems sketch dN/dt versus N , determine the equilibrium points and classify each one as stable or unstable:

19. $\frac{dN}{dt} = aN + bN^2, \quad a, b > 0.$ 20. $\frac{dN}{dt} = N(N - 1)(N - 2).$

21. **Semistable Equilibrium Solutions:** Sometimes a constant equilibrium solution has the property that solutions lying on one side of the equilibrium solution tend to approach it, whereas solutions lying on the other side recede from it. In this case the equilibrium solution is said to be **semistable**.

(a) Consider the equation

$$\frac{dN}{dt} = k(1 - N)^2, \quad (1)$$

where k is a positive constant. Show that $N = 1$ is the only equilibrium point, with the corresponding equilibrium solution $\phi(t) = 1$.

(b) Sketch dN/dt versus N . Show that N is increasing as a function of t for $N < 1$ and also for $N > 1$. Thus solutions below the equilibrium solution approach it while those above it grow further away. Thus $\phi(t) = 1$ is semistable.

(c) Solve Equation (1) subject to the initial condition $N(0) = N_0$, and confirm the conclusion reached in part (b).

In each of the following two problems sketch dN/dt versus N and determine the equilibrium points. Also classify each equilibrium point as stable, unstable, or semistable:

22. $\frac{dN}{dt} = N(1 - N^2).$

23. $\frac{dN}{dt} = N^2(4 - N^2).$

24. Consider the following model of a fishery. Let us assume that the fish are caught at a constant rate h independent of the size of the fish population $N(t)$. Then N satisfies the equation

$$\frac{dN}{dt} = r \left(1 - \frac{N}{K}\right) N - h, \quad (2)$$

where r and K are some positive constants.

(a) If $h < rK/4$, show that (2) has two equilibrium points N_1 and N_2 with $N_1 < N_2$; determine these points.

(b) Show that N_1 is unstable and N_2 is stable.

(c) From a plot of dN/dt versus N show that if the initial population $N_0 > N_1$, then $N(t) \rightarrow N_2$ as $t \rightarrow \infty$, but if $N_0 < N_1$, then $N(t)$ decreases as t increases. Note that $N = 0$ is not an equilibrium point, so if $N_0 < N_1$, the extinction will be reached in a finite time.

(d) If $h > rK/4$, show that $N(t)$ decreases to zero as t increases regardless of the value of N_0 .

(e) If $h = rK/4$, show that there is a single equilibrium point $N = K/2$, and that this equilibrium point is semistable. Notice that $h_m = rK/4$ is the maximal sustainable yield of the fishery corresponding to the equilibrium value of $N = K/2$. The fishery is considered overexploited if N is reduced to a level below $K/2$.

In each of the following problems find the general solution of the given differential equation:

25. $y'' + 2y' - 3y = 0.$

26. $y'' + 5y' = 0.$

27. $y'' - 2y' - 2y = 0.$

28. Find the solution of the initial value problem

$$y'' + 8y' - 9y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Sketch the graph of this solution and describe its behavior as x increases.

In each of the following problems find the general solution of the given differential equation:

29. $y'' - 2y' + 6y = 0.$ 30. $y'' + 2y' + 2y = 0.$ 31. $y'' + 6y' + 13y = 0.$

32. Find the solution of the initial value problem

$$y'' + 4y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Sketch the graph of this solution and describe its behavior as x increases.

In each of the following two problems find the general solution of the given differential equation:

33. $y'' - 2y' + y = 0.$ 34. $y'' - 6y' + 9y = 0.$

Find the solution of the following initial value problems:

35. $y'' + y' - 2y = 2x, \quad y(0) = 0, \quad y'(0) = 1.$

36. $y'' + 4y = x^2 + 3e^x, \quad y(0) = 0, \quad y'(0) = 2.$

37. $y'' - 2y' + y = xe^x + 4, \quad y(0) = 1, \quad y'(0) = 1.$

In the following problems, determine the suitable **form** for the particular solution. You do not need to evaluate the constants.

38. $y'' + 3y' = 2x^4 + x^2e^{-x} + \sin 3x.$

39. $y'' + 2y' + 2y = 3e^{-x} + 2e^{-x} \cos x + 4e^{-x}x^2 \sin x.$

40. $y'' - 4y' + 4y = 2x^2 + 4xe^{2x} + x \sin 2x.$

In the following three problems, find the general solution to the given differential equation:

41. $y'' + 2y' = 3 + 4 \sin 2x.$ 42. $2y'' + 3y' + y = x^2 + 3 \sin x.$ 43. $\ddot{u} + \omega_0^2 u = \cos \omega_0 t.$

In the following problems, determine the suitable **form** for the particular solution. You do not need to evaluate the constants.

44. $y'' + y = x(1 + \sin x)$.

45. $y'' - 5y' + 6y = e^x \cos 2x + e^{2x}(3x + 4) \sin x$.

In each of the following problems, given one solution, use reduction of order to find the general solution of the given differential equation:

46. $x^2y'' + 2xy' - 2y = 0$, $y_1(x) = x$.

47. $xy'' - x(x + 2)y' + (x + 2)y = 0$, $y_1(x) = x$.

48. $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$, $y_1(x) = \frac{\sin x}{\sqrt{x}}$.

Find the general solutions of the following Euler's equations

49. $x^2y'' + 4xy' + 2y = 0$.

50. $(x - 1)^2y'' + 8(x - 1)y' + 12y = 0$.

51. $2x^2y'' - 4xy' + 6y = 0$.

52. $x^2y'' - 5xy' + 9y = 0$.

53. Find the solution of the initial-value problem

$$4x^2y'' + 8xy' + 17y = 0, \quad y(1) = 2, \quad y'(1) = -3.$$

Sketch this solution, and discuss its behavior as $x \rightarrow 0$.

In each of the following problems use variation of parameters to find the general solution of the given differential equation:

54. $y'' + y = \tan x$.

55. $y'' + 9y = 9 \sec^2 3x$.

56. $y'' + 4y = 3 \csc 2x$.

57. Find the general solution of the differential equation

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 3x^{\frac{3}{2}} \sin x, \quad x > 0,$$

given that two linearly independent solutions of the homogeneous equations are

$$y_1(x) = \frac{\sin x}{\sqrt{x}}, \quad y_2(x) = \frac{\cos x}{\sqrt{x}}.$$

In each of the following two problems determine ω_0 , R , and δ so as to write the given expression in the form $u = R \cos(\omega_0 t - \delta)$.

58. $u = 3 \cos 2t + 4 \sin 2t.$

59. $u = -\cos t + \sqrt{3} \sin t.$

60. A mass weighing 16 lb stretches a spring 3 in. The mass is attached to a viscous damper with a damping constant of 2 lb-sec/ft. If the mass is set in motion from its equilibrium position with a downward velocity of 3 in/sec, find its position u at any time t . Draw u versus t . Determine when the mass first returns to its equilibrium position. Also, find the time τ such that $|u(t)| < 0.01$ in for all $t > \tau$. (Recall that $g = 32 \text{ ft/sec}^2$.)

The positions of certain mass-spring systems satisfy the following initial value problems

61. $\ddot{u} + 2u = 0, \quad u(0) = 0, \quad \dot{u}(0) = 2.$

62. $\ddot{u} + \frac{1}{4}\dot{u} + 2u = 0, \quad u(0) = 0, \quad \dot{u}(0) = 2.$

(a) Find the solutions of these initial value problems.

(b) Draw u versus t and \dot{u} versus t on the same axes. (c) Draw \dot{u} versus u ; that is, draw $u(t)$ and $\dot{u}(t)$ parametrically with t as the parameter. What is the direction on this curve as t increases? Identify several corresponding points on the curves in parts (b) and (c).

63. A mass of 5 kg stretches a spring 10 cm. The mass is acted on by an external force of $10 \sin(t/2)$ N (newtons= $\text{kg}\cdot\text{m}/\text{sec}^2$) and moves in a medium that imparts a viscous force of 2 N when the speed is of the mass is 4 cm/sec.

(a) If the mass is set in motion from its equilibrium position with an initial velocity of 3 cm/sec, formulate the initial value problem describing the motion of the mass, and find its solution.

(b) Identify the transient and steady-state parts of the solution.

(c) Draw the graph of the solution, as well as the steady-state solution.

(d) If the given external source is replaced by a force $2 \cos \omega t$ of frequency ω , find the value of ω for which the amplitude of the forced response is maximum.

64. Consider a vibrating system described by the initial value problem

$$\ddot{u} + \frac{1}{4}\dot{u} + 2u = 2 \cos \omega t, \quad u(0) = 0, \quad \dot{u}(0) = 2.$$

(a) Determine the steady part of the solution of this problem.

(b) Find the amplitude R of the steady state solution in terms of ω .

(c) Find the maximum value of R and the frequency ω for which it occurs.

(d) Draw R versus ω .

In each of the following problems, find the eigenvalues and eigenfunctions of the given boundary-value problem. Assume that all eigenvalues are real.

$$65. y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(\pi) = 0. \quad 66. y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(L) = 0.$$

In each of the following problems, find the Fourier series corresponding to the given function:

$$67. f(x) = \begin{cases} 1, & -l \leq x < 0, \\ 0, & 0 \leq x < l, \end{cases} \quad f(x+2l) = f(x).$$

$$68. f(x) = x, \quad -1 \leq x < 1, \quad f(x+2) = f(x).$$

$$69. f(x) = \begin{cases} x+1, & -1 \leq x < 0, \\ x, & 0 \leq x < 1, \end{cases} \quad f(x+2) = f(x).$$

$$70. f(x) = \begin{cases} x+1, & -1 \leq x < 0, \\ 1-x, & 0 \leq x < 1, \end{cases} \quad f(x+2) = f(x).$$

$$71. f(x) = \begin{cases} x+l, & -l \leq x \leq 0, \\ l, & 0 < x < l, \end{cases} \quad f(x+2l) = f(x).$$

72. If $f(x) = -x$ for $-l < x < l$ and $f(x+2l) = f(x)$, find a formula for $f(x)$ in the interval $l < x < 2l$, and in the interval $-3l < x < -2l$.

Find the Fourier series for the following problems. Assume that the functions are periodically extended outside the original interval. Sketch the function to which each series converges outside the original interval.

$$73. f(x) = \begin{cases} 1, & 0 \leq x < s < 1, \\ 0, & s \leq x < 2-s, \\ 1, & 2-s \leq x < 2. \end{cases}$$

$$74. f(x) = 1 - x^2, \quad -1 < x < 1.$$

$$75. f(x) = \begin{cases} 0, & -1 \leq x < 0, \\ x^2, & 0 \leq x < 1. \end{cases}$$

In each of the following problems find the required Fourier series for the given function; sketch the graph of the function to which the function converges over two or three periods.

$$76. f(x) = \begin{cases} 1-x, & 0 < x \leq 1, \\ 0, & 1 < x \leq 2. \end{cases} \quad \text{Both even and odd extensions, period 4.}$$

$$77. f(x) = \begin{cases} x, & 0 \leq x < 1, \\ 1, & 1 \leq x < 2. \end{cases} \quad \text{Sine series, period 4.}$$

$$78. f(x) = 1, \quad 0 \leq x \leq \pi. \quad \text{Cosine series, period } 2\pi.$$

$$79. f(x) = 1, \quad 0 < x < \pi. \quad \text{Sine series, period } 2\pi.$$

$$80. f(x) = l - x, \quad 0 \leq x \leq l. \quad \text{Cosine series, period } 2l.$$

$$81. f(x) = l - x, \quad 0 < x < l. \quad \text{Sine series, period } 2l.$$

82. State exactly the boundary value problem determining the temperature in a silver rod 2 meters long if the ends are held at the temperatures 30° Celsius and 50° Celsius, respectively. The thermal diffusivity of silver is $\alpha^2 = 1.71 \text{ cm}^2/\text{sec}$. Assume that the initial temperature in the bar is given by a quadratic function of the distance along the bar consistent with the preceding boundary conditions, and with the condition that the temperature at the center of the rod is 60° Celsius.

83. Find the solution of the heat conduction problem

$$\begin{aligned} u_{xx} &= 4u_t, & 0 < x < 2, & \quad t > 0 \\ u(0, t) &= 0, & u(2, t) &= 0, & \quad t > 0 \\ u(x, 0) &= 2 \sin \frac{\pi x}{2} - \sin \pi x + 4 \sin 2\pi x. \end{aligned}$$

In each of the following problems determine whether the method of separation of variables can be used to replace the given partial differential equation by a pair of ordinary differential equations. If so, find the equations.

$$84. u_{xx} + u_{xt} + u_t = 0.$$

$$85. xu_{xx} + u_t = 0.$$

$$86. u_{xx} + (x + y)u_{yy} = 0.$$

87. Consider the conduction of heat in a copper rod 100 cm in length whose ends are maintained at 0° Celsius for all $t > 0$. Find an expression for the temperature $u(x, t)$ if the initial temperature distribution in the rod is given by

$$u(x, 0) = \begin{cases} 0, & 0 \leq x < 25 \\ 50, & 25 \leq x \leq 75 \\ 0, & 75 < x \leq 100 \end{cases}$$

88. Let a metallic rod 20 cm long be heated to a uniform temperature of 100° Celsius. Suppose that at $t = 0$ the ends of the bar are plunged into an ice bath at 0° Celsius, and thereafter maintained at this temperature, but that no heat is allowed to escape through the lateral surface. Find an expression for the temperature at any point in the bar at any later time. Use two terms in the series expansion to determine approximately the temperature at the center of the bar at time $t = 30$ sec if the bar is made of (a) silver, (b) aluminum, or (c) cast iron. Thermal diffusivities of silver, aluminum, and cast iron are $1.71 \text{ cm}^2/\text{sec}$, $0.86 \text{ cm}^2/\text{sec}$, and $0.12 \text{ cm}^2/\text{sec}$, respectively. Also, use just one term in the series expansion for $u(x, t)$ to find the time that will elapse before the center of the bar cools to a temperature of 25° Celsius for each of the three metals.

89. Let an aluminum rod of length l be initially at the uniform temperature of 25° Celsius. Suppose that at time $t = 0$ the end $x = 0$ is cooled to 0° Celsius, while the end $x = l$ is heated to 60° Celsius, and both are thereafter maintained at those temperatures.

(a) Find the temperature distribution in the rod at any time t .

Assume in the remaining parts of this problem that $l = 20$ cm.

(b) Use only the first term in the series for the temperature $u(x, t)$ to find the approximate temperature at $x = 5$ cm when $t = 30$ sec; when $t = 60$ sec.

(c) Use the first two terms in the series for $u(x, t)$ to find an approximate value of $u(5, 30)$. What is the percentage difference between the one- and -two-term approximations? Does the third term have any appreciable effect for this value of t ?

(d) Use the first term in the series for $u(x, t)$ to estimate the time interval that must elapse before the temperature at $x = 5$ cm comes within 1 percent of its steady state value.

90. Consider a uniform rod of length l with an initial temperature given by $\sin(\pi x/l)$, $0 \leq x \leq l$. Assume that both ends of the bar are insulated. Find the temperature $u(x, t)$, and the steady-state temperature as $t \rightarrow \infty$.

In each of the following problems find the steady-state solution of the heat conduction equation $u_t = \alpha^2 u_{xx}$ that satisfies the given set of boundary conditions.

91. $u_x(0, t) = 0$, $u(l, t) = T$. 92. $u(0, t) = 30$, $u(40, t) = -20$. 93. $u(0, t) = T$, $u_x(l, t) = 0$.

94. Find the steady-state solution of the partial differential equation

$$u_{xx} - u_x + u_t = 0$$

that satisfies the boundary conditions

$$u(0, t) = 0, \quad u_x(l, t) = e.$$

95. Consider a uniform bar of length l having an initial temperature distribution given by $f(x)$, $0 \leq x \leq l$. Assume that the temperature at the end $x = 0$ is held at 0° Celsius, while

the end $x = l$ is insulated so that no heat passes through it.

(a) Show that the fundamental solutions of the partial differential equation and boundary conditions are

$$u_n(x, t) = e^{-(2n-1)^2\pi^2\alpha^2t/4l^2} \sin [(2n - 1)\pi x/2l], \quad n = 1, 2, 3, \dots$$

(b) Find a formal series expansion for the temperature $u(x, t)$,

$$u(x, t) = \sum_{n=1}^{\infty} c_n u_n(x, t),$$

that also satisfies the initial condition $u(x, 0) = f(x)$.

96. Find the displacement $u(x, t)$ in an elastic string, fixed at both ends, that is set in motion with no initial velocity from the initial position $u(x, 0) = f(x)$, where

$$f(x) = \begin{cases} Ax, & 0 \leq x \leq l/4 \\ Al/4, & l/4 < x < 3l/4 \\ A(l - x), & 3l/4 \leq x \leq l. \end{cases}$$

97. Find the displacement $u(x, t)$ in an elastic string of length l , fixed at both ends, that is set in motion from its straight equilibrium position with the initial velocity g defined by

$$g(x) = \begin{cases} Ax, & 0 \leq x \leq l/2 \\ A(l - x), & l/2 \leq x \leq l. \end{cases}$$

98. If an elastic string is free at one end, the boundary condition there is that $u_x = 0$. Find the displacement in an elastic string of length l , fixed at $x = 0$ and free at $x = l$, set in motion with no initial velocity from the initial position $u(x, 0) = f(x)$, where f is a given function.

HINT: Show that the fundamental solutions for this problem, satisfying all conditions except the inhomogeneous initial condition, are

$$u_n(x, t) = \sin [(2n - 1)\pi x/2l] \cos [(2n - 1)\pi at/2l], \quad n = 1, 2, 3, \dots$$

99. A vibrating string moving in an elastic medium satisfies the equation

$$a^2 u_{xx} - \alpha^2 u = u_{tt},$$

where α^2 is proportional to the coefficient of elasticity of the medium. Suppose that the string is fixed at the ends, and is released with no initial velocity from the initial position $u(x, t) = f(x)$, $0 < x < l$. Find the displacement $u(x, t)$.

100. The motion of a circular elastic membrane, such as a drum head, is governed by the two-dimensional wave equation in polar coordinates

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = \frac{1}{a^2}u_{tt}.$$

Assuming that $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$, find ordinary differential equations satisfied by $R(r)$, $\Theta(\theta)$ and $T(t)$. (Do not solve them.)

In each of the following problems find the general solution of the given system of equations. Also, sketch a number of representative trajectories.

$$101. \dot{\mathbf{x}} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x}. \quad 102. \dot{\mathbf{x}} = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mathbf{x}. \quad 103. \dot{\mathbf{x}} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}.$$

$$104. \dot{\mathbf{x}} = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix} \mathbf{x}. \quad 105. \dot{\mathbf{x}} = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \mathbf{x}. \quad 106. \dot{\mathbf{x}} = \begin{pmatrix} 1 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}.$$

$$107. \dot{\mathbf{x}} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x}. \quad 108. \dot{\mathbf{x}} = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \mathbf{x}. \quad 109. \dot{\mathbf{x}} = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \mathbf{x}.$$

In each of the following problems find all the equilibrium points, determine their type and stability, and sketch local phase portraits near them. If possible, also try to sketch the global phase portraits in the whole $x - y$ plane.

$$110. \dot{x} = x - x^2 - xy, \quad \dot{y} = 3y - xy - 2y^2.$$

$$111. \dot{x} = 1 - y, \quad \dot{y} = x^2 - y^2.$$

$$112. \dot{x} = x \left(\frac{3}{2} - x - \frac{1}{2}y \right), \quad \dot{y} = y \left(2 - y - \frac{3}{4}x \right).$$

$$113. \dot{x} = x \left(\frac{3}{2} - x - \frac{1}{2}y \right), \quad \dot{y} = y \left(2 - \frac{1}{2}y - \frac{3}{2}x \right).$$

$$114. \dot{x} = x(1 - x - y), \quad \dot{y} = y \left(\frac{3}{2} - y - x \right).$$

$$115. \dot{x} = x \left(1 - x - \frac{1}{2}y \right), \quad \dot{y} = y \left(\frac{5}{2} - \frac{3}{2}y - \frac{1}{4}x \right).$$