Math 4210: Homework Problems

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1. Derive the formula

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^{n-1}x^{n-1} + \frac{(-1)^n x^n}{1+x},$$

for $x \neq -1$ and deduce that

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots$$

on $-1 < x \leq 1$. What is $\log 2$?

HINT: First show that

$$\log(1+x) = \int_0^x \frac{dt}{1+t}$$

for $x > -1$. Use this to obtain

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n} + \int_0^x \frac{(-1)^n t^n \, dt}{1+t}.$$

Estimate the remainder directly in a way similar to deriving Lagrange's form of the remainder in the general Taylor series derivation. It is straight forward on $0 \leq x \leq 1$. For $-1 < x \leq t \leq 0$, use

$$\frac{1}{1+t} \leq \frac{1}{1-|x|}.$$

2. Show that for any real α and $0 < |x| < 1$,

$$(1+x)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!} x^n,$$

by completing the following outline:

First, show that the n-th derivative of $(1+x)^\alpha$ at $x=0$ is equal to $\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)$. Use Cauchy's form of the remainder to obtain

$$R_n = \frac{(1-\theta)^n}{n!}\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n)x^{n+1}(1+\theta x)^{\alpha-n-1}$$

with some $0 \le \theta \le 1$. Since $|x| < 1$, show that

$$0 \le \frac{(1-\theta)}{(1+\theta x)} \le 1,$$

and deduce that

$$|R_n| \le (1+\theta x)^{\alpha-1}|\alpha x| \left|\left(1-\frac{\alpha}{1}\right)x\right| \left|\left(1-\frac{\alpha}{2}\right)x\right| \cdots \left|\left(1-\frac{\alpha}{n}\right)x\right|.$$

There exists a number q such that $|x| < q < 1$. Convince yourself that

$$\left|\left(1-\frac{\alpha}{m}\right)x\right| < q$$

for all sufficiently large m, say $m > N$. Deduce that for $n > N$,

$$|R_n| \le (1+\theta x)^{\alpha-1}|\alpha|(1+|\alpha|)^N q^{n-N}.$$

Show that the factor $(1+\theta x)^{\alpha-1}$ is bounded by $2^{\alpha-1}$ when $\alpha \ge 1$ and by $(1-q)^{\alpha-1}$ when $\alpha < 1$, and thus conclude the proof.

3. Let $f(x)$ have a continuous derivative in the interval $[a,b]$, and let $f''(x) \ge 0$ for every $x \in [a,b]$. Then if ξ is any point in the interval $[a,b]$, show that the curve nowhere falls below its tangent at the point $x=\xi$, $y=f(\xi)$. Draw a picture.

HINT: Use a three-term Taylor expansion.

4. Use Taylor's formula to show that if $f'(x_0)=0$, the sign of $f''(x_0)$ determines whether x_0 is a maximum or a minimum. What happens if $f''(x_0)=0$?

5. Suppose $a \in \mathbb{R}$, f is a twice-differentiable function on (a,∞), and M_0, M_1, M_2 are the least upper bounds of $|f(x)|$, $|f'(x)|$, and $|f''(x)|$, respectively, on (a,∞). Prove that $M_1^2 \le 4M_0M_2$.

HINT: If $h > 0$, use Taylor's theorem to show that

$$f'(x) = \frac{f(x+2h)-f(x)}{2h} - hf''(\xi)$$

for some $\xi \in (x, x+2h)$. Hence, show that

$$|f'(x)| \leq hM_2 + \frac{M_0}{h}.$$

To show that $M_1^2 = 4M_0M_2$ can actually happen, take $a = -1$, define

$$f(x) = \begin{cases} 2x^2 - 1, & -1 < x < 0, \\ \dfrac{x^2-1}{x^2+1}, & 0 \leq x < \infty, \end{cases}$$

and show that $M_0 = 1$, $M_1 = 4$, $M_2 = 4$.

6. **Alternative derivation of Taylor's formula:** Let $f(x)$ be $(n+1)$-times continuously differentiable on an interval containing the points a and b. Consider a as the independent variable and keep b fixed. Differentiate the expression

$$f(b) = f(a) + (b-a)f'(a) + \cdots + \frac{(b-a)^n}{n!} f^{(n)}(a) + R_n(a)$$

on a sufficiently many times to show that

$$0 = \frac{(b-a)^n}{n!} f^{(n+1)}(a) + R_n'(a).$$

Deduce that

$$R_n(a) = \int_a^b \frac{(b-t)^n}{n!} f^{(n+1)}(t)\, dt.$$

7. Prove that the function

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0, \end{cases}$$

is infinitely many times differentiable everywhere, yet it cannot be expanded in a Taylor series about $x = 0$. Nevertheless, write down a suitable series expansion for $f(x)$ valid for all $x \neq 0$.

HINT: For the first part, compute that $f^{(n)}(0) = 0 = \lim_{x \to 0} f^{(n)}(x)$ for every n.

8. **Asymptotic Property of the Taylor Expansion:** For simplicity, consider a function $f(x)$, which is $(n+1)$-times continuously differentiable on the symmetric interval $[-a, a]$. Show that Taylor's formula is an *asymptotic formula* in the following sense: If

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2} f''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + R_n(x) \equiv f_n(x) + R_n(x),$$

then

$$\lim_{x\to 0}\frac{f(x)-f_n(x)}{x^n}=\lim_{x\to 0}\frac{R_n(x)}{x^n}=0,$$

regardless of whether $R_n(x)\to 0$ as $n\to\infty$ or not. Also, interpret the result of problem 7 in view of this fact.

9. Consider the series

$$S=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}.$$

Does this series converge? If yes, why, and what is its sum?

10. (i) Let $a_1\geq a_2\geq a_3\geq\cdots\geq 0$. Show that the series $S=\sum\limits_{n=1}^{\infty}a_n$ converges if and only if the series $\Sigma=\sum\limits_{m=1}^{\infty}2^m a_{2^m}$ does.

HINT: If S_n and Σ_m are the respective partial sums, show that for $n<2^m$, $S_n\leq\Sigma_m$, and that for $n>2^m$, $2S_n\geq\Sigma_m$.

(ii) Use part (i) to conclude that $\sum\limits_{n=1}^{\infty}\frac{1}{n^\alpha}$ converges for $\alpha>1$ and diverges for $\alpha\leq 1$.

11. Investigate for convergence or divergence of the series $\sum a_n$ with the general term

(i) $a_n=\sqrt{n+1}-\sqrt{n}$,

(ii) $a_n=\dfrac{\sqrt{n+1}-\sqrt{n}}{n}$.

HINT: Use 10 (ii) for (ii).

12. Let all $a_n\geq 0$. Show that the convergence of the series $\sum a_n$ implies the convergence of the series $\sum\limits_{n=1}^{\infty}\frac{\sqrt{a_n}}{n}$.

HINT: Use the Cauchy-Schwartz inequality for sums.

13. If the series $\sum a_n$ converges and the sequence $\{b_n\}$ is monotonic and bounded, show that $\sum a_n b_n$ converges.

HINT: Show that there is no loss of generality in assuming that $\{b_n\}$ is increasing. Let $b = \lim_{n\to\infty} b_n$. (Show that b exists!) Use Abel's test proven in class to show that the series $\sum a_n(b - b_n)$ converges, and thus conclude the validity of the claim you had to prove.

14. Show that the series $\sum_{n=1}^{\infty} \sin\frac{\pi}{n}$ diverges, but the series $\sum_{n=1}^{\infty} \sin\frac{\pi}{n^2}$ converges.

HINT: First, from the graph of $\sin x$, find the estimate $2x/\pi \leq \sin x \leq x$ on $0 \leq x \leq \pi$.

15. (i) For what values of α does the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^\alpha}$ converge?

(ii) For what values of α does it converge absolutely?

HINT: Use the alternating series and problem 10 or the integral test.

16. Find the sums of the following rearrangements of the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

for $\log 2$:

(i) $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + - - \cdots,$

HINT: Insert pairs of parentheses according to some appropriate simple pattern, and evaluate the sum in each pair of the parentheses explicitly.

(ii) $1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} + + + \cdots.$

HINT: Look carefully at blocks of length 6.

17. Show that the series $\sum_{n=1}^{\infty} \frac{\cos nx}{n}$ converges for all x which are not integer multiples of 2π.

HINT: Restrict your analysis to $x \in [0, \pi]$. (Why can you do it?) Multiply the sum

$$\sigma_n(x) = \frac{1}{2} + \cos x + \cos 2x + \cdots + \cos nx \tag{1}$$

by $\sin\frac{1}{2}x$ and use appropriate trigonometric identities to show that

$$\sigma_n(x) = \frac{\sin\left(n + \frac{1}{2}\right)x}{2\sin\frac{1}{2}x}. \tag{2}$$

For $x \in [0, \pi]$, show that $\sin(x/2) \geq x/\pi$. Deduce that $|\sigma_n(x)| < \pi/2x$ for $x \neq 0$, then use Abel's test.

18. Show that if n is an arbitrary integer greater than 1,

$$\sum_{m=1}^{\infty} \frac{a_m(n)}{m} = \log n,$$

where $a_m(n)$ is defined as

$$a_m(n) = \begin{cases} 1, & \text{if } n \text{ is not a factor of } m, \\ -(n-1), & \text{if } n \text{ is a factor of } m. \end{cases}$$

HINT: If γ is the Euler-Mascheroni constant, then

$$\gamma = \lim_{M\to\infty} \left(\sum_{m=1}^{M} \frac{1}{m} - \log M \right) = \lim_{M\to\infty} \left(\sum_{m=1}^{nM} \frac{1}{m} - \log nM \right).$$

19. Show that the series

$$1 - \frac{1}{2^\alpha} + \frac{1}{3} - \frac{1}{4^\alpha} + \frac{1}{5} - \frac{1}{6^\alpha} + \frac{1}{7} - + \cdots$$

only converges for $\alpha = 1$.

HINT: For $\alpha > 1$, show that it is the sum of a convergent and a divergent series. For $0 < \alpha < 1$, write the series in the form

$$1 - \frac{1}{2} + \left(\frac{1}{2} - \frac{1}{2^\alpha} \right) + \frac{1}{3} - \frac{1}{4} + \left(\frac{1}{4} - \frac{1}{4^\alpha} \right) + \frac{1}{5} - \frac{1}{6} + \left(\frac{1}{6} - \frac{1}{6^\alpha} \right) + \frac{1}{7} - \cdots$$

and show that the series

$$\left(\frac{1}{2} - \frac{1}{2^\alpha} \right) + \left(\frac{1}{4} - \frac{1}{4^\alpha} \right) + \left(\frac{1}{6} - \frac{1}{6^\alpha} \right) + \cdots$$

diverges. What happens for $\alpha \leq 0$.

20. Show that the series $\displaystyle\sum_{n=1}^{\infty} \left(1 - \frac{1}{\sqrt{n}} \right)^n$ converges.

HINT: Write $\left(1-\frac{1}{\sqrt{n}}\right)^n$ in terms of an exponential and use the series for $\log(1-x)$ with small x to show that

$$\left(1-\frac{1}{\sqrt{n}}\right)^n \leq e^{-\sqrt{n}}.$$

Then use the integral test.

21. By comparison with $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$, prove Raabe's test:

The series $\sum |a_n|$ converges or diverges according as

$$n\left(\frac{|a_n|}{|a_{n+1}|}-1\right)$$

is greater than $1+\epsilon$ or less than $1-\epsilon$ for every sufficiently large n and for some $\epsilon > 0$ independent of n.

HINT: First show that the binomial series for $(1+x)^\alpha$ converges absolutely for $|x| < 1$. Conclude that for $|x| \leq q < 1$, the estimate

$$\left|\sum_{k=2}^{\infty} \binom{\alpha}{k} x^k\right| \leq C|x|^2$$

holds for some constant C depending only on q and α.

Infer that, for sufficiently large n,

$$1+\frac{1+\epsilon}{n} \geq \left(1+\frac{1}{n}\right)^{1+\epsilon/2} = \left(\frac{n+1}{n}\right)^{1+\epsilon/2}$$

and

$$1+\frac{1-\epsilon}{n} \leq \left(1+\frac{1}{n}\right)^{1-\epsilon/2} = \left(\frac{n+1}{n}\right)^{1-\epsilon/2}.$$

You will need these inequalities at some appropriate points in your proof of the test.

22. Show that $\sum_{n=1}^{\infty} \frac{n!}{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}$ converges if $\alpha > 1$ and diverges if $\alpha \leq 1$.

23. (i) If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E and $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, then show that $\{f_n g_n\}$ converges uniformly on E.

(ii) Construct sequences $\{f_n\}$ and $\{g_n\}$ which converge uniformly on some set E, but such that $\{f_n g_n\}$ converges only pointwise on E.

24. Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2 x}.$$

For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?

25. Let

$$f_n(x) = \begin{cases} 0, & x < \dfrac{1}{n+1}, \\ \sin^2 \dfrac{\pi}{x}, & \dfrac{1}{n+1} \le x \le \dfrac{1}{n}, \\ 0, & \dfrac{1}{n} < x. \end{cases}$$

Show that $\{f_n\}$ converges to a continuous function, but not uniformly. Use the series $\sum f_n$ to show that absolute convergence, even for all x, does not imply uniform convergence.

26. Show that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

27. Let

$$f_n(x) = \frac{x}{1+nx^2}, \qquad n = 1, 2, \cdots .$$

Show that f_n converges uniformly to a function f, and that the equation

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

holds for $x \neq 0$ and does not hold for $x = 0$.

28. Let

$$H(x) = \begin{cases} 0, & x \le 0, \\ 1, & x > 0, \end{cases}$$

let $\{x_n\}$ be a sequence of distinct points in (a, b), and let $\sum c_n$ converge absolutely. Show that the series

$$f(x) = \sum_{n=1}^{\infty} c_n H(x - x_n), \qquad a \leq x \leq b,$$

converges uniformly, and that f is continuous at every $x \neq x_n$.

29. Let f_n be a sequence of continuous functions which converges uniformly to a function f on a set D. Show that

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

for every sequence of points $x_n \in D$ such that $x_n \to x$, and $x \in D$. By finding a counter example, show that the converse is not true if D is not compact.

30. Let f_n be Riemann integrable on $[a, b]$ for $n = 1, 2, 3, \ldots$, and let $f_n \to f$ uniformly on $[a, b]$. Then f is Riemann integrable on $[a, b]$, and

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \int_a^b f_n(x)\, dx.$$

HINT: Let

$$\epsilon_n = \sup_{a\leq x\leq b} |f_n(x) - f(x)|.$$

Let $U(f, a, b)$ and $L(f, a, b)$ be the upper and lower Riemann integrals, respectively, defined as

$$U(f, a, b) = \sup \sum_{k=0}^{K-1} M_k (x_{k+1} - x_k), \qquad L(f, a, b) = \inf \sum_{k=0}^{K-1} m_k (x_{k+1} - x_k)$$

where the supremum and infimum are taken over all possible partitions $a = x_0 < x_1 < \cdots < x_{K-1} < x_K = b$ of the interval $[a, b]$, and

$$M_k = \sup_{x_k\leq x\leq x_{k+1}} f(x), \qquad m_k = \inf_{x_k\leq x\leq x_{k+1}} f(x).$$

Show that $f_n - \epsilon_n < f < f_n + \epsilon_n$ implies both

$$0 \leq U(f, a, b) - L(f, a, b) \leq 2\epsilon_n(b - a)$$

and

$$\left| \int_a^b f(x)\, dx - \int_a^b f_n(x)\, dx \right| \leq \epsilon_n(b - a).$$

31. Suppose $\{f_n(x)\}$ and $\{g_n(x)\}$ are defined on an interval I and

(a) $\sum f_n(x)$ has uniformly bounded partial sums;

(b) $g_n(x) \to 0$ uniformly on I;

(c) $g_1(x) \geq g_2(x) \geq g_3(x) \geq \cdots$ at every $x \in I$.

Show that $\sum f_n(x) g_n(x)$ converges uniformly on I.

32. On what intervals of x does the series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos nx}{n}$ converge uniformly?

HINT: Use the solutions of problems 17 and 31.

33. From the appropriate geometric and binomial (see problem 2) series, derive the series expansions in powers of x of the functions $\arctan x$ and $\arcsin x$. What are their respective radii of convergence? Show that the series for $\arctan x$ also converges at the endpoints of the convergence interval.

34. Show that

$$\frac{\log(1+x)}{1+x} = \sum_{n=1}^{\infty} (-1)^{n+1} \left(\sum_{k=1}^{n} \frac{1}{k} \right) x^n.$$

35. Let

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

Using complex variables, one can show that the radius of convergence of this series is 2π.

(i) Multiply the above series by $e^x - 1$ to show that the Bernoulli numbers B_n satisfy the equation

$$\binom{n+1}{1} B_n + \binom{n+1}{2} B_{n-1} + \binom{n+1}{3} B_{n-2} + \cdots + \binom{n+1}{n+1} B_0 = 0$$

for $n > 0$, where $B_0 = 1$. Conclude that these numbers are rational.

(ii) Show that

$$\frac{x}{e^x - 1} + \frac{x}{2} = \frac{x}{2} \coth \frac{x}{2}$$

and thus that, for $n > 0$, $B_{2n+1} = 0$ and

$$x \coth x = \sum_{n=0}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!} x^{2n}.$$

What is the radius of convergence of this series?

(iii) Replace x by ix, to find

$$x \cot x = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} x^{2n}.$$

(iv) Derive the formula $2 \cot 2x = \cot x - \tan x$ to conclude that

$$\tan x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n} \left(2^{2n} - 1\right) B_{2n}}{(2n)!} x^{2n-1}.$$

Where does this series converge?

36. (i) Integrate by parts to obtain

$$\int_0^{\frac{\pi}{2}} \sin^m x \, dx = \frac{m-1}{m} \int_0^{\frac{\pi}{2}} \sin^{m-2} x \, dx$$

for all integer $m > 1$. Deduce that

$$\int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx = \frac{2n-1}{2n} \frac{2n-3}{2n-2} \cdots \frac{1}{2} \frac{\pi}{2}.$$

(ii) Let $|x| < 1$, and

$$K(x) = \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1 - x^2 \sin^2 t}}.$$

Find the power series expansion for $K(x)$ in powers of x. Where does this series converge?

HINT: Use problem 2 or 33.

(iii) The Bessel function of order zero is given by the formula

$$J_0(x) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos(x \sin t) \, dt.$$

By expanding the integrand in a power series and carrying out the integration term-by-term (justify it!), show that $J_0(x)$ has a power series expansion

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}.$$

Where does this series converge?

37. Use the result of problem 2 to show that

$$\sqrt{1-x} = 1 - \frac{1}{2}x - \frac{1}{2\cdot 4}x^2 - \frac{1\cdot 3}{2\cdot 4\cdot 6}x^3 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4 - \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10}x^5 - \cdots,$$

where the series converges for $-1 < x < 1$ and all the coefficients a_n, $n \geq 1$, are negative.

(i) Show that this power series still converges to $\sqrt{1-x}$ at $x = 1$ by completing the following outline:

Denote $g(x) = \sqrt{1-x}$, and let $S_n(x)$ be its n-th partial sum.

(a) For finite n and $0 \leq x < 1$, show that

$$S_n(x) = \sum_{k=0}^{n} a_k x^k \geq g(x) > 0.$$

(b) Conclude that $S_n(1) \geq 0$ and that $S_n(1) \to S \geq 0$.

(c) Show that, for fixed n, $S_n(1) < S_n(x)$.

(d) Given $\epsilon > 0$, choose x so close to 1 that $g(x) < \epsilon/2$ and n so large that $R_n(x) = S_n(x) - g(x) < \epsilon/2$. Conclude that $0 \leq S_n(1) < S_n(x) < \epsilon$.

(ii) Show that for every $\epsilon > 0$, there exists an n such that $\left|\sqrt{1-x} - S_n(x)\right| < \epsilon$ uniformly on $0 \leq x \leq 1$.

HINT: $0 \leq R_n(x) \leq R_n(1) \to 0$.

(iii) Replace x by $1 - x^2$ in (ii), and show that there exist polynomials $P_n(x)$ that converge to $|x|$ uniformly on $-1 \leq x \leq 1$.

(iv) Replace x by $1 - (x-a)^2/A^2$ in (ii) to generalize the result of (iii) to the function $|x-a|$ on the interval $[a-A, a+A]$.

38. (i) Show that if $f(x)$ is continuous on $[a,b]$, then given $\epsilon > 0$, there exists a piecewise linear function $\phi(x)$ such that $|f(x) - \phi(x)| < \epsilon$ on $[a,b]$.

(ii) Show that every polygonal function $\phi(x)$ can be represented as

$$\phi(x) = c_0 + \sum_{i=1}^{n} c_i \left(x - x_i + |x - x_i|\right) = a + bx + \sum_{i=1}^{n} c_i \left|x - x_i\right|.$$

HINT: First investigate the behavior of function $x + |x|$.

(iii) Prove the **Weierstrass approximation theorem**: For every continuous function $f(x)$ on $[a,b]$ and every $\epsilon > 0$, there exists a polynomial $P(x)$ such that $|f(x) - P(x)| < \epsilon$ for all $x \in [a,b]$.

39. Prove the following *existence and uniqueness theorem* for second-order linear differential equations with power-series coefficients: Let

$$p(x) = \sum_{n=0}^{\infty} p_n x^n, \qquad q(x) = \sum_{n=0}^{\infty} q_n x^n,$$

where both series converge at least for $|x| < R$. Then the initial-value problem

$$y'' + p(x)y' + q(x)y = 0, \qquad y(0) = y_0, \quad y'(0) = y_1$$

has a unique power-series solution converging at least for $|x| < R$.

HINT: (i) Assume $y(x) = \sum_{n=0}^{\infty} y_n x^n$. By differentiation and multiplication of power series, show that the coefficients y_n, $n \geq 2$, can be computed recursively, provided y_0 and y_1 are given.

(ii) From the recursion formulas for the coefficients y_n, show that if you can find an auxiliary differential equation $y'' + P(x)y' + Q(x)y = 0$ such that the coefficients of $-P(x)$ and $-Q(x)$ are all positive and larger than the corresponding $|p_n|$ and $|q_n|$, then the coefficients of its solution series will also be positive and larger than the corresponding $|y_n|$. The series for $y(x)$ will therefore converge at least as far as this auxiliary solution.

(iii) Using the ratio test, show that an appropriate auxiliary equation is

$$y'' - M\left(1 - \frac{x}{\rho}\right)^{-1} y' - N\left(1 - \frac{x}{\rho}\right)^{-2} y = 0$$

for some appropriate constants M and N. Here, ρ is is any number $0 < \rho < R$.

(iv) Show that the general solution of the auxiliary equation is

$$Y(x) = Y_0\left(1 - \frac{x}{\rho}\right)^{\alpha_0} + Y_1\left(1 - \frac{x}{\rho}\right)^{\alpha_1},$$

where α_0 and α_1 are the two roots of the quadratic equation

$$\alpha(\alpha - 1) - \alpha M - N = 0.$$

Show that its power series expansion about $x = 0$ has radius of convergence ρ. Since $\rho < R$ is arbitrary, the radius of convergence of the series for $y(x)$ is R.

40. Let M be a metric space. Show that both M and the null set are open. Show that an arbitrary union of open sets and the intersection of a finite number of open sets are open.

41. Let M be a metric space and $N \subset M$. Show that $A \subset N$ is open in N if and only if $A = N \cap B$, where B is some (not necessarily unique) open set in M.

42. Show that if $f : M \to N$ is continuous, then every preimage of an open set is open, and every preimage of a closed set is closed.

43. (i) Prove that the interval $I = [0, 1]$ is compact by carrying out the following argument: Assume that it is not compact. Then there exists a cover $\{G_\alpha\}$ of I which does not contain a finite subcover. One of the intervals $\left[0, \frac{1}{2}\right]$ and $\left[\frac{1}{2}, 1\right]$ cannot be covered by any finite sub-collection of $\{G_\alpha\}$. Repeat the argument, until you arrive at a single point not covered by any finite sub-collection of $\{G_\alpha\}$. Argue that this is a contradiction.

(ii) Generalize this result to any rectangle $\{\mathbf{x} \mid a_k \leq x_k \leq b_k, k = 1, \ldots, n\} \subset \mathbb{R}^n$.

44. Show that every closed subset of a compact set is compact. Then use the result of problem 43 (ii) to show that any closed and bounded subset of $\mathbb{R}^n$ is compact.

45. Show that if A is closed and B is compact, then $A \cap B$ is compact.

46. Let M be a metric space and let $\{K_\alpha\}$ be a collection of its compact subsets. Show that if the intersection of every finite sub-collection of $\{K_\alpha\}$ is nonempty, then $\cap_\alpha K_\alpha$ is nonempty.

HINT: The complement of each K_α is an open set. If no point of some K_β belongs to every other K_α, their complements form an open cover of K_β.

47. The closure $\bar{A}$ of a set A is the union of A and all its limit points. Show that $\bar{A}$ is closed, that $A = \bar{A}$ precisely when A is closed, and that $\bar{A}$ is the smallest closed set containing A, that is, if $A \subset B$ and B is closed then $\bar{A} \subset B$.

48. A point x is an interior point of the set A if some open ball $B_r(x) \subset A$. The set $\mathring{A}$ of all the interior points if A is called the interior of A. Show that $\mathring{A}$ is open, that $\mathring{A} = A$ precisely when A is open, and that $\mathring{A}$ is the largest open set contained in A. (Formulate the last statement precisely!)

49. Show that if M is compact and $f : M \to N$ is continuous and one-to-one, then its inverse f^{-1} is continuous.

50. If A is connected and f is continuous, show that $f(A)$ is connected.

51. (i) Show that any open connected subset of the real line is an open interval.

(ii) Show that every open subset of the real line is a union of (at most) countably many disjoint open intervals.

52. Show that a continuous real function f on $[a, b]$ achieves its maximum, minimum, and every point in-between.

53. Show that the closed unit ball in $C[0, 1]$ is not compact.

HINT: Look at all the powers x^n.

54. Let

$$f_n(x) = \frac{x^2}{x^2 + (1 - nx)^2}, \qquad 0 \le x \le 1, \quad n = 1, 2, 3, \dots .$$

Show that $\{f_n\}$ is uniformly bounded on $[0, 1]$, $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0, 1]$, but

$$f_n\left(\frac{1}{n}\right) = 1, \qquad n = 1, 2, 3, \dots,$$

so that no subsequence of $\{f_n\}$ can converge uniformly on $[0, 1]$. Show that $\{f_n'\}$ is unbounded, and so $\{f_n\}$ cannot be equicontinuous.

HINT: For the last statement, compute $f_n'\left(\frac{1}{n} - \frac{1}{n^2}\right)$.

55. Suppose f is a real continuous function on $\mathbb{R}$, $f_n(x) = f(nx)$ for $n = 1, 2, \dots$, and $\{f_n\}$ is equicontinuous on $[0, 1]$. What conclusion can you draw about f?

56. Let $\{f_n\}$ be a uniformly bounded sequence of continuous function on $[a, b]$. Show that the sequence $\{F_n\}$ of functions given by

$$F_n(x) = \int_a^x f_n(t)\, dt, \qquad a \le x \le b,$$

has a subsequence which converges uniformly on $[a, b]$.

57. Use the Arzelà-Ascoli Theorem to prove **Peano's existence theorem**: Let the function $f(t, x)$ be continuous and bounded on the strip defined by $0 \le t \le 1$, $-\infty < x < \infty$. Then there exists at least one continuously differentiable solution of the initial-value problem

$$\dot{x} = f(t, x), \qquad x(0) = x_0$$

on the interval $0 \le t \le 1$.

HINT: Fix n. For $i = 0, \dots, n$ put $t_i = i/n$. Let ϕ_n be a continuous function on $0 \le t \le 1$ such that $\phi_n(0) = x_0$,

$$\dot{\phi}_n(t) = f(t_i, \phi_n(t_i)) \qquad \text{if } \ t_i < t < t_{i+1},$$

and put

$$\Delta_n(t) = \dot{\phi}_n(t) - f(t, \phi_n(t)),$$

except at the points t_i, where $\Delta_n(t) = 0$. Then

$$\phi_n(t) = x_0 + \int_0^t [f(\tau, \phi_n(\tau)) + \Delta_n(\tau)] \, d\tau.$$

Choose M so that $f < M$. Verify the following assertions:

(i) $|\dot{\phi}_n| \le M$, $|\Delta_n| \le 2M$, Δ_n Riemann integrable, and $|\phi_n| \le |x_0| + M = M_1$, say, on $0 \le t \le 1$, for all n.

(ii) $\{\phi_n\}$ is equicontinuous on $0 \le t \le 1$, since $|\dot{\phi}_n| \le M$.

(iii) Some $\{\phi_{n_k}\}$ converges to some ϕ, uniformly on $0 \le t \le 1$.

(iv) Since f is uniformly continuous on the rectangle $0 \le t \le 1$, $|x| \le M_1$,

$$f(t, \phi_{n_k}(t)) \to f(t, \phi(t))$$

uniformly on $0 \le t \le 1$.

(v) $\Delta_n(t) \to 0$ uniformly on $0 \le t \le 1$ since

$$\Delta_n(t) = f(t_i, \phi_n(t_i)) - f(t, \phi_n(t))$$

for $t_i < t < t_{i+1}$.

(vi) Hence

$$\phi(t) = x_0 + \int_0^t f(\tau, \phi(\tau)) \, d\tau.$$

This ϕ is the solution of the given problem.

58. Consider the space ℓ_2 of real sequences $\{a_n\}$ such that $\sum a_n^2 < \infty$. If $\mathbf{a} = \{a_n\}$ and $\mathbf{b} = \{b_n\}$ define their sum to be $\mathbf{a} + \mathbf{b} = \{a_n + b_n\}$, and if $\alpha \in \mathbb{R}$ define $\alpha\mathbf{a} = \{\alpha a_n\}$.

(i) Show that ℓ_2 is a vector space.

(ii) If $\mathbf{a}, \mathbf{b} \in \ell_2$, let $\langle \mathbf{a}, \mathbf{b} \rangle = \sum a_n b_n$. Show that $|\langle \mathbf{a}, \mathbf{b} \rangle| < \infty$ and that $\langle \mathbf{a}, \mathbf{b} \rangle$ defines an inner product on ℓ_2.

(iii) Show that the induced norm and metric are

$$\|\mathbf{a}\| = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle} = \sqrt{\sum_{n=1}^{\infty} a_n^2} \quad \text{and} \quad d(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} - \mathbf{b}\| = \sqrt{\sum_{n=1}^{\infty} (a_n - b_n)^2},$$

respectively.

(iv) Show that ℓ_2 is complete: if $\mathbf{a}_n$ is a Cauchy sequence (of sequences) in ℓ_2, that is, if $\|\mathbf{a}_n - \mathbf{a}_m\| \to 0$ as $n, m \to \infty$, then there exists a sequence $\mathbf{a} \in \ell_2$ such that $\|\mathbf{a}_n - \mathbf{a}\| \to 0$.

HINT: Define the limit component-wise. Use $\|\mathbf{a}_m\| \le \|\mathbf{a}_n - \mathbf{a}_m\| + \|\mathbf{a}_n\|$ at some opportune moment.

(v) Show that the closed unit ball $\{\mathbf{a} \mid \|\mathbf{a}\| \le 1\}$ is not compact.

HINT: Consider the sequences $\mathbf{e}_n = \{0, \ldots, 0, 1, 0, \ldots\}$, $n = 1, 2, \ldots$, in which 1 is in the n-th spot.

(vi) Show that the Hilbert cube, $\{\mathbf{a} \mid 0 \le a_n \le 1/n\}$, is compact.

HINT: Proceed component-wise and mimic the proof of the Arzelà-Ascoli theorem.

(vii) Show that the Hilbert cube has no interior points. In other words, it is not a neighborhood of any of its points.

(viii) Show that sequences with rational terms are dense in ℓ_2, so that ℓ_2 is separable.

(ix) Show that the vectors $\mathbf{e}_n$, defined in (v), form a *complete orthonormal set*. In other words,

$$\langle \mathbf{e}_n, \mathbf{e}_m \rangle = \begin{cases} 0, & m \ne n \\ 1, & m = n \end{cases},$$

and any sequence $\mathbf{a} \in \ell_2$ can be expressed as

$$\mathbf{a} = \sum_{n=1}^{\infty} \langle \mathbf{a}, \mathbf{e}_n \rangle \mathbf{e}_n.$$

Here, the sum of the series is to be interpreted as the limit in the ℓ_2-norm of its partial sums.

The space ℓ_2 is a prototypical *Hilbert space*.

59. Suppose f is a real function on $(-\infty, \infty)$. Call x a *fixed point* of f if $f(x) = x$.

(i) If f is differentiable and $f'(t) \neq 1$ for all real t, show that f has at most one fixed point.

HINT: Mean-value theorem.

(ii) Show that the function f defined by

$$f(t) = t + \frac{1}{1 + e^t}$$

has no fixed point although $0 < f'(t) < 1$ for all real t.

(iii) However, if there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t, prove that a fixed point x of f exists, and that $x = \lim_{n\to\infty} x_n$, where x_1 is an arbitrary number and

$$x_{n+1} = f(x_n)$$

for $n = 1, 2, 3, \dots$.

A proof without using the contraction mapping theorem will bring you extra points.

(vi) Show that the process described in (iii) can be visualized by the zig-zag path

$$(x_1, x_2) \to (x_2, x_2) \to (x_2, x_3) \to (x_3, x_3) \to (x_3, x_4) \to \cdots .$$

60. Use the contraction principle to prove **Picard's existence theorem**: Let the function $f(t, x)$ be continuous in t for $0 \leq t \leq 1$ and let it satisfy the Lipschitz continuity condition $|f(t, x) - f(t, y)| < L|x - y|$ for $-\infty < x, y < \infty$. Then there exists a unique continuously differentiable solution of the initial-value problem

$$\dot{x} = f(t, x), \qquad x(0) = x_0, \tag{3}$$

on some interval $0 \leq t \leq T$ with $T \leq 1$.

HINT: (i) Show that every continuous solution of the the integral equation

$$x(t) = x_0 + \int_0^t f(\tau, x(\tau))\, d\tau. \tag{4}$$

must be continuously differentiable, and so (3) and (4) are equivalent.

(ii) Set up the *Picard iteration* procedure: Let $x_0(t) \equiv x_0$, and let

$$x_{n+1}(t) = x_0 + \int_0^t f\left(\tau, x_n(\tau)\right)\, d\tau \equiv x_0 + Ax_n(t).$$

Using the Lipschitz continuity condition, derive the estimate

$$|Ax(t) - Ay(t)| \le Lt \sup_{0<\tau<t} |x(\tau) - y(\tau)| \le LT \sup_{0<\tau<T} |x(\tau) - y(\tau)|$$

for $0 \le t \le T \le 1$.

(iii) Let $C[0,T]$ denote the space of continuous functions on the interval $[0,T]$ with the distance induced by the norm $\|f\|_T = \sup_{0<\tau<T} |f(t)|$. Show that for a sufficiently small T, the mapping A maps $C[0,T]$ into itself, and is a contraction.

(iv) Deduce that the integral equation (4) has a unique continuous solution on $[0,T]$.

61. (i) Let $f(x)$ be piecewise smooth and periodic with period 2π. Let its Fourier expansion be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right). \tag{5}$$

Show that if $f(x)$ is even, $b_n = 0$, and (5) becomes a *Fourier cosine series* with

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx. \tag{6}$$

Likewise, if $f(x)$ is odd, $a_n = 0$, and (5) becomes a *Fourier sine series* with

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx\, dx. \tag{7}$$

(ii) Let $f(x)$ be a piecewise smooth real function on the interval $[0,\pi]$. Show that for $0 < x < \pi$,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \sum_{n=1}^{\infty} b_n \sin nx,$$

with a_n and b_n given by fomulas (6) and (7) respectively. Do these two series converge outside of the interval $[0,\pi]$, and if yes, to what functions?

62. Let $f(x)$ be periodic with period $2L$. Show that its Fourier series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}\right),$$

with

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx.$$

63. (i) Show the complex counterpart of the orthogonality relation for trigonometric functions:

$$\int_{-\pi}^{\pi} e^{inx}e^{-imx}\,dx = \begin{cases} 0, & m \neq n, \\ 2\pi, & m = n. \end{cases}$$

(ii) Show that the complex form of the Fourier series for a piecewise-smooth, 2π-periodic, real function $f(x)$ is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx, \qquad c_{-n} = \bar{c}_n,$$

where the overbar denotes the complex conjugate.

(iii) If the real form of the Fourier series for $f(x)$ is (5), what is the connection between the two sets of coefficients $\{a_n, b_n\}$ and c_n?

64. **Differentiation of Fourier Series:** Let $f(x)$ be 2π-periodic. Let it also have continuous derivatives up to order k and a piecewise continuous derivative of order $k+1$.

(i) Show that there exists a constant B, depending only on f and k, such that the Fourier coefficients of f satisfy

$$|a_n|, |b_n| < \frac{B}{n^{k+1}}.$$

HINT: If c_n is the n-the complex Fourier coefficient of f, then integrate by parts to show

$$2\pi c_n = \left(\frac{-i}{n}\right)^{k+1} \int_{-\pi}^{\pi} f^{(k+1)}(x)e^{-inx}\,dx.$$

(ii) Use the (i) to conclude that for $k > 2$ the Fourier series for $f(x)$ can be differentiated $k-1$ times and yields the Fourier series for the differentiated function.

65. Let x not be an integer, and let $f(t) = \cos xt$ for $-\pi < t < \pi$. Extend $f(t)$ periodically in t outside the interval $-\pi < t < \pi$, and then expand it in a Fourier series.

(i) Show that this series is

$$\cos xt = \frac{2x\sin\pi x}{\pi}\left(\frac{1}{2x^2} - \frac{\cos t}{x^2-1^2} + \frac{\cos 2t}{x^2-2^2} - \frac{\cos 3t}{x^2-3^2} + \cdots\right).$$

Convince yourself that this series represents a continuous function near $t = \pm\pi$. Setting $t = \pi$ thus conclude that

$$\cot \pi x = \frac{2x}{\pi}\left(\frac{1}{2x^2} + \frac{1}{x^2 - 1^2} + \frac{1}{x^2 - 2^2} + \frac{1}{x^2 - 3^2} + \cdots\right).$$

This is the so-called *partial fraction decomposition* of the cotangent.

(ii) Let $0 \le x \le q < 1$ for some q. Write the above formula as

$$\cot \pi x - \frac{1}{\pi x} = -\frac{2x}{\pi}\left(\frac{1}{1^2 - x^2} + \frac{1}{2^2 - x^2} + \frac{1}{3^2 - x^2} + \cdots\right).$$

Using the Weierstrass M-test, convince yourself that the series on the right-hand side converges uniformly. Integrate term-by-term between 0 and x to conclude that

$$\log \frac{\sin \pi x}{\pi x} = \sum_{n=1}^{\infty} \log\left(1 - \frac{x^2}{n^2}\right).$$

Interpret the series on the right-hand side as the limit of its partial sums, and invoke the continuity of the exponential function to show that

$$\frac{\sin \pi x}{\pi x} = \lim_{N\to\infty} \prod_{n=1}^{N}\left(1 - \frac{x^2}{n^2}\right) = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}\right).$$

Therefore,

$$\sin \pi x = \pi x \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}\right).$$

This is the *infinite product* expansion of the sine function. It can be shown that it is not only valid for $0 \le x < 1$, by for all complex x.

66. Use Fourier analysis to derive the formal solution of the heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \pi, \quad t > 0, \tag{8}$$

with the boundary conditions

$$u(0,t) = 0, \qquad u(\pi,t) = 0, \qquad t > 0, \tag{9}$$

and the initial condition

$$u(x,0) = f(x), \qquad 0 < x < \pi. \tag{10}$$

HINT: (i) Assume solutions of the form $u(x,t) = X(x)T(t)$, substitute into (8), and divide by $X(x)T(t)$. Argue that a function of x can only be equal to a function of t for all x and t if both functions are equal to the same constant. In this way, derive the two equations

$$X''(x) + \lambda X(x) = 0, \qquad \dot{T}(t) + \alpha^2 \lambda T(t) = 0, \qquad \lambda = \text{const.}$$

(ii) Show that the boundary conditions (9) translate into the boundary conditions $X(0) = X(\pi) = 0$.

(iii) Show that all possible nonzero solutions $X(x)$ are proportional to $X_n(x) = \sin nx$, $n = 1, 2, 3, \ldots$, with the corresponding $\lambda_n = n^2$.

(iv) Show that the corresponding nonzero solutions $T(t)$ are proportional to $T_n(t) = e^{-n^2\alpha^2 t}$.

(v) Conclude that $u_n(x,t) = e^{-n^2\alpha^2 t} \sin nx$ satisfy both the heat equation (8) and the boundary conditions (9), and, since both are homogeneous, so does any sum $\sum_n c_n e^{-n^2\alpha^2 t} \sin nx$.

(vi) At $t = 0$, the initial condition (10) becomes $f(x) = \sum_n c_n \sin nx$. Compute c_n to find the solution $u(x,t)$ of the original problem.

(vii) Let $f(x)$ be piecewise smooth on $0 < x < \pi$. By comparing it to the series $\sum e^{-n^2\alpha^2\epsilon}$ and its derivatives, show that the series solution $u(x,t)$ you just obtained converges absolutely and uniformly for $0 < x < \pi$ and $t \geq \epsilon$, for any $\epsilon > 0$. Show that it can be differentiated an arbitrary number of times, and is therefore a true solution of the heat equation (and satisfies the boundary conditions).

(viii) If $f(x)$ is continuous and piecewise smooth on $0 < x < \pi$, and also $f(0) = f(\pi) = 0$, show that $u(x, t \to 0) \to f(x)$. Therefore, $u(x,t)$ also truly satisfies the initial condition.

HINT: Derive the estimate $|f(x) - u(x,t)| \leq \sum_{n=1}^{\infty}(1 - e^{-n^2\alpha^2 t})|c_n| \leq \sum_{n=1}^{\infty} |c_n|$, then use the result of problem 64 (i) to show that the last series converges. Show that this implies uniform convergence of the first series, so that $\lim_{t\to 0} \sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} \lim_{t\to 0}$ in that series.

67. (i) Derive the formal solution of the heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \pi, \quad t > 0,$$

with the boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad \frac{\partial u}{\partial x}(\pi,t) = 0, \qquad t > 0,$$

and the initial condition

$$u(x,0) = f(x), \qquad 0 < x < \pi.$$

(ii) If $f(x)$ is piecewise smooth on $0 < x < \pi$, show that the formal solution $u(x,t)$ is a true solution of the heat equation. Moreover, if $f(x)$ is also continuous, show that it also truly satisfies the initial condition.

HINT: Proceed along the lines of 66 (vii) and (viii).

68. (i) Derive the formal solution of the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < \pi, \quad t > 0,$$

with the boundary conditions

$$u(0,t) = 0, \qquad u(\pi,t) = 0, \qquad t > 0,$$

and the initial condition

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x), \qquad 0 < x < \pi.$$

(ii) Try to obtain a rigorous justification for the formal Fourier series solution $u(x,t)$ as in parts (vii) and (viii) of problem 66. Just how smooth must $f(x)$ be for this justification to succeed?

69. (i) Derive the formal solution of Laplace's equation on a rectangle

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < a, \quad 0 < y < b,$$

with the boundary values

$$\begin{aligned} u(x,0) &= 0, & u(x,b) &= 0, & 0 < x < a, \\ u(0,y) &= 0, & u(a,y) &= f(y), & 0 < y < b. \end{aligned}$$

HINT: The solution is

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b},$$

where

$$c_n = \frac{2}{b}\left(\sinh\frac{n\pi a}{b}\right)^{-1}\int_0^b f(y)\sin\frac{n\pi y}{b}\,dy.$$

(ii) Obtain a rigorous justification for the formal Fourier series solution $u(x,t)$ as in parts (vii) and (viii) of problem 66. State precisely the smoothness assumptions on the function $f(y)$.

(iii) Write down the solution of Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < a, \quad 0 < y < b,$$

with the boundary values

$$\begin{aligned} u(x, 0) &= f_1(x), \qquad & u(x, b) &= f_2(x), \qquad & 0 < x < a, \\ u(0, y) &= f_3(y), \qquad & u(a, y) &= f_4(y), \qquad & 0 < y < b. \end{aligned}$$

70. Derive the formal solution of Laplace's equation on a circle

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 \le x^2 + y^2 = r^2 < a^2,$$

with the boundary values

$$u(r = a, \theta) = f(\theta).$$

HINT: (i) In polar cordinates, $x = r\cos\theta$, $y = r\sin\theta$, Laplace's equation becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

(ii) Show that the boundary conditions in polar coordinates become

$$u(a, \theta) = f(a), \qquad u(0, \theta) \text{ is bounded}, \qquad u(r, \theta + 2\pi) = u(r, \theta).$$

(iii) After separating variables $u(r, \theta) = R(r)\Theta(\theta)$, show that the resulting equations become

$$r^2 R''(r) + rR'(r) - \lambda R(r) = 0, \qquad \Theta''(\theta) + \lambda\Theta(\theta) = 0,$$

with the conditions

$$R(0) \text{ is bounded}, \qquad \Theta(\theta + 2\pi) = \Theta(\theta).$$

(iv) Show that the eventual solution has the form

$$u(r, \theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n \left(c_n \cos n\theta + d_n \sin n\theta\right).$$

What are the expressions for the coefficients c_n and d_n?

71. Derive the formal solution of Laplace's equation on an annulus

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad a \le r < b,$$

with the boundary values

$$u(a, \theta) = f(\theta), \qquad u(b, \theta) = g(\theta).$$

HINT: The solution has the form

$$u(r, \theta) = \frac{c_0 + e_0 \log r}{2} + \sum_{n=1}^{\infty} r^n \left(c_n \cos n\theta + d_n \sin n\theta \right)$$
$$+ \sum_{n=1}^{\infty} \frac{1}{r^n} \left(e_n \cos n\theta + f_n \sin n\theta \right).$$

72. (i) Draw the graph of the function $t(u) = u - \epsilon \sin u$ for $0 \le \epsilon \le 1$, and convince yourself from the graph, as well as analytically, that the following statements are true:

(a) $t(u)$ is odd, monotonically increasing, and continuously differentiable for all real u,

(b) $t(u + 2\pi) = t(u) + 2\pi$,

(ii) Consider the equation

$$t = u - \epsilon \sin u \tag{11}$$

(a) Given $\epsilon \le 1$, show that there is a unique function $u(t)$ that solves equation (11), which is odd and monotonically increasing for all u. If $\epsilon < 1$ show that this function is continuously differentiable for all t, and if $\epsilon = 1$, it is continuously differentiable at $t \neq 2n\pi$ with integer n. (What happens at $t = 2n\pi$ with integer n?)

(b) Use part (b) of (i) to show that $u(t)$ can be written as $u(t) = t + f(t)$, where $f(t + 2\pi) = f(t)$ for all real t.

HINT: To show (a), use monotonicity and continuous differentiablity of $t(u)$.

(iii) If $0 \le \epsilon < 1$, show that the function $f(t)$ can be expanded in a Fourier sine series,

$$f(t) = \sum_{n=1}^{\infty} c_n(\epsilon) \sin nt,$$

where

$$c_n(\epsilon) = \frac{2}{\pi} \int_0^{\pi} f(t) \sin nt \, dt. \tag{12}$$

Calculate $c_n(\epsilon)$ in the following way: Write $f(t) = u(t) - t$, and split the integrand in (12) into a sum of two terms, the first of which is

$$d_n(\epsilon) = \frac{2}{\pi} \int_0^\pi u(t) \sin nt \, dt.$$

You will be able to calculate the second term easily, but for the first term, integrate by parts, use periodicity of $u(t)$, and make a substitution $t = t(u)$ to get

$$\begin{aligned} d_n(\epsilon) &= \frac{2(-1)^{n+1}}{n} + \frac{2}{n\pi} \int_0^\pi \cos nt \; u'(t) \, dt \\ &= \frac{2(-1)^{n+1}}{n} + \frac{2}{n\pi} \int_0^\pi \cos nt(u) \, du \\ &= \frac{2(-1)^{n+1}}{n} + \frac{2}{n\pi} \int_0^\pi \cos n(u - \epsilon \sin u) \, du. \end{aligned}$$

(Justify all your steps.) This integral involves a standard special function, called the Bessel function $J_n(x)$, given by the formula

$$J_n(x) = \frac{2}{\pi} \int_0^\pi \cos(nu - x \sin u) \, du.$$

Thus, express $c_n(\epsilon)$ in terms of algebraic expressions and a Bessel function, and $u(t)$ as the sum of t and a Fourier series in t with coefficients involving Bessel functions.

73. **Integration of Fourier Series:** Show that if $f(x)$ is a piecewise continuous function in $-\pi \leq x \leq \pi$ having the formal Fourier expansion

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

then for any two points x_1 and x_2,

$$\int_{x_1}^{x_2} f(x) \, dx = \int_{x_1}^{x_2} \frac{a_0}{2} \, dx + \sum_{n=1}^{\infty} \int_{x_1}^{x_2} (a_n \cos nx + b_n \sin nx) \; dx,$$

that is, the formal Fourier series can be integrated termwise. Moreover, the series on the right converges uniformly in x_2 for fixed x_1.

HINT: The function

$$F(x) = \int_{-\pi}^{x} \left[f(t) - \frac{a_0}{2} \right] dt$$

is continuous and piecewise smooth. Compute its Fourier coefficients and compare them with those of the termwise-integrated series.

74. Show that

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

by completing the following outline:

(i) For any nonnegative integer n, let

$$a_n = \int_{n\pi}^{(n+1)\pi} \frac{\sin x}{x}\,dx.$$

Show that $a_{2m} > 0$ and $a_{2m+1} < 0$ for every nonnegative integer m. By introducing the substitution $x = t + \pi$ in the integral for a_{n+1}, show that $|a_{n+1}| < |a_n|$, therefore $\{|a_n|\}$ is a monotonically decreasing sequence. Introducing the substitution $x = t + n\pi$, show that $a_n \to 0$ as $n \to \infty$, and therefore that the series $\sum_{n=0}^\infty a_n$ converges. Now let $n\pi \le A < (n+1)\pi$. Then

$$\int_0^A \frac{\sin x}{x}\,dx = \int_0^{n\pi} \frac{\sin x}{x}\,dx + \int_{n\pi}^A \frac{\sin x}{x}\,dx.$$

Introduce the substitution $x = t + n\pi$ to show that the last integral tends to zero as $A \to \infty$, and deduce that the integral

$$\int_0^\infty \frac{\sin x}{x}\,dx$$

converges.

(ii) Integrate by parts to show that

$$\lim_{\lambda\to\infty} \int_0^a f(x)\sin \lambda x\,dx = 0$$

for any continuously differentiable function $f(x)$ on $[0, a]$. (We showed this in class for piecewise continuous functions!) Thus

$$\lim_{\lambda\to\infty} \int_0^\pi \sin \lambda x \left(\frac{1}{x} - \frac{1}{2\sin\frac{x}{2}}\right) dx = 0.$$

(iii) Integrate formulas (1) and (2) to show that

$$\int_0^\pi \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\frac{x}{2}}\,dx = \frac{\pi}{2}.$$

(iv) Show that

$$\int_0^a \frac{\sin \lambda x}{x}\,dx = \int_0^{\lambda a} \frac{\sin x}{x}\,dx.$$

Let $a = \pi$ and $\lambda = n + \frac{1}{2}$ to deduce that

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

75. (i) Show that, for $0 < x < \pi$,

$$\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{\sin nx}{n} \equiv \phi(x).$$

What is $\phi(x)$ for values of x outside this interval?

(ii) Show that $\phi(x)$ has a jump discontinuity at $x = 0$. What is its size? What is $\phi(0)$?

(iii) Integrate formulas (1) and (2) to show that

$$S_n(x) = \sum_{k=1}^{n} \frac{\sin kx}{k} = -\frac{x}{2} + \int_0^x \frac{\sin\left(n + \frac{1}{2}\right)t}{2\sin\frac{1}{2}t} dt.$$

(iv) Show that $\phi(x) - S_n(x) = \sigma_n(x) + \rho_n(x)$, with

$$\sigma_n(x) = \frac{\pi}{2} - \int_0^x \frac{\sin\left(n + \frac{1}{2}\right)t}{t} dt, \qquad \rho_n(x) = \int_0^x \frac{2\sin\frac{1}{2}t - t}{2\sin\frac{1}{2}t} \sin\left(n + \tfrac{1}{2}\right) t\; dt.$$

(v) Show that $\rho_n(x) \to 0$ uniformly on $0 < x < \pi$ as $n \to \infty$.

(vi) Show that

$$\sigma_n(x) = \frac{\pi}{2} - \int_0^{\left(n+\frac{1}{2}\right)x} \frac{\sin t}{t} dt.$$

Since, by problem 74,

$$\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2},$$

$\sigma_n(x) \to 0$ as $n \to \infty$ for fixed x.

(vii) Show that $\sigma_n(x)$ has extrema at the points $x_k = 2k\pi/(2n+1)$, for $k = 1, 2, 3, \ldots$, minima at x_1, x_3, x_5, $\ldots$ and maxima at x_2, x_4, $\ldots$.

(viii) Show that the values $\sigma_n(x_{2k+1})$ at the minima form an increasing sequence, and that thus the biggest oscillation of $\sigma_n(x)$ is at x_1. Show that

$$\begin{aligned}\sigma_n(x_1) &= \frac{\pi}{2} - \int_0^{\pi} \frac{\sin t}{t} dt \\ &= \pi \left(\frac{1}{2} - 1 + \frac{\pi^2}{2 \cdot 3 \cdot 3} - \frac{\pi^4}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 5} \right. \\ &\qquad \left. + \frac{\pi^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 7} \right) \\ &\approx -0.090 \cdots \pi.\end{aligned}$$

This overshoot is called the Gibbs phenomenon.

76. Suppose f is a continuous function on $\mathbb{R}$, $f(x+2\pi) = f(x)$, and α/π is irrational. Show that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x+n\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, dt.$$

HINT: Do it first for $f(x) = e^{ikx}$, k integer. Then use the Weierstrass approximation theorem for trigonometric functions.

77. (i) Show that

$$\langle f, g \rangle = \frac{1}{b-a} \int_a^b f(x)\bar{g}(x)\, dx$$

is an inner product on the space of piecewise continuous, complex-valued functions. What is the corresponding induced norm $\|f\|_2$?

(ii) Let $f(x)$ be a piecewise continuous function and let α_n be its complex Fourier coefficients. For any set of complex numbers β_n, $n = -N, \ldots, N$, show that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f(x) - \sum_{n=-N}^{N} \beta_n e^{inx} \right|^2 dx = \|f\|_2^2 - \sum_{n=-N}^{N} |\alpha_n|^2 + \sum_{n=-N}^{N} |\alpha_n - \beta_n|^2.$$

Conclude that its N-th order Fourier partial sum minimizes the distance in the $\|\cdot\|_2$ norm between f and N-th order trigonometric polynomials.

(iii) Prove *Parseval's equality*: For any continuous, 2π-periodic real function $f(x)$, if $\{a_n, b_n\}$ are its Fourier coefficients, show that

$$\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) = \|f\|_2^2.$$

In the process, also prove that if $S_n(x)$ is the n-th Fourier partial sum of the function $f(x)$,

$$\lim_{n\to\infty} \|f - S_n\|_2^2 \to 0.$$

In other words, the Fourier series converges to f in the $\|\cdot\|_2$ norm.

HINT: By the Weierstrass approximation theorem for trigonometric polynomials (How did we prove this theorem in class?), there exist trigonometric polynomials $T_n(x)$ such that $f(x) - T_n(x) \to 0$ uniformly in x. Use this fact and (ii).

(iv) Show that Parseval's equality and the convergence of the Fourier series in the $\|\cdot\|_2$ norm remain valid if f has a finite number of jump discontinuities.

HINT: Put a sufficiently steep straight line through each discontinuity to approximate f with a continuous function in the $\|\cdot\|_2$ norm.

78. (i) Let $f(x)$ be piecewise continuous and 2π-periodic, and let $\{a_n, b_n\}$ be its Fourier coefficients. Use Parseval's equality to show that the mapping

$$f \to \left\{\frac{a_0}{\sqrt{2}}, a_1, b_1, a_2, b_2, \ldots\right\}$$

defines an *isometry* (distance-preserving mapping) from the space of piecewise continuous functions equipped with the norm $\|.\|_2$ into the Hilbert space ℓ_2, discussed in problem 58.

(ii) Show that this mapping is 1-1. Speculate whether it is onto or not.

79. If f is continuous on $[0,1]$ and if

$$\int_0^1 f(x)x^n\,dx = 0, \qquad n = 0, 1, \ldots,$$

show that $f(x) = 0$ on $[0,1]$.

HINT: The integral of the product of f with any polynomial is zero. Use the Weierstrass theorem to show that

$$\int_0^1 f^2(x)\,dx = 0.$$

80. Following the outline below, provide another proof of the **Weierstrass approximation theorem**: For every continuous function $f(x)$ on $[a,b]$ and every $\epsilon > 0$, there exists a polynomial $P(x)$ such that $|f(x) - P(x)| < \epsilon$ for all $x \in [a,b]$.

(i) Show that, with no loss of generality, $[a, b] = [0, 1]$, and $f(0) = f(1) = 0$.

Define $f(x) = 0$ for x outside $[0, 1]$.

(ii) Let $g(x) = (1 - x^2)^n - 1 + nx^2$. Show that $g(0) = 0$ and $g'(x) > 0$ on $(0, 1)$ to conclude that $g(x) \geq 0$ on $[0, 1]$. Conclude that

$$\int_{-1}^{1} (1 - x^2)^n dx \geq 2 \int_{0}^{1/\sqrt{n}} (1 - nx^2)\, dx \geq \frac{1}{\sqrt{n}}.$$

(iii) Let

$$Q_n(x) = c_n(1 - x^2)^n, \qquad c_n = \left(\int_{-1}^{1} (1 - x^2)^n dx \right)^{-1}, \qquad n = 1, 2, \ldots .$$

Given $\delta > 0$, show that $Q_n(x) \to 0$ uniformly on $\delta \leq |x| \leq 1$.

HINT: $Q_n(x) \leq \sqrt{n}(1 - \delta^2)^n$ there.

(iv) Let

$$P_n(x) = \int_{-1}^{1} f(x + t) Q_n(t)\, dt, \qquad 0 \leq x \leq 1.$$

Change the integration variable $t \to t - x$ to show that $P_n(x)$ is a polynomial.

HINT: $f(x) = 0$ for x outside $[0, 1]$.

(v) Given $\epsilon > 0$, estimate

$$|P_n(x) - f(x)| = \left| \int_{-1}^{1} \left[f(x + t) - f(x) \right] Q_n(t)\, dt \right|$$

to show it is $< \epsilon$.

HINT: Use $\int_{-1}^{1} Q_n(t)\, dt = 1$, then consider the integral on three intervals $[-1, -\delta]$, $[-\delta, \delta]$, and $[1, \delta]$. Use (iii) for the outer two intervals, and uniform continuity of f on the middle interval.

REMARK: Reading pages 296 through 307 of the Strichartz book will be very illuminating.

81. Following the outline below, provide yet another proof of the **Weierstrass approximation theorem**: For every continuous function $f(x)$ on $[0, 1]$ and every $\epsilon > 0$, there exists a polynomial $B_n(x)$ such that $|f(x) - B_n(x)| < \epsilon$ for all $x \in [0, 1]$. In fact, one can take

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k}\, f\left(\frac{k}{n}\right) \tag{13}$$

for some large enough n.

(i) Show that

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1. \tag{14}$$

Differentiate (14) on x and multiply by $x(1-x)$; differentiate again and use (14); finally multiply by $x(1-x)/n^2$. You should obtain

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \left(x - \frac{k}{n}\right)^2 = \frac{x(1-x)}{n}. \tag{15}$$

(ii) Use (14) to show

$$f(x) - B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \left[f(x) - f\left(\frac{k}{n}\right)\right]. \tag{16}$$

Given $\epsilon > 0$, argue that there exists a $\delta > 0$ such that

$$\left|f(x) - f\left(\frac{k}{n}\right)\right| < \frac{\epsilon}{2}$$

if

$$\left|x - \frac{k}{n}\right| < \delta.$$

Taking appropriate absolute values, estimate the size of (16) by two sums, $\sum_1$ and $\sum_2$. The sum $\sum_1$ runs over all the terms for which

$$\left|x - \frac{k}{n}\right| < \delta.$$

Show that $\sum_1 < \epsilon/2$.

Now show that $\sum_2$ can be made less than $\epsilon/2$ as follows: Let $K = \max|f(x)|$ on $[0,1]$. Then

$$\sum\nolimits_2 \leq 2K \sum \binom{n}{k} x^k (1-x)^{n-k} \equiv 2K \sum\nolimits_3,$$

where $\sum_3$ is taken over all k such that

$$\left|x - \frac{k}{n}\right| \geq \delta.$$

Use (15) to show that

$$\sum\nolimits_3 \leq \frac{x(1-x)}{\delta^2 n},$$

and conclude that for n large enough $\sum_3 < \epsilon/4K$. This should let you finish the proof.

82. Let K be the unit circle in the complex plane (i.e., the set of all z with $|z| = 1$) and let $\mathcal{A}$ be the algebra of all functions of the form

$$f(e^{i\theta}) = \sum_{n=0}^{N} c_n e^{in\theta}, \qquad \theta \in \mathbb{R}.$$

Show that $\mathcal{A}$ separates points, yet there are continuous functions on K which are not in the uniform closure of $\mathcal{A}$. What are those functions?

HINT: What is $\int_0^{2\pi} f(e^{i\theta})e^{i\theta}\,d\theta$?

83. (i) If $f(0,0) = 0$ and

$$f(x,y) = \frac{xy}{x^2+y^2}, \qquad (x,y) \neq (0,0)$$

show that $D_1 f(x,y)$ and $D_2 f(x,y)$ exist at every point of $\mathbb{R}^2$, although f is not continuous at $(0,0)$.

(ii) If f is a real-valued function defined in an open set $E \subset \mathbb{R}^2$, and if the partial derivatives $D_1 f$ and $D_2 f$ are bounded in E, then f is continuous.

HINT: $f(x+h, y+k) - f(x,y) = f(x+h, y+k) - f(x+h, y) + f(x+h, y) - f(x,y)$.

84. If f and g are differentiable real functions in $\mathbb{R}^n$, show that

$$\nabla(fg) = f\nabla g + g\nabla f,$$

and that

$$\nabla\left(\frac{1}{f}\right) = -\frac{1}{f^2}\nabla f$$

whenever $f \neq 0$.

85. Suppose $\mathbf{f}$ is a differentiable mapping of $\mathbb{R}^1$ into $\mathbb{R}^n$ such that $\|\mathbf{f}(t)\| = 1$ for every t. Show that $\mathbf{f}'(t) \cdot \mathbf{f}(t) = 0$ for every t. Interpret this result geometrically for $n = 2, 3$.

86. Define $f(0,0) = 0$ and

$$f(x,y) = \frac{x^3}{x^2+y^2}, \qquad (x,y) \neq (0,0).$$

(i) Show that $D_1 f$ and $D_2 f$ are bounded functions in $\mathbb{R}^2$. (Hence f is continuous by problem 83 (ii).)

(ii) Let $\mathbf{u}$ be any unit vector in $\mathbb{R}^2$. Show that the directional derivative $D_{\mathbf{u}} f(0,0)$ exists, and that its absolute value is at most 1.

(iii) Let γ be a differentiable mapping of $\mathbb{R}^1$ into $\mathbb{R}^2$ (in other words, γ is a differentiable curve in $\mathbb{R}^2$), with $\gamma(0) = (0,0)$ and $\|\gamma'(0)\| > 0$. Put $g(t) = f(\gamma(t))$ and show that g is differentiable for every $t \in \mathbb{R}^1$. If γ is continuously differentiable, show that so is g.

HINT: Dividing the tops and bottoms of fractions by some appropriate power of t will help in the limit as $t \to 0$.

(iv) Despite of this, show that $D_{\mathbf{u}} f(0,0) \neq D_1 f(0,0) u_1 + D_2 f(0,0) u_2$, so that f is not differentiable at $(0,0)$.

87. Define $f(0,0) = 0$ and

$$f(x,y) = \frac{xy\,(x^2 - y^2)}{x^2 + y^2}, \qquad (x,y) \neq (0,0).$$

Show that

(i) f, $D_1 f$, and $D_2 f$ are continuous in $\mathbb{R}^2$.

(ii) $D_{12} f$ and $D_{21} f$ exist at every point in $\mathbb{R}^2$, and are continuous except at $(0,0)$.

(iii) $D_{12} f(0,0) = 1$ and $D_{21} f(0,0) = -1$.

88. (i) Let f and g be twice continuously differentiable. Show that the function $u(x,t) = f(x - ct) + g(x + ct)$ solves the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}.$$

(ii) Let ϕ be twice and ψ be once continuously differentiable. Show that the solution of the wave equation satisfying the initial conditions

$$u(x,0) = \phi(x), \qquad \frac{\partial u}{\partial t}(x,0) = \psi(x),$$

is

$$u(x,t) = \frac{1}{2}\Big[\phi(x - ct) + \phi(x + ct)\Big] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s)\, ds.$$

89. A smooth function $f : \mathbb{R}^n \to \mathbb{R}$ is called *homogeneous* of degree h if $f(t\mathbf{x}) = t^h f(\mathbf{x})$ for every $t \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$. Show that f is homogeneous of degree h if and only if it satisfies the differential equation $\mathbf{x} \cdot \nabla f(\mathbf{x}) = hf(\mathbf{x})$.

HINT: To show the only if part, derive a differential equation for the function $g(t) = f(t\mathbf{x}) - t^h f(\mathbf{x})$.

90. Let $\mathbf{f} = (x, y, z) : A \to \mathbb{R}^3$, with $A \subset \mathbb{R}^2$, be continuously differentiable, i.e., a smooth parametrization of a surface. Let I_j, $j = 1, 2$, be two intervals, and let $\gamma_j : I_j \to A$ be two smooth curves in A.

(i) At any point in A where the curves γ_1 and γ_2 cross, i.e., $\gamma_1(t) = \gamma_2(s)$, show that the angle θ between the two image curves $\mathbf{f}(\gamma_1(t))$ and $\mathbf{f}(\gamma_2(s))$ in $\mathbb{R}^3$ is given by the formula

$$\cos\theta = \frac{\gamma_1'(t) \cdot M(u,v)\gamma_2'(s)}{\sqrt{\gamma_1'(t) \cdot M(u,v)\gamma_1'(t)}\sqrt{\gamma_2'(s) \cdot M(u,v)\gamma_2'(s)}},$$

where (u, v) are the coordinates in A, and $M(u, v)$ is the matrix

$$M(u,v) = \begin{pmatrix} E & F \\ F & G \end{pmatrix},$$

with

$$E = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 = D_u\mathbf{f} \cdot D_u\mathbf{f}\ ,$$

$$F = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} = D_u\mathbf{f} \cdot D_v\mathbf{f}\ ,$$

$$G = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = D_v\mathbf{f} \cdot D_v\mathbf{f}\ .$$

(ii) A continuously differentiable planar map

$$x = \phi(u,v), \qquad y = \psi(u,v)$$

is called *conformal* if it maps two intersecting curves into two others enclosing the same angle as the original ones. Show that the necessary and sufficient condition that a planar map is conformal is is that the *Cauchy-Riemann equations*

$$\frac{\partial \phi}{\partial u} - \frac{\partial \psi}{\partial v} = 0, \qquad \frac{\partial \phi}{\partial v} + \frac{\partial \psi}{\partial u} = 0$$

or

$$\frac{\partial \phi}{\partial u} + \frac{\partial \psi}{\partial v} = 0, \qquad \frac{\partial \phi}{\partial v} - \frac{\partial \psi}{\partial u} = 0$$

hold.

Use the result of problem 91 below to show that in the first case the direction of the angles is preserved, while in the second case it is reversed.

HINT: Adapt part (i) to planar maps. If $(u, v) \to (x, y)$ is conformal, it must map orthogonal curves into orthogonal curves. Choose a pair of straight lines parallel to the (u, v) coordinate axes and the same pair rotated by $\pi/4$ to show that $F = E - G = 0$, and infer the Cauchy-Riemann equations. The converse is straight forward.

91. Let $\mathbf{f} : A \to \mathbb{R}^2$, with $A \subset \mathbb{R}^2$, be given in components as $x = \phi(u, v)$ and $y = \psi(u, v)$. Show that $\mathbf{f}$ preserves or reverses orientation, depending on whether the Jacobian

$$\det \mathbf{f}'(u, v) = \frac{\partial(\phi, \psi)}{\partial(u, v)}$$

is positive or negative, by carrying out the following outline:

(i) Let $\gamma(t) = (u(t), v(t))$ be a curve in A. Argue that its slope is $m(t) = v'(t)/u'(t)$.

(ii) Show that the slope of the curve $\mathbf{f}(\gamma(t))$ is given by

$$\mu(t) = \frac{c + dm}{a + bm},$$

where the quantities a, b, c and d are the partial derivatives of the function $\mathbf{f}$, that is,

$$\mathbf{f}'(u, v) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(iii) Compute $d\mu/dm$ to show that μ increases or decreases with m depending on whether $\det \mathbf{f}'$ is positive or negative. Argue that this implies the counterclockwise or clockwise rotation of the curve $\mathbf{f}(\gamma(t))$ if the curve $\gamma(t)$ is rotated counterclockwise, which is the preservation or reversal of orientation.

92. (i) Let $\phi_{ij}(t)$, $i, j = 1, \dots, n$, be continuously differentiable, and let $W(t)$ be the determinant

$$W(t) = \begin{vmatrix} \phi_{11}(t) & \dots & \phi_{1n}(t) \\ \vdots & \ddots & \vdots \\ \phi_{n1}(t) & \dots & \phi_{nn}(t) \end{vmatrix}.$$

Show that

$$W'(t) = \sum_{i=1}^{n} \begin{vmatrix} \phi_{11}(t) & \dots & \phi_{1n}(t) \\ \vdots & \ddots & \vdots \\ \phi'_{i1}(t) & \dots & \phi'_{in}(t) \\ \vdots & \ddots & \vdots \\ \phi_{n1}(t) & \dots & \phi_{nn}(t) \end{vmatrix}.$$

HINT: First use known facts from linear algebra to show that

$$\frac{\partial W}{\partial \phi_{ij}} = (-1)^{i+j+1} W_{ij},$$

where W_{ij} is the cofactor of the element ϕ_{ij}, i.e., the determinant obtained by erasing the i-th row and j-th column from the determinant W.

(ii) Let $A(t)$ be and $n \times n$ matrix with continuous entries $a_{ij}(t)$. Let the $n \times n$ matrix $\Phi(t)$ with entries $\phi_{ij}(t)$ be a solution of the matrix differential equation $\Phi'(t) = A(t)\Phi(t)$, i.e., each column of $\Phi(t)$ solves the linear system $\mathbf{x}' = A(t)\mathbf{x}$. Use (i) to show that $W(t) = \det \Phi(t)$ satisfies the equation

$$W(t) = W(t_0) \exp\left(\int_{t_0}^{t} \operatorname{trace} A(s)\, ds \right), \qquad \operatorname{trace} A = \sum_{i=1}^{n} a_{ii}.$$

HINT: Derive a differential equation for $W(t)$.

93. (i) If $\mathbf{f}$ is a differentiable mapping of a *connected* open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$, and if $\mathbf{f}'(\mathbf{x}) = 0$ for every $\mathbf{x} \in E$, show that $\mathbf{f}$ is constant in E.

HINT: Apply the mean-value theorem to the appropriate directional derivative to show that if $\mathbf{f}(\mathbf{x}_0) = \mathbf{c}$ at some point $\mathbf{x} = \mathbf{x}_0$, then $\mathbf{f}(\mathbf{x}) = \mathbf{c}$ in some open ball $B_r(\mathbf{x}_0)$, so that the set of points on which $\mathbf{f}(\mathbf{x}) = \mathbf{c}$ is open. On the other hand, show that the set of points on which $\mathbf{f}(\mathbf{x}) \neq \mathbf{c}$ must also be open, and thus empty. (Why?)

(ii) A subset of $\mathbb{R}^n$ is called *convex* if, together with every pair of its points, it also contains the straight line connecting them.

If f is a real function defined in a convex open set $E \subset \mathbb{R}^n$, such that $D_1 f(\mathbf{x}) = 0$ for every $\mathbf{x} \in E$, show that $f(\mathbf{x})$ depends only on $x_2, \dots, x_n$.

Show that the convexity of E can be replaced by a weaker condition, but that some condition is required. For example, if $n = 2$ and E is shaped like a horseshoe, the statement may be false.

HINT: The condition is that points in E with the same coordinate x_1 be connected by a straight line contained in E.

94. Define $f(0,0) = 0$ and

$$f(x,y) = x^2 + y^2 - 2x^2y - \frac{4x^6y^2}{(x^4+y^2)^2}, \qquad (x,y) \neq (0,0).$$

(i) Show, for all $(x,y) \in \mathbb{R}^2$, that

$$4x^4y^2 \leq \left(x^4 + y^2\right)^2.$$

Conclude that f is continuous.

(ii) For $0 \leq \theta \leq 2\pi$, $-\infty < t < \infty$, define

$$g_\theta(t) = f(t\cos\theta, t\sin\theta).$$

Show that $g_\theta(0) = 0$, $g'_\theta(0) = 0$, $g''_\theta(0) = 2$. Each g_θ has therefore a strict local minimum at $t = 0$. (In other words, the restriction of f to each line through $(0,0)$ has a strict local minimum at $(0,0)$.)

(iii) Show that $(0,0)$ is nevertheless not a local minimum for f, since $f(x,x^2) = -x^4$.

95. (i) Let $f(\mathbf{r}) = 1/r$, with $\mathbf{r} = (x,y,z)$ and $r = \|\mathbf{r}\| = \sqrt{x^2+y^2+z^2}$. Find the four-term (including the remainder) Taylor expansion of the function $f(\mathbf{r}-\mathbf{h})$, with $\mathbf{h} = (h_1, h_2, h_2)$ for small values of $\|\mathbf{h}\|$. Show that the remainder can be bounded by $C\|\mathbf{h}\|^3/r^4$ for some appropriate constant C. Conclude that, up to a scale difference, the small $\|\mathbf{h}\|$ expansion gives the same result as the large $\|\mathbf{r}\|$ expansion.

HINT: For sufficiently small $\|\mathbf{h}\|/r$, we have $\|\mathbf{r} - \theta\mathbf{h}\| \geq r/2$ for any $0 \leq \theta \leq 1$.

(ii) Let e_j, $j = 1, \ldots, N$, be electrostatic charges forming a neutral charge cloud, $\sum_{j=1}^N e_j = 0$. Let the charge e_j be fixed at the position $\mathbf{r}_j = (x_j, y_j, z_j)$. Using the result of (i), show that the electrostatic potential $U(\mathbf{r})$ produced by these charges is given by the formula

$$U(\mathbf{r}) \equiv \sum_{j=1}^N \frac{e_j}{\|\mathbf{r}-\mathbf{r}_j\|} = \frac{\mathbf{p}\cdot\mathbf{r}}{r^3} + \frac{\mathbf{r}\cdot Q\mathbf{r}}{2r^5} + \mathcal{O}\left(\frac{R^3}{r^4}\right),$$

where $R = \max\limits_{j=1,\ldots,N} r_j$,

$$\mathbf{p} = \sum_{j=1}^N e_j \mathbf{r}_j,$$

and

$$Q = \sum_{j=1}^{N} e_j \left(3\mathbf{r}_j \otimes \mathbf{r}_j - r_j^2 \right).$$

Here $\mathbf{r} \otimes \mathbf{r}$ is the "tensor product," i.e., the matrix product $\mathbf{r}\mathbf{r}^T$, with the $\mathbf{r}$ being a column and $\mathbf{r}^T$ a row vector.

REMARK: The vector $\mathbf{p}$ is known as the dipole moment of the charge cloud, and Q as the quadrupole matrix.

96. Fix two real numbers, $0 < a < b$. Define a mapping $\mathbf{f} = (f_1, f_2, f_3)$ of $\mathbb{R}^2$ into $\mathbb{R}^3$ by

$$\begin{aligned} f_1(s,t) &= (b + a\cos s)\cos t \\ f_2(s,t) &= (b + a\cos s)\sin t \\ f_3(s,t) &= a\sin s. \end{aligned}$$

(i) Show that the range K of this mapping is a torus in $\mathbb{R}^3$.

(ii) Show that there are exactly four points on this torus for which ∇f_1 vanishes. Find these points, and show that one corresponds to a local maximum of f_1, one to a local minimum, and two to saddles.

(iii) Determine the set of points on the torus for which ∇f_3 vanishes. Which of these points correspond to maxima, minima, or saddles?

97. Show that the continuity of $\mathbf{f}'$ at the point $\mathbf{a}$ is needed in the inverse function theorem, even in the case $n = 1$: If

$$f(t) = t + 2t^2 \sin\left(\frac{1}{t}\right)$$

for $t \neq 0$ and $f(0) = 0$, then $f'(0) = 1$, f' is bounded in $(-1, 1)$, but f is not one-to-one in any neighborhood of 0.

98. Let $\mathbf{f} = (f_1, f_2)$ be the mapping $\mathbb{R}^2$ into $\mathbb{R}^2$ given by

$$f_1(x,y) = e^x \cos y, \qquad f_2(x,y) = e^x \sin y.$$

(i) What is the range of $\mathbf{f}$?

(ii) Show that the Jacobian of $\mathbf{f}$ is nonzero at any point of $\mathbb{R}^2$. Thus every point of $\mathbb{R}^2$ has a neighborhood in which $\mathbf{f}$ is one-to-one. Nevertheless, $\mathbf{f}$ is not one-to-one on $\mathbb{R}^2$.

(iii) Put $\mathbf{a} = (0, \pi/3)$, $\mathbf{b} = \mathbf{f}(\mathbf{a})$, and let $\mathbf{g}$ be the continuous inverse of $\mathbf{f}$, defined in a neighborhood of $\mathbf{b}$, such that $\mathbf{g}(\mathbf{b}) = \mathbf{a}$. Find an explicit formula for $\mathbf{g}$, compute $\mathbf{f}'(\mathbf{a})$ and $\mathbf{g}'(\mathbf{b})$, and verify that they are each other's inverses.

(iv) What are the images under $\mathbf{f}$ of lines parallel to the coordinate axes?

99. Show that the system of equations

$$\begin{aligned} 3x + y - z + u^2 &= 0 \\ x - y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0 \end{aligned}$$

can be solved for x, y, u, in terms of z; for x, z, u, in terms of y; for y, z, u, in terms of x; but not for x, y, z, in terms of u.

100. Define f in $\mathbb{R}^2$ by

$$f(x, y) = 2x^3 - 3x^2 + 2y^3 + 3y^2.$$

(i) Find the four points at which the gradient of f is zero. Show that f has exactly one local maximum and one local minimum in $\mathbb{R}^2$.

(ii) Let S be the set of all $(x, y) \in \mathbb{R}^2$ at which $f(x, y) = 0$. Show that S is the union of a straight line and an ellipse. Find those points of S that have no neighborhoods in which the equation $f(x, y) = 0$ can be solved for y in terms of x or x in terms of y.

HINT: Rewrite

$$f(x, y) = (x + y)(2x^2 - 2xy + 2y^2 - 3x + 3y).$$

Diagonalize the quadratic form in the second factor and complete a square to find the ellipse.

101. Define f in $\mathbb{R}^3$ by

$$f(x, y_1, y_2) = x^2 y_1 + e^x + y_2.$$

Show that $f(0, 1, -1) = 0$, $D_1 f(0, 1, -1) \neq 0$, and that there exists therefore a differentiable function g in some neighborhood of $(1, -1)$ in $\mathbb{R}^2$ such that $g(1, -1) = 0$ and

$$f(g(y_1, y_2), y_1, y_2) = 0.$$

Find $D_1 g(1, -1)$ and $D_2 g(1, -1)$.

102. By following the outline below, prove the one-dimensional version of the **implicit function theorem**: Let $F : A \to \mathbb{R}$, with $A \subset \mathbb{R}^2$, be continuously differentiable. Let the equation $F(x_0, y_0) = 0$ hold at some point (x_0, y_0), and let $D_y F(x_0, y_0) \neq 0$. Then

there exists an open interval $x_1 < x < x_2$ containing x_0 on which the equation $F(x, y) = 0$ defines a unique function $y = f(x)$ with $f(x_0) = y_0$ and $F(x, f(x)) = 0$. Moreover, $f(x)$ is continuously differentiable on $x_1 < x < x_2$, with

$$f'(x) = -\frac{D_x F(x, f(x))}{D_y F(x, f(x)).}$$

(i) Assume $D_y F(x_0, y_0) > 0$ with no loss of generality. (Why?) Using the continuous differentiability of F, conclude that, for all x and y in some rectangle $x_1 < x < x_2$, $y_1 < y < y_2$ around (x_0, y_0), the function $F(x, y)$ increases monotonically in y along every line $x =$ constant. Use $F(x_0, y_0) = 0$ to show that $F(x, y_1) < 0$ and $F(x, y_2) > 0$ on $x_1 < x < x_2$. Infer that for every x in $x_1 < x < x_2$, there is a unique value $y = f(x)$ such that $F(x, f(x)) = 0$.

(ii) Let x and $x + h$ be two points in $x_1 < x < x_2$, let $y = f(x)$ and $y + k = f(x + h)$. Use the two term Taylor formula to show that

$$\frac{k}{h} = -\frac{D_x F(x + \theta h, y + \theta k)}{D_y F(x + \theta h, y + \theta k)}. \tag{17}$$

Bound the right-hand side of this equation and conclude that $|k| < C|h|$ for some constant C, and therefore $f(x)$ is continuous.

(iii) Use (17) again to show that

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = -\frac{D_x F(x, f(x))}{D_y F(x, f(x))}.$$

Conclude that $f'(x)$ exists and is continuous.

103. (i) Let $\mathbf{r} = (x, y, z)$, and let $\mathbf{g} : A \to \mathbb{R}^3$, with $A \subset \mathbb{R}^2$ open, be a smooth parametrization of a surface S, i.e., rank $(g') = 2$. Show that the tangent plane $T_{\mathbf{g}(u,v)}S$ of the surface S at the point $\mathbf{g}(u, v)$ is spanned by the vectors $\mathbf{g}_u$ and $\mathbf{g}_v$. (Here the subscripts denote the partial derivatives on the variable in the subscript.) Find a normal to S at $\mathbf{g}(u, v)$, and show that the equation of the tangent plane is

$$\mathbf{g}_u \times \mathbf{g}_v \cdot [\mathbf{r} - \mathbf{g}(u, v)] = 0.$$

(ii) Repeat the discussion of (i) for the explicit parametrization of the surface S given by $\mathbf{g}(x, y) = (x, y, f(x, y))$. Find the explicit expressions all the vectors involved in terms of f_x and f_y.

(iii) Find the equation of the tangent plane and the normal at any point of the surface S described implicitly by $F(x, y, z) = 0$.

(iv) Following (i), compute two basis vectors of the tangent plane, the equation of the tangent plane, and a normal at any point of the torus given in problem 96.

104. Let $f : M_m \to \mathbb{R}$ be a C^1 function, where $M_m \subset \mathbb{R}^n$ is a C^1 surface. Show that every point in M_m lies in a neighborhood $U \subset \mathbb{R}^n$ such that there exists a C^1 function $F : U \to \mathbb{R}$ that extends f, i.e., $F(y) = f(y)$ for y in $M_m \cap U$.

HINT: Use the local explicit (graph) representation of the surface M_m.

105. Let M_2 be any compact two-dimensional surface in $\mathbb{R}^3$. Show that for any two dimensional vector subspace V in $\mathbb{R}^3$, there exists a point x on M_2 whose tangent space equals V.

HINT: If u is a vector perpendicular to V, what happens at points on M_2 where $x \cdot u$ achieves a maximum or a minimum?

106. A matrix $M \in \mathbb{R}^{n\times n}$ is orthogonal if $M^T M = I$, where ^{T} denotes the transpose and I the $n \times n$ identity matrix. (Therefore also $MM^T = I$.) Show that the orthogonal $n \times n$ matrices form a C^1 surface of dimension $n(n-1)/2$ in $\mathbb{R}^{n\times n}$. How many connected components does it consists of?

HINT: $\det M^T = \det M$.

107. A linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ is self adjoint if $\langle T\mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, T\mathbf{y}\rangle$ for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

(i) Let T be a self-adjoint linear map, with matrix $A = (a_{ij})$, which is symmetric, so that $a_{ij} = a_{ji}$. If $f(\mathbf{x}) = \langle T\mathbf{x}, \mathbf{x}\rangle = \sum a_{ij}x_i x_j$, show that $D_k f(\mathbf{x}) = 2\sum_{j=1}^{n} a_{kj}x_j$. Then, by considering the maximum of $\langle T\mathbf{x}, \mathbf{x}\rangle$ on the unit sphere $S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| = 1\} \subset \mathbb{R}^n$ show that there is $\mathbf{x} \in S^{n-1}$ and $\lambda \in \mathbb{R}$ with $T\mathbf{x} = \lambda\mathbf{x}$.

(ii) If $V = \{\mathbf{y} \in \mathbb{R}^n : \langle \mathbf{x}, \mathbf{y}\rangle = 0\}$, show that $T(V) \subset V$ and $T : V \to V$ is self-adjoint.

(iii) Show that T has a basis of eigenvectors.

HINT: In any orthonormal basis on V, the matrix of $T : V \to V$ is symmetric.

108. (i) Find the maximum of the function $f(x, y, z) = x^2y^2z^2$ on the sphere $x^2+y^2+z^2 = c^2$. Conclude the inequality

$$\left(x^2y^2z^2\right)^{\frac{1}{3}} \le \frac{x^2 + y^2 + z^2}{3},$$

which states that the geometric mean of three nonnegative numbers x^2, y^2, z^2 is never greater than their arithmetic mean.

(ii) Prove the same result in $\mathbb{R}^n$.

109. (i) Let two positive numbers α and β be such that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

Find the minimum of the expression

$$f(u, v) = \frac{u^\alpha}{\alpha} + \frac{v^\beta}{\beta}$$

subject to the condition $uv = 1$. Conclude that

$$uv \leq \frac{u^\alpha}{\alpha} + \frac{v^\beta}{\beta}. \tag{18}$$

HINT: If $uv \neq 1, 0$, consider $ut^{\frac{1}{\alpha}}$ and $vt^{\frac{1}{\beta}}$, where $t = 1/uv$.

(ii) Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be nonnegative numbers and let at least one x_j and at least one y_k be nonzero. Prove Hölder's inequality

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^\alpha \right)^{\frac{1}{\alpha}} \left(\sum_{i=1}^{n} y_i^\beta \right)^{\frac{1}{\beta}}.$$

HINT: Let

$$u = \frac{x_j}{\left(\sum_{i=1}^{n} x_i^\alpha \right)^{\frac{1}{\alpha}}}, \qquad v = \frac{y_j}{\left(\sum_{i=1}^{n} y_i^\beta \right)^{\frac{1}{\beta}}}, \qquad j = 1, \ldots, n,$$

in (18) and sum over j.

110. (i) Show that the point on the closed surface $\phi(x, y, z) = 0$ that is the closest to (or farthest from) the given point (ξ, η, ζ) lies on the straight line

$$\frac{(x - \xi)}{\phi_x} = \frac{(y - \eta)}{\phi_y} = \frac{(x - \zeta)}{\phi_z},$$

normal to the surface.

(ii) Extend the result of (i) to m-dimensional surfaces in $\mathbb{R}^n$: Let M_m be a C^1 surface in $\mathbb{R}^n$ and let $\mathbf{y}$ be a in $\mathbb{R}^n$ not on M_n. If $\mathbf{x}$ is a point on M_m that minimizes or maximizes the distance to $\mathbf{y}$, prove that the line joining $\mathbf{x}$ and $\mathbf{y}$ is perpendicular to the surface M_m, i.e., its tangent space at $\mathbf{x}$.

HINT: It is easier to consider the square of the distance.

111. Let $f, g : A \to \mathbb{R}$, where A is a rectangle in $\mathbb{R}^n$, be integrable.

(i) For any partition and P and subrectangle S, show that

$$m_S(f) + m_S(g) \leq m_S(f+g) \quad \text{and} \quad M_S(f+g) \leq M_S(f) + M_S(g),$$

and therefore

$$L(f,P) + L(g,P) \leq L(f+g,P) \quad \text{and} \quad U(f+g,P) \leq U(f,P) + U(g,P).$$

(ii) Show that $f+g$ is integrable and $\int_A f + g = \int_A f + \int_A g$.

(iii) For any constant c, show that $\int_A cf = c\int_A f$.

(iv) If $f \leq g$, show that $\int_A f \leq \int_A g$.

(v) Show that $|f|$ is integrable and $\left|\int_A f\right| \leq \int_A |f|$.

112. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} 1 & x \text{ irrational}, \\ 1 & x \text{ rational}, \ y \text{ irrational}, \\ 1 - 1/q & x = p/q \text{ in lowest terms}, \ y \text{ rational}. \end{cases}$$

(i) Show that f is integrable and $\int_{[0,1]\times[0,1]} f = 1$.

HINT: Show that f is only discontinuous when x is rational.

(ii) Show that $\int_0^1 f(x,y)\,dy = 1$ if x is irrational and does not exist if x is rational. In general, a common "cure" for $h(x)$ not being defined at a few isolated points is to assign an arbitrary number to be $h(x)$ at all those points, and then proceed with the integration on x. Show that this is not necessarily possible here: h is not integrable if $h(x) = \int_0^1 f(x,y)\,dy$ is set arbitrarily to any number other than 1 when the integral does not exist. However, compute the lower and upper y-integrals of $f(x,y)$ for any x, show that they are integrable, and that they integrate to 1.

HINT: If x is rational, $f(x,y)$ is discontinuous at every y. Also, show that the lower y-integral of $f(x,y)$ is

$$L\int_0^1 f(x,y)\,dy = \begin{cases} 1 & x \text{ irrational,} \\ 1-1/q & x = p/q \text{ in lowest terms,} \end{cases}$$

which is only discontinuous at rationals.

113. Let $f : C \to \mathbb{R}$, where $C \subset \mathbb{R}^n$ is connected and Jordan measurable, be integrable. If $m = \inf_C f$ and $M = \sup_C f$, show that $\int_C f = \mu v(C)$, where $m \le \mu \le M$ and $v(C)$ is the volume of the set C. If C is compact and f continuous, show that $\mu = f(\xi)$ for some $\xi \in C$.

114. Let $f : [a,b] \to \mathbb{R}$ be integrable and non-negative and let

$$A_f = \{(x,y) : a \le x \le b \text{ and } 0 \le y \le f(x)\}.$$

Show that A_f is Jordan measurable and has area $\int_a^b f(x)\,dx$.

HINT: Let $P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$ be a partition of $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$. (Why does such a partition exist?) What is the total area of all the rectangles of the form $[x_i, x_{i+1}] \times \big[m_{[x_i,x_{i+1}]}(f), M_{[x_i,x_{i+1}]}(f)\big]$?

115. Use Fubini's theorem to derive an expression for the volume of a subset of $\mathbb{R}^3$ obtained by revolving a Jordan-measurable set in the yz-plane about the z-axis.

116. Show **Cavalieri's Principle**: Let A and B be Jordan measurable subsets of $\mathbb{R}^3$. Let $A_c = \{(x,y) : (x,y,c) \in A\}$ and define B_c similarly. Suppose each A_c and B_c are Jordan-measurable and have the same area. Then A and B have the same volume.

117. Let $A = [a_1,b_1] \times \cdots \times [a_n,b_n]$ and let $f : A \to \mathbb{R}$ be continuous. Define $I_A(f)$ to be the n-fold integral

$$I_A(f) = \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} f(x_1,\ldots,x_n)\,dx_1\ldots dx_n,$$

carried out in the order so that x_1 is integrated first, then x_2, and so on, until x_n. Use the Stone-Weierstrass theorem to show that this order is immaterial, and can thus be interchanged arbitrarily.

HINT: This is clear for functions of the form $h(x_1,\ldots,x_n) = h_1(x_1)\cdots h_n(x_n)$ and their sums. Show that the latter form an algebra $\mathcal{A}$ on the space of continuous real functions on A that separates points.

118. Let $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable and suppose that $D_1 g_2 = D_2 g_1$. Let

$$f(x,y) = \int_0^x g_1(t,0)\,dt + \int_0^y g_2(x,t)\,dt.$$

Show that $D_1 f(x,y) = g_1(x,y)$.

119. Let f be a continuously differentiable function that vanishes outside of a bounded interval, and let g be continuous. Let their *convolution* $f * g$ be defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy.$$

Show that $f * g$ is continuously differentiable and that $(f * g)' = f' * g$. Show the analogous result if f is n-times continuously differentiable.

120. Consider again the Weierstrass approximation theorem as proved in Problem 80. Let $f(x)$ be the function on $[0,1]$ that is to be uniformly approximated by polynomials. Suppose that f is k-times continuously differentiable.

(i) Show that we can assume f and all its derivatives up to $f^{(k)}$ to vanish at 0 and 1 with no loss of generality.

(ii) Use problem 119 to show that you can approximate f and all its derivatives up to $f^{(k)}$ by polynomials P through $P^{(k)}$.

121. (i) Suppose that $f : (0,1) \to \mathbb{R}$ is a non-negative continuous function. Show that $\int_{(0,1)} f$ exists if and only if $\lim_{\epsilon \to 0} \int_\epsilon^{1-\epsilon} f$ exists.

HINT: Recall from class that if Φ is a partition of unity on $(0,1)$, then every finite sum $\sum \varphi$ vanishes outside some $[a,b] \subset (0,1)$, and that for every $[c,d] \subset (0,1)$, there is a finite sum $\sum \varphi$ such that $\sum \varphi = 1$ on $[c,d]$.

(ii) Let $A_n = [1 - 1/2^n, 1 - 1/2^{n+1}]$. Suppose that $f : (0,1) \to \mathbb{R}$ satisfies $\int_{A_n} f = (-1)^n/n$ and $f(x) = 0$ for $x \notin$ any A_n. Show that $\int_{(0,1)} f$ does not exist, but $\lim_{\epsilon \to 0} \int_\epsilon^{1-\epsilon} f = \log 2$. HINT: $\int_{(0,1)} f = \sum \int_{(0,1)} \varphi f$ regardless of the order in the sum.

122. For $(x,y) \in \mathbb{R}^2$, define

$$\mathbf{F}(x,y) = (e^x \cos y - 1, e^x \sin y).$$

Show that $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$, where

$$\mathbf{G}_1(x,y) = (e^x \cos y - 1, y)$$
$$\mathbf{G}_2(u,v) = (u, (1+u)\tan v)$$

in some neighborhood of $(0,0)$. Compute the Jacobians of $\mathbf{G}_1$, $\mathbf{G}_2$, and $\mathbf{F}$ at $(0,0)$.

Define

$$\mathbf{H}_2(x,y) = (x, e^x \sin y)$$

and find

$$\mathbf{H}_1(u,v) = (h(u,v), v)$$

so that $\mathbf{F} = \mathbf{H}_1 \circ \mathbf{H}_2$ in some neighborhood of $(0,0)$.

123. (i) Derive the formula

$$\int_0^\infty \frac{dy}{x^2+y^2} = \frac{\pi}{2}\frac{1}{x},$$

and by repeated differentiations show that

$$\int_0^\infty \frac{dy}{(x^2+y^2)^n} = \frac{\pi}{2}\frac{1\cdot 3\cdots(2n-3)}{2\cdot 4\cdots(2n-2)}\frac{1}{x^{2n-1}}.$$

(ii) Use (i) to show that

$$\int_0^\infty \frac{dy}{(1+y^2/n)^n} = \frac{\pi}{2}\frac{1\cdot 3\cdots(2n-3)}{2\cdot 4\cdots(2n-2)}\sqrt{n}.$$

(iii) Write the integral

$$\int_0^\infty \left[e^{-y^2} - \frac{1}{(1+y^2/n)^n}\right] dy$$

as $\int_0^T + \int_T^\infty$. Use $(1+y^2/n)^n > y^2$ and the growth/decay properties of the exponential to show that the second integral is smaller in magnitude than $\epsilon/2$ for large enough T. Use the property of the alternating series that, in absolute value, the remainder is smaller than the first omitted term, to show that

$$x - n\log\left(1+\frac{x}{n}\right) \to 0$$

as $n \to \infty$ uniformly in $0 \le x \le X$. Conclude that

$$e^{-y^2} - \frac{1}{(1+y^2/n)^n} \to 0$$

as $n \to \infty$ uniformly in $0 \le y \le T$ so that the magnitude of $\int_0^T$ can also be brought below $\epsilon/2$. Thus conclude that

$$\lim_{n\to\infty} \frac{1\cdot 3\cdots(2n-3)}{2\cdot 4\cdots(2n-2)}\sqrt{n} = \frac{1}{\sqrt{\pi}}.$$

124. (i) Let I^k be the set of all $\mathbf{u} = (u_1, \ldots, u_k) \in \mathbb{R}^k$ with $0 \le u_i \le 1$ for all i; let Q^k be the set of all $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$ with $x_i \ge 0$ and $\sum_{i=1}^n x_i \le 1$. (I^k is the unit cube and Q^k is the standard simplex in $\mathbb{R}^k$.) Define $\mathbf{x} = T(\mathbf{u})$ by

$$\begin{aligned} x_1 &= u_1 \\ x_2 &= (1-u_1)u_2 \\ &\vdots \\ x_k &= (1-u_1)\cdots(1-u_{k-1})u_k. \end{aligned}$$

Show that

$$\sum_{i=1}^k x_i = 1 - \prod_{i=1}^k (1-u_i).$$

Show that T maps I^k onto Q^k, that T is 1-1 in the interior of I^k, and that its inverse S is defined in the interior of Q^k by $u_1 = x_1$ and

$$u_i = \frac{x_i}{1 - x_1 - \cdots - x_{i-1}}$$

for $k = 2, \ldots, k$. Show that

$$J_T \equiv \frac{\partial(x_1, \ldots, x_k)}{\partial(u_1, \ldots, u_k)} = (1-u_1)^{k-1}(1-u_2)^{k-2}\ldots(1-u_{k-1}),$$

and

$$J_S \equiv \frac{\partial(u_1, \ldots, u_k)}{\partial(x_1, \ldots, x_k)} = \frac{1}{(1-x_1)(1-x_1-x_2)\cdots(1-x_1-\cdots-x_{k-1})}.$$

HINT: To compute the Jacobians J_T and J_s, note that the matrices $T'(\mathbf{u})$ and $S'(\mathbf{x})$ are triangular, so that their determinants are the products of their diagonal elements.

(ii) Let $r_1, \ldots, r_k$ be nonnegative integers, and prove that

$$\int_{Q^k} x_1^{r_1} \ldots x_k^{r_k}\, dx = \frac{r_1! \cdots r_k!}{(k + r_1 + \cdots r_k)!}.$$

HINT: Use $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$.

125. Derive the formula for the volume of the n-dimensional ball in the following way.

(i) Consider the hyperspherical coordinates

$$\begin{aligned} x_1 &= r\cos\phi_1 \\ x_2 &= r\sin\phi_1\cos\phi_2 \\ &\vdots \\ x_k &= r\prod_{j=1}^{k-1}\sin\phi_j\cos\phi_k \\ &\vdots \\ x_{n-1} &= r\prod_{j=1}^{n-2}\sin\phi_j\cos\phi_{n-1} \\ x_n &= r\prod_{j=1}^{n-1}\sin\phi_k \end{aligned}$$

Argue that, since the transformation is linear in r,

$$\frac{\partial(x_1,\dots,x_n)}{\partial(r,\phi_1,\dots,\phi_{n-1})} = r^{n-1}\Omega_n(\phi_1,\dots,\phi_{n-1})$$

for some function $\Omega_n(\phi_1,\dots,\phi_{n-1})$. (No need to compute this Ω_n.)

(ii) Show that for any function $f(r)$ of r only,

$$\int_{B_R(0)} f(r)\,dx_1\cdots dx_n = \omega_n\int_0^R f(r)r^{n-1}\,dr \tag{19}$$

where

$$\omega_n = \int_0^\pi d\phi_1\dots\int_0^\pi d\phi_{n-2}\int_0^{2\pi}\Omega_n(\phi_1,\dots,\phi_{n-1})\,d\phi_{n-1}.$$

Here $B_R(0)$ is the ball of radius R around the origin.

(iii) Put $f(r) = e^{-r^2}$, and let $R\to\infty$ in (19) to show that

$$\left(\int_{-\infty}^\infty e^{-x^2}\,dx\right)^n = \omega_n\int_0^\infty e^{-r^2}r^{n-1}\,dr.$$

Conclude that

$$\omega_n = \frac{2\sqrt{\pi^n}}{\Gamma\left(\frac{n}{2}\right)}.$$

(iv) Use (19) to show that the volume of the ball $B_R(0)$ equals

$$v_n(R) = \frac{R^n\sqrt{\pi^n}}{\Gamma\left(\frac{n+2}{2}\right)}.$$

Express this volume explicitly for odd and even n.

126. For any nonnegative integer index n the Bessel function $J_n(x)$ may be defined by

$$J_n(x) = \frac{x^n}{1\cdot 3\cdot 5\cdots(2n-1)\pi}\int_{-1}^{1}(1-t^2)^{n-1/2}\cos xt\,dt.$$

Show that

$$J_n'' + \frac{1}{x}J_n' + \left(1-\frac{n^2}{x^2}\right)J_n = 0, \qquad n \geq 0.$$

HINT: Use integration by parts to get the same trigonometric function in all parts of the integrand.

127. Using Fourier transforms show that

$$\frac{2}{\pi}\int_0^\infty \frac{\sin\tau\cos\tau x}{\tau}d\tau = \begin{cases} 1 & \text{for} \quad |x|<1 \\ \frac{1}{2} & \text{for} \quad x=\pm 1 \\ 0 & \text{for} \quad |x|>1. \end{cases}$$

128. Find the Fourier transform of $J_n(x)/x^n$, with J_n defined as in problem 126.

HINT: Do not perform any integrals.

129. Assume that $f:\mathbb{R}\to\mathbb{R}$ is as smooth as you need.

(i) Show that if

$$\int_{-\infty}^{\infty}|x|^n|f(x)|\,dx < \infty$$

and

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx$$

is the Fourier transform of $f(x)$, then

$$F^{(n)}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-ix)^n f(x) e^{-ikx}\, dx.$$

(ii) If

$$\int_{-\infty}^{\infty} |f^{(j)}(x)|\, dx < \infty, \qquad j = 1, \dots n,$$

then

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^{(n)}(x) e^{-ikx}\, dx = (ik)^n F(k).$$

130. Find the solution to the heat equation

$$u_t = \alpha^2 u_{xx}, \qquad -\infty < x < \infty, \quad t > 0,$$

with the initial condition

$$u(x, 0) = f(x), \qquad \int_{-\infty}^{\infty} |f(x)|\, dx < \infty,$$

with $f(x)$ also being continuous and piecewise smooth, by completing the following outline:

(i) Let

$$U(t, k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t) e^{-ikx}\, dx$$

be the Fourier transform of $u(x, t)$. Show that it satisfies the initial-value problem

$$U_t = -\alpha^2 k^2 U, \qquad U(0, k) = F(k), \qquad F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx,$$

and solve this problem for every k.

(ii) Deduce from (i) and the Fourier integral theorem that

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} f(\xi) e^{-\alpha^2 k^2 t + ik(x-\xi)}\, d\xi.$$

Show that the repeated integral exists as an integral over $\mathbb{R}^2$ for all $t \geq t_0 > 0$ and reverse the order of integration.

(iii) Use (ii) and the formula

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-y^2/2} \cos \lambda y\, dy = e^{-\lambda^2/2},$$

proven in class, to show that

$$u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}} \int_{-\infty}^{\infty} f(\xi) \exp\left(-\frac{(x-\xi)^2}{4\alpha^2 t}\right) d\xi \tag{20}$$

for all x and all $t > 0$. Show uniform convergence of the integral and its partial derivatives for all $t \geq t_0 > 0$ and thus verify directly that (20) indeed satisfies the heat equation. In fact, show that $u \in C^\infty(\mathbb{R}, t \geq t_0 > 0)$.

(iv) Show that

$$\lim_{t\to 0+} \frac{1}{2\alpha\sqrt{\pi t}} \exp\left(-\frac{y^2}{4\alpha^2 t}\right) = 0, \qquad y \neq 0,$$

$$\frac{1}{2\alpha\sqrt{\pi t}} \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{4\alpha^2 t}\right) dy = 1, \qquad t > 0,$$

and

$$\lim_{t\to 0+} \frac{1}{2\alpha\sqrt{\pi t}} \int_{-\rho}^{\rho} \exp\left(-\frac{y^2}{4\alpha^2 t}\right) dy = 1$$

for any $\rho > 0$.

(v) Consider the integral

$$\frac{1}{2\alpha\sqrt{\pi t}} \int_{-\infty}^{\infty} \Big[f(\xi) - f(x)\Big] \exp\left(-\frac{(x-\xi)^2}{4\alpha^2 t}\right) d\xi.$$

Break the integral up into $\int_{-\infty}^{x-\rho} + \int_{x-\rho}^{x+\rho} + \int_{x+\rho}^{\infty}$ and carefully estimate each of these terms using (iv). Thus, deduce that, with $u(x,t)$ as in (20),

$$\lim_{t\to 0+} u(x,t) = f(x).$$

In other words, $u(x,t)$ indeed satisfies the initial value problem.

131. Consider the curve $\gamma : [a,b] \to \mathbb{R}^n$. For each partition $P = \{a = t_0 < t_1 < \cdots < t_N = b\}$ of $[a,b]$ define

$$\Lambda(\gamma, P) = \sum_{i=1}^{N} \|\gamma(t_i) - \gamma(t_{i-1})\|.$$

(i) What does $\Lambda(\gamma, P)$ represent geometrically?

(ii) Define the *length* of γ as

$$\Lambda(\gamma) = \sup_P \Lambda(\gamma, P).$$

and call γ rectifiable if $\Lambda(\gamma) < \infty$.

By carrying out the outline below, prove the following **Theorem:** If $\gamma \in C^1[a,b]$, then γ is rectifiable, and

$$\Lambda(\gamma) = \int_a^b \|\gamma'(t)\|\, dt.$$

(a) Show that

$$\|\gamma(t_i) - \gamma(t_{i-1})\| \le \int_{t_{i-1}}^{t_i} \|\gamma'(t)\|\, dt,$$

and conclude that

$$\Lambda(\gamma) \le \int_a^b \|\gamma'(t)\|\, dt.$$

(b) To show the opposite inequality, first choose $\epsilon > 0$. Prove and use the uniform continuity of γ' on $[a,b]$ to show that for every sufficiently fine partition P,

$$\|\gamma'(t_i) - \gamma'(t)\| < \epsilon \quad \text{and} \quad \|\gamma'(t)\| \le \|\gamma'(t_i)\| + \epsilon$$

whenever $t_{i-1} \le t \le t_i$. Thus, writing in some appropriate place $\gamma'(t_i) = \gamma'(t) + \gamma'(t_i) - \gamma'(t)$, derive the estimate

$$\int_{t_{i-1}}^{t_i} \|\gamma'(t)\|\, dt \le \|\gamma(t_i) - \gamma(t_{i-1})\| + 2\epsilon(t_i - t_{i-1}).$$

Conclude that

$$\int_a^b \|\gamma'(t)\|\, dt \le \Lambda(\gamma) + 2\epsilon(b-a),$$

and thus the statement of the theorem.

(iii) For $a \le t \le b$, define the *arclength*, $s(t)$, of γ as

$$s(t) = \int_a^t \|\gamma'(\tau)\|\, d\tau.$$

Compute $s'(t)$and deduce that $s(t)$ is monotonically increasing. Conclude that the curve γ can be re-parametrized in terms of the arclength by $\sigma(s) = \gamma(t(s))$, where s runs through the interval $[0, \Lambda(\gamma)]$. Show that $\|\sigma'(s)\| = 1$, and so the integral for $\Lambda(\sigma)$ becomes trivial.

132. (i) Consider the curve $\gamma : [a,b] \to \mathbb{R}^3$ and the line integral

$$I_\gamma = \int_\gamma f\, dx + g\, dy + h\, dz = \int_a^b \left[f(\gamma(t))\gamma_1'(t) + g(\gamma(t))\gamma_2'(t) + h(\gamma(t))\gamma_3'(t) \right] dt.$$

Show that I_γ can be written as

$$I_\gamma = \int_\gamma \mathbf{F} \cdot \mathbf{t}\, ds,$$

where $\mathbf{F} = (f, g, h)$, $\mathbf{t}$ is the unit tangent to the curve $\gamma(t)$, and ds is the differential of the arclength.

(ii) What must $\mathbf{F} = (f, g, h)$ be so that you can write

$$\Lambda(\gamma) = \int_\gamma f\, dx + g\, dy + h\, dz \ ?$$

(iii) Generalize the result of (i) and (ii) to curves in $\mathbb{R}^n$.

133. (i) What are the arclength and the length of one turn of the helix

$$\gamma(t) = (a\cos t, a\sin t, bt)?$$

(ii) Compute the arclength of the curve of intersection between the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $(x-1)^2 + y^2 = 1$. What is its total length?

134. Let Σ be a smooth surface in $\mathbb{R}^3$, expressed in terms of a single coordinate patch with parameters u and v in the parameter domain Δ. Show that the surface integral

$$I_\Sigma = \iint_\Sigma f\, dy \wedge dz + g\, dz \wedge dx + h\, dx \wedge dy$$

can be written in the form

$$I_\Sigma = \iint_\Sigma \mathbf{F} \cdot \mathbf{n}\, dA = \iint_\Delta \mathbf{F}\Big(\mathbf{r}(u,v)\Big) \cdot \mathbf{n}\Big(\mathbf{r}(u,v)\Big)\sqrt{EG - F^2}\, du\, dv.$$

Here $\mathbf{F} = (f, g, h)$, $\mathbf{r}(u,v) = \Big(x(u,v), y(u,v), z(u,v)\Big)$ is the parametrization of the surface Σ,

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}$$

is the unit normal to the surface Σ,

$$E = \mathbf{r}_u \cdot \mathbf{r}_u, \quad F = \mathbf{r}_u \cdot \mathbf{r}_v, \quad G = \mathbf{r}_v \cdot \mathbf{r}_v,$$

and

$$dA = \sqrt{EG - F^2}\, du\, dv$$

is the *surface area element*. (The subscripts denote partial derivatives.) Observe that

$$EG - F^2 = \|\mathbf{r}_u \times \mathbf{r}_v\|^2 .$$

Choose $\mathbf{F} = \mathbf{n}$; what is the result?

135. Compute the volume and the area of the torus parametrized by

$$\begin{aligned} x(r,s,t) &= (b + r\cos s)\cos t \\ y(r,s,t) &= (b + r\cos s)\sin t \\ z(r,s,t) &= r\sin s, \end{aligned}$$

with $0 \le t, s \le 2\pi$ and $0 \le r \le a$, with $0 < a < b$ constants.

136. Let E be an open rectangle in $\mathbb{R}^3$, with edges parallel to the coordinate axes. Let $(a,b,c) \in E$ and $f_i \in C^1(E)$ for $i = 1,2,3$. Consider

$$\omega = f_1\, dy \wedge dz + f_2\, dz \wedge dx + f_2\, dx \wedge dy,$$

and assume that $d\omega = 0$ in E. Define

$$\lambda = g_1\, dx + g_2\, dy,$$

where

$$g_1(x,y,z) = \int_c^z f_2(x,y,s)\, ds - \int_b^y f_3(x,t,c)\, dt$$

$$g_2(x,y,z) = -\int_c^z f_1(x,y,s)\, ds$$

for $(x,y,z) \in E$. Prove that $d\lambda = \omega$ in E.

137. **Vector analysis:** Let $\mathbf{F} = (F_1, F_2, F_3)$ be a smooth mapping of a star-shaped open set $E \subset \mathbb{R}^3$ into $\mathbb{R}^3$, which we will now call a *vector field* in E. With every such vector field $\mathbf{F}$, we associate a 1-form

$$\lambda_{\mathbf{F}} = F_1\, dx + F_2\, dy + F_3\, dz,$$

and a 2-form

$$\omega_{\mathbf{F}} = F_1\, dy \wedge dz + F_2\, dz \wedge dx + F_3\, dx \wedge dy.$$

For any smooth function $u : E \to \mathbb{R}$, define its *gradient* as the vector

$$\nabla u = (D_1 u, D_2 u, D_3 u).$$

For a smooth vector field $\mathbf{F}$ in E define its *curl*

$$\nabla \times \mathbf{F} = (D_2F_3 - D_3F_2, D_3F_1 - D_1F_3, D_1F_2 - D_2F_1),$$

and its *divergence*

$$\nabla \cdot \mathbf{F} = D_1F_1 + D_2F_2 + D_3F_3.$$

Use Poincaré's lemma to show:

(i) $\mathbf{F} = \nabla u$ for some smooth function u if and only if $\nabla \times \mathbf{F} = 0$ in E.

(ii) $\mathbf{F} = \nabla \times \mathbf{G}$ for some smooth vector field $\mathbf{G}$ if and only if $\nabla \cdot \mathbf{F} = 0$.

138. Let $E = \mathbb{R}^2 - \{0\}$, the plane with the origin removed.

(i) Show that the 1-form

$$\eta = \frac{x\,dy - y\,dx}{x^2 + y^2}$$

is closed in E.

(ii) Fix $r > 0$ and consider the circle

$$\gamma(t) = (r\cos t, r\sin t), \qquad 0 \le t \le 2\pi.$$

Since $\gamma(0) = \gamma(2\pi)$, we have $\partial\gamma = 0$. Show directly that

$$\int_\gamma \eta = 2\pi.$$

Use Stokes' theorem to show that:

(a) η is not exact in E,

(b) γ is not the boundary of any 2-chain in E.

(iii) Let Γ be a smooth curve in $\mathbb{R}^2$ with parameter interval $[0, 2\pi]$ such that no straight line segment $[\gamma(t), \Gamma(t)]$, with $0 \le t \le 2\pi$, contains the origin. Prove that

$$\int_\Gamma \eta = 2\pi.$$

HINT: For $0 \le t \le 2\pi$, $0 \le u \le 1$, define

$$\Phi(t, u) = (1 - u)\Gamma(t) + u\gamma(t).$$

Then Φ is a 2-surface in $\mathbb{R}^2 - \{0\}$ whose parameter domain is the indicated rectangle. Show that $\partial\Phi = \Gamma - \gamma$ and use Stokes' theorem to deduce that

$$\int_\Gamma \eta = \int_\gamma \eta$$

because $d\eta = 0$.

(iv) Take the ellipse $\Gamma(t) = (a\cos t, b\sin t)$, where $a > 0$ and $b > 0$ are fixed. Use part (iii) to show that

$$\int_0^{2\pi} \frac{ab}{a^2\cos^2 t + b^2\sin^2 t}\, dt = 2\pi.$$

(v) Show that

$$\eta = d\left(\arctan\frac{y}{x}\right)$$

in any convex open set in which $x \neq 0$, and that

$$\eta = d\left(-\arctan\frac{x}{y}\right)$$

in any convex open set in which $y \neq 0$.

Explain why this justifies the notation $\eta = d\theta$ despite the fact that η is not exact in $\mathbb{R}^2 - \{0\}$.

(vi) Show that (iii) can be derived from (v).

139. (i) Let $\omega = \sum a_i(\mathbf{x})\, dx_i$ be a 1-form in a convex open set $E \subset \mathbb{R}^n$. (See Problem 93 (ii) for the definition of a convex set. Note that a convex set is always star-shaped.) Assume $d\omega = 0$ and prove that ω is exact in E by completing the following outline:

Fix $\mathbf{p} \in E$, and let $[\mathbf{p}, \mathbf{x}]$ be the straight line segment between the points $\mathbf{p}$ and $\mathbf{x}$. Define

$$f(\mathbf{x}) = \int_{[\mathbf{p},\mathbf{x}]} \omega, \qquad \mathbf{x} \in E.$$

Apply Stokes' theorem to the appropriately oriented triangles $[\mathbf{p}, \mathbf{x}, \mathbf{y}]$ in E, with the vertices $\mathbf{p}$, $\mathbf{x}$, and $\mathbf{y}$. Deduce that

$$f(\mathbf{y}) - f(\mathbf{x}) = \sum_{i=1}^{n} (y_i - x_i) \int_0^1 a_i\Big((1-t)\mathbf{x} + t\mathbf{y}\Big)\, dt$$

for $\mathbf{x}, \mathbf{y} \in E$. Hence $D_i f(\mathbf{x}) = a_i(\mathbf{x})$.

(ii) Assume that ω is a smooth 1-form in an open set E such that

$$\int_{\gamma} \omega = 0$$

for every smooth closed curve γ in E. Prove that ω is exact in E by imitating part of the argument sketched in (i).

(iii) Assume ω is a smooth 1-form in $\mathbb{R}^3 - \{0\}$ and $d\omega = 0$. Prove that ω is exact in $\mathbb{R}^3 - \{0\}$.

HINT: Every closed smooth curve in $\mathbb{R}^3 - \{0\}$ is the boundary of a 2-surface in $\mathbb{R}^3 - \{0\}$. Apply Stokes' theorem and (ii).

140. State conditions under which the formula

$$\int_{\Phi} f\, d\omega = \int_{\partial\Phi} f\omega \; - \int_{\Phi} df \wedge \omega$$

is valid and show that it generalizes the formula for integration by parts.

HINT: $d(f\omega) = df \wedge \omega + f\, d\omega$.

141. Using the notation of the problems 132, 134, and 137, as well as $dV = dx\, dy\, dz$, formulate precisely and prove the two classical formulas in $\mathbb{R}^3$:

$$\int_{\Sigma} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \;\; dA = \int_{\partial\Sigma} \mathbf{F} \cdot \mathbf{t} \;\; ds, \qquad \text{(Stokes)},$$

and

$$\int_{\Omega} \nabla \cdot \mathbf{F} \;\; dV = \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} \;\; dA, \qquad \text{(Gauss)}.$$

HINT: Simply translate them in the language of differential forms and use the results shown in class.

142. Let $E \subset \mathbb{R}^3$ be open, and $g, h : E \to \mathbb{R}$ smooth. Consider the vector field

$$\mathbf{F} = g\nabla h.$$

(i) Show that

$$\nabla \cdot \mathbf{F} = g\nabla^2 h + \nabla g \cdot \nabla h,$$

where

$$\nabla^2 h = \nabla \cdot (\nabla h) = \sum_{i=1}^{3} \frac{\partial^2 h}{\partial x_i^2}$$

is the Laplacian of h.

(ii) If Ω is a 3-dimensional manifold-with-boundary in E with positively oriented boundary $\partial\Omega$, show that

$$\int_\Omega \Big(g\nabla^2 h + \nabla g \cdot \nabla h\Big)\, dV = \int_{\partial\Omega} g\frac{\partial h}{\partial n}\, dA,$$

where we have written $\partial h/\partial n$ in place of $\nabla h \cdot \mathbf{n}$. (Thus $\partial h/\partial n$ is the directional derivative of h in the direction of the outward normal to $\partial\Omega$, the so-called *normal derivative* of h.) Interchange g and h and subtract the resulting formula from the first one, to obtain

$$\int_\Omega \Big(g\nabla^2 h - h\nabla^2 g\Big)\, dV = \int_{\partial\Omega} \left(g\frac{\partial h}{\partial n} - h\frac{\partial g}{\partial n}\right)\, dA.$$

These formulas are called *Green's identities.*

(iii) Assume h is *harmonic* in E; this means that $\nabla^2 h = 0$. Take $g = 1$ and conclude that

$$\int_{\partial\Omega} \frac{\partial h}{\partial n}\, dA = 0.$$

Take $g = h$ and conclude that $h = 0$ in Ω if $h = 0$ on $\partial\Omega$.

(iv) Show that, with appropriate changes, Green's identities are also valid in $\mathbb{R}^2$.

143. Let Σ be a "tube" in $\mathbb{R}^3$, that is, a surface parametrized by a function $\mathbf{r}(t,z) = \Big(x(t,z), y(t,z), z\Big)$ defined on the rectangle $0 \le t \le 1$, $a \le z \le b$, such that $\mathbf{r}(0,z) = \mathbf{r}(1,z)$ for every $z \in [a,b]$. In other words, each z-slice through the surface Σ is a closed curve. Use Stokes' theorem to show that

$$\int_\Sigma dx \wedge dy = A(b) - A(a),$$

where $A(z)$ is the area enclosed by the curve $\Big(x(t,z), y(t,z)\Big)$ in the xy-plane.

144. The physical principles of electricity and magnetism can be stated in the following way:

(i) Faraday's law: The total electromotive force induced in a closed loop $\partial\Sigma$ equals minus the time rate of change of the magnetic flux through this loop. In the appropriate units, this law reads

$$\oint_{\partial\Sigma} \mathbf{E}\cdot\mathbf{t}\, ds = -\frac{1}{c}\frac{d}{dt}\iint_\Sigma \mathbf{B}\cdot\mathbf{n}\, dA.$$

(ii) Ampère's law: The total magnetic force induced in a loop $\partial\Sigma$ equals the total of the enclosed currents and the time rate of change of the electric displacement flux through the loop:

$$\oint_{\partial\Sigma} \mathbf{H}\cdot\mathbf{t}\,ds = \frac{4\pi}{c}\iint_{\Sigma}\mathbf{J}\cdot\mathbf{n}\,dA + \frac{1}{c}\frac{d}{dt}\iint_{\Sigma}\mathbf{D}\cdot\mathbf{n}\,dA.$$

(iii) Coulomb's law: The electric displacement flux through any closed surface $\partial\Omega$ equals the enclosed charge:

$$\iint_{\partial\Omega}\mathbf{D}\cdot\mathbf{n}\,dA = \iiint_{\Omega}\rho\,dv.$$

(iv) Absence of magnetic monopoles: There is no flux of the magnetic induction through any closed surface:

$$\iint_{\partial\Omega}\mathbf{B}\cdot\mathbf{n}\,dA = 0.$$

Show that these laws result in Maxwell's equations

$$\nabla\times\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} = 0, \qquad \nabla\cdot\mathbf{B} = 0,$$

$$\nabla\times\mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}, \qquad \nabla\cdot\mathbf{D} = \rho.$$