Extreme Fluctuations in Small Worlds with Relaxational Dynamics

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Synchronization in Coupled Multi-Component Systems

Collective dynamics on the network

Examples:
- Internet (packet traffic/flux)
- Load-balancing schemes (job allocation among processors)
- Electric power grid (power transmission and phase synchronization)
- High-performance or grid-computing networks (progress of the processors in distributed simulations)

Fluctuations of the “load” in the network
- Average size of the fluctuations
- Typical size of the largest fluctuations
  failures/delays are often triggered by extreme events
Critical phenomena in small-world networks

- Monasson (1999): diffusion on SW networks
- These systems typically exhibit (strict or anomalous) mean-field-like phase transitions
- Hastings (2003): general criterion for mean-field scaling
Synchronization in Parallel Discrete-Event Simulations (PDES)

Parallelization for asynchronous dynamics of continuum-time processes

Examples:
- Cellular communication networks (call arrivals)
- Magnetization dynamics in condensed matter (Ising model: spin flips with Glauber dynamic)
- Spatial epidemic models, contact process (infections)

Paradoxical task:
- (algorithmically) parallelize (physically) non-parallel dynamics

Must use local simulated times \( \{h_i\} \) on each processor and a synchronization scheme to preserved causality
local synchronization rules for the processing elements (PE)

- one-site-per PE, $N_{PE} = N = L^d$
- $t=0,1,2,…$ parallel steps
- $h_i(t)$ local simulated time
- local time increments are iid exponential random variables (Poisson asynchrony)
- advance only if $h_i \leq \min\{h_{nn}\}$

(Lubachevsky (1987)

\{h_i\} : "virtual time horizon"
Synchronization landscape (virtual time horizon)

\[ w^2(t) = \frac{1}{N} \sum_{i=1}^{N} [h_i(t) - \bar{h}(t)]^2 \]

width (measure of de-synchronization)

\[ \bar{h} = \frac{1}{N} \sum_{i=1}^{N} h_i \]

progress of the simulation

\[ \xi \sim N \]

"rough" landscape

correlation length

\[ w \sim N^\alpha \]

roughness exponent
local synchronization/communication rules in PDES ("microscopic dynamics")


effective equation of motion form the virtual time horizon

\[ \partial_t h_i = (h_{i+1} + h_{i-1} - 2h_i) + \ldots + \eta_i(t) \]

(Edward-Wilkinson model, kinetic roughening)

prototypical synchronization problem for unbounded local variables with local relaxation and short tailed noise

\[ \langle \eta_i(t)\eta_j(t') \rangle = 2\delta_{ij}\delta(t-t') \]

Gaussian noise

\[ \partial_t h_i = -\sum_{j=1}^{N} \Gamma_{ij}^o h_j + \eta_i(t) \]

\[ \langle w^2 \rangle_L \sim N^{2\alpha} \]

\[ \alpha = 1/2 \]

roughness exponent
Suppressing roughness in virtual time horizons

- Watts & Strogatz (1998):
  “... enhanced signal-propagation speed, computational power, and synchronizability”. 
Alternative SW constructions:

“soft” SW network (ER on top of ring)  
\[
J_{ij} = \begin{cases} 
1 & \text{with probability } \frac{p}{N} \\
0 & \text{with probability } 1 - \frac{p}{N} 
\end{cases}
\]
\[
\left[ \sum_l J_{il} \right] = \sum_l [J_{il}] = p \quad \text{average degree}
\]
(in addition to n.n.)

“hard” SW network
\[
J_{ij} = 0 \text{ or } p: \quad N/2 \text{ random links are selected, such that each site has exactly one random link of strength } p
\]
\[
\sum_l J_{il} = p \quad \text{no fluctuations in the individual degree}
\]
Edward-Wilkinson model on small-world networks

\[ \partial_t h_i = (h_{i+1} + h_{i-1} - 2h_i) - \sum_{j=1}^{N} J_{ij} (h_i - h_j) + \eta_i(t) \]

\[ \Gamma_{ij}^0 = 2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1} \]  
- Laplacian on regular network (ring)

\[ V_{ij} = \delta_{ij} \sum_l J_{il} - J_{ij} \]  
- Laplacian on random part of the network

\[ \partial_t h_i = - \sum_{j=1}^{N} \Gamma_{ij} h_j + \eta_i(t) \]

\[ \Gamma = \Gamma^0 + V \]

disorder-averaged width:

\[ \langle w^2 \rangle_N = \left[ (\Gamma^{-1})_{ii} \right] \]
Impurity-averaged perturbation theory


\[ \partial_t h_i = (h_{i+1} + h_{i-1} - 2h_i) - \sum (h_i - \bar{h}) + \eta_i(t) \]

effective interaction due to random links (self-energy)

**SW network induces effective relaxation to the mean**

**“soft” network:** \[ \Sigma \sim p^2 + \ldots \] (anomalous mean-field)

**“hard” network:** \[ \Sigma \sim p - \frac{1}{2} p^{3/2} + \ldots \] (asymptotically strict mean-field)

\[ \xi \overset{N \to \infty}{\sim} \frac{1}{\sqrt{\Sigma}} \]
finite correlation length

\[ \left< w^2 \right>_N \overset{N \to \infty}{\sim} \frac{1}{2\sqrt{\Sigma}} \]
finite width for any \( p > 0 \)
Implementation to PDES:

- **regular lattice (ring) topology**
  
  \( N \rightarrow 0.0 = p_{\text{regular}} \)

- **random connections:**
  used with probability \( p > 0 \)

\[ w \sim N^\alpha \]

\[ N = 10^4 \]

\[ N \rightarrow \infty \]

\[ w \sim \text{const.} \]
Steady-state height structure factors

\[ S(k) = \langle \tilde{h}_k \tilde{h}_{-k} \rangle / N = \langle |\tilde{h}_k|^2 \rangle / N \]

1d ring

\[ S(k) \sim \frac{1}{k^2} \]

SW network

\[ S(k) \sim \frac{1}{k^2 + \Sigma} \]

\[ p = 0.0 \quad N_{PE} = L \]

\[ p = 0.10 \]
utilization trade-off/scalable data management

fully scalable high-performance or grid-computing networks

roughness

utilization = fraction of non-idling PEs

speedup = utilization × N_{PE}

![Graphs showing roughness, utilization, and speedup with varying parameters.](image-url)
Relationship between extremal statistics and universal fluctuations

\[ P(m), P(w^2), \text{etc.} \]

- Bramwell et al. (1998): turbulence, \( XY \), SOC, etc.
- Dahlstedt & Jensen (2001): SOC (Sneppen model)
- Aji & Goldenfeld (2001)
- Antal, Györgyi, Rácz ... (2001, 2003): \( 1/f \) and other types of interfaces
Extreme height fluctuations in synchronization landscape

\[
\partial_t h_i = (h_{i+1} + h_{i-1} - 2h_i) - \sum_{j=1}^{N} J_{ij} (h_i - h_j) + \ldots + \eta_i(t)
\]

\[
w = \sqrt{\left< \frac{1}{N} \sum_{i=1}^{N} (h_i - \overline{h})^2 \right>}
\]

\[
\langle \Delta_{\text{max}} \rangle = \langle h_{\text{max}} - \overline{h} \rangle
\]
Synchronizability of coupled multi-component systems with “local” relaxation and short-tailed noise

regular 1d (ring)

\[ w \sim N^\alpha \]

\[ \langle \Delta_{\text{max}} \rangle \sim N^\alpha \]

Raychaudhuri et al., *PRL* (2001)

small world

\[ w \sim \text{const.} \]

\[ \langle \Delta_{\text{max}} \rangle \sim w[\ln(N)]^{1/\delta} \]

virtual time horizon: \[ \Delta_i \equiv h_i - \bar{h} \]
finite correlation length \( \rightarrow \) quasi-independent blocks

\[ N \rightarrow N / \xi \]

Galambos (1978)
Bouchaud & Mézard (1997)
Baldassarri (2000)

\[ \delta = 1 \quad \text{(measured)} \]

\[ < \Delta_{\text{max}} > \sim w (\ln(N / \xi))^{1/\delta} \xrightarrow{N \to \infty} w \ln(N) \]
Extreme height fluctuations in the virtual time horizon

\[ \Delta_{\text{max}} = h_{\text{max}} - \overline{h} \]

small worlds:

\[ N \to \infty \]

\[ w \sim \text{const.} \]

\[ \Delta_{\text{max}} \sim \ln(N) \]

ring:

\[ w \sim N^\alpha \]

same exponent

\[ \Delta_{\text{max}} \sim N^\alpha \]

\[(p = 0.0)\]

\[ \langle w^2(\infty) \rangle_L \]

\[ \langle \Delta_{\text{max}}^2(\infty) \rangle_L \]

\[ 2\alpha = 1 \]

\[ \langle \Delta_{\text{max}} \rangle \]
$p(\Delta_{\text{max}})$

FTG
Summary

- SW links can facilitate synchronization of parallel computing networks with many nodes.
- Relevant node-to-node process is *relaxation* in the presence of *short-tailed noise* → FTG distribution for extreme fluctuations (weak logarithmic divergence for the typical size of the largest fluctuations).
- In progress: extreme fluctuations of the “load” in other types of networks with other types of noise.

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