Homework 7

Problem 1. [Haberman 8.3.1 (e)] Solve the initial value problem for the heat equation with time-dependent sources

\[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + Q(x, t), \]
\[ u(x, 0) = f(x), \]

subject to the following boundary conditions:

\[ \frac{\partial u}{\partial x}(0, t) = A(t), \quad \frac{\partial u}{\partial x}(L, t) = B(t). \]

We can either subtract out the boundary conditions and differentiate term by term or use the “discrete cosine transform.” We let

\[ u(x, t) = v(x, t) + r(x, t), \]

where \( r(x, t) \) satisfies the boundary conditions and \( v(x, t) \) satisfies the heat equation with a source and homogeneous boundary conditions. Let

\[ r_x(x, t) = A(t) + [B(t) - A(t)] \frac{x}{L}, \]

interpolate between the boundary conditions, such that

\[ r(x, t) = A(t)x + [B(t) - A(t)] \frac{x^2}{2L}. \]

Note that we could add an additional constant (with respect to \( x \), \( D(t) \)) but it will ultimately have no effect in the end. Then we find that \( v(x, t) \) solves

\[ \frac{\partial v}{\partial t} = k \frac{\partial^2 v}{\partial x^2} + \tilde{Q}(x, t), \]
\[ v(x, 0) = \tilde{f}(x), \]
\[ \frac{\partial v}{\partial x}(0, t) = \frac{\partial v}{\partial x}(L, t) = 0, \]

where

\[ \tilde{f}(x) = f(x) - r(x, 0) = f(x) - A(0)x + [B(0) - A(0)] \frac{x^2}{2L}, \]
\[ \tilde{Q}(x, t) = Q(x, t) + \frac{1}{L} [B(t) - A(t)] - A'(t)x - [B'(t) - A'(t)] \frac{x^2}{2L}. \]

The associated homogeneous problem is

\[ \frac{\partial w}{\partial t} = k \frac{\partial^2 w}{\partial x^2}, \]
\[ \frac{\partial w}{\partial x}(0, t) = \frac{\partial w}{\partial x}(L, t) = 0, \]
which has eigenfunctions \( \phi_n(x) = \cos(n\pi x/L) \) and eigenvalues \( \lambda_n = (n\pi/L)^2 \) for \( n \geq 0 \). We take the expansion

\[
v(x, t) = \sum_{n=0}^{\infty} C_n(t) \cos \left( \frac{n\pi x}{L} \right) = C_0(t) + \sum_{n=1}^{\infty} C_n(t) \cos \left( \frac{n\pi x}{L} \right).
\]

Since this is a cosine series for a continuous function, we can differentiate term by term to obtain

\[
v_x(x, t) = -\sqrt{\lambda_n} \sum_{n=1}^{\infty} C_n(t) \sin \left( \frac{n\pi x}{L} \right).
\]

We know that \( v_x \) satisfies homogeneous boundary conditions, so this sine series may also be differentiated term by term, to give

\[
v_{xx}(x, t) = -\lambda_n \sum_{n=1}^{\infty} C_n(t) \cos \left( \frac{n\pi x}{L} \right).
\]

The PDE for \( v \) now gives the ODEs

\[
C'_n(t) + k\lambda_n C_n(t) = \tilde{q}_n(t),
\]

for \( n = 0, 1, 2, 3 \ldots \), where

\[
\tilde{q}_0(t) = \frac{1}{L} \int_0^L \tilde{Q}(x, t) dx,
\]

\[
\tilde{q}_n(t) = \frac{2}{L} \int_0^L \tilde{Q}(x, t) \cos \left( \frac{n\pi x}{L} \right) dx.
\]

We solve the ODE to obtain

\[
C_n(t) = e^{-k\lambda_n t} \left( \int_0^t e^{k\lambda_n s} \tilde{q}_n(s) ds + C_n(0) \right),
\]

where

\[
C_0(0) = \frac{1}{L} \int_0^L \tilde{f}(x) dx,
\]

\[
C_n(0) = \frac{2}{L} \int_0^L \tilde{f}(x) \cos \left( \frac{n\pi x}{L} \right) dx.
\]

**Problem 2.** [Haberman 8.5.2] Consider a vibrating string with time-dependent forcing

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + Q(x, t),
\]

\[
u(0, t) = 0, \quad u(x, 0) = f(x),
\]

\[
u(L, t) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = 0.
\]

(a) Solve the initial value problem.

(b) Solve the initial value problem if \( Q(x, t) = g(x) \cos \omega t \). For what values of \( \omega \) does resonance occur?

(a) This is the wave equation with homogeneous Dirichlet boundary conditions and forcing \( Q(x, t) \). We know the homogeneous wave equation with homogeneous Dirichlet boundary conditions has eigenfunctions

\[
\lambda_n = \left( \frac{n\pi}{L} \right)^2, \quad \phi_n(x) = \sin \left( \frac{n\pi x}{L} \right), \quad n = 1, 2, 3 \ldots.
\]
We therefore take the eigenfunction expansion

\[ u(x, t) = \sum_{n=1}^{\infty} C_n(t) \sin \left( \frac{n\pi x}{L} \right). \]

The homogeneous boundary conditions also mean we can differentiate the sine series term by term, so the PDE gives

\[ C_n''(t) + \lambda_n c^2 C_n(t) = q_n(t), \]

where

\[ q_n(t) = \frac{2}{L} \int_0^L Q(x, t) \sin \left( \frac{n\pi x}{L} \right) \, dx. \]

Variation of parameters gives the full solution

\[ C_n(t) = A_n \cos \sqrt{\lambda_n} ct + B_n \sin \sqrt{\lambda_n} ct + \int_0^t q_n(s) \frac{\sin \sqrt{\lambda_n} c(t - s)}{\sqrt{\lambda_n} c} \, ds, \]

such that the initial conditions give (see section 8.5 of the book)

\[ A_n = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx, \]

\[ \sqrt{\lambda_n} c B_n = \frac{2}{L} \int_0^L 0 \cdot \sin \left( \frac{n\pi x}{L} \right) \, dx = 0. \]

(b) For \( Q(x, t) = g(x) \cos \omega t \), we obtain

\[ q_n(t) = \frac{2 \cos \omega t}{L} \int_0^L g(x) \sin \left( \frac{n\pi x}{L} \right) \, dx \equiv \gamma_n \cos \omega t. \]

Thus,

\[ C_n''(t) + \lambda_n c^2 C_n(t) = \gamma_n \cos \omega t. \]

The solution for \( \omega \neq \sqrt{\lambda_n} c \) is given by

\[ C_n(t) = A_n \cos \sqrt{\lambda_n} ct + B_n \sin \sqrt{\lambda_n} ct + \frac{\gamma_n}{c^2 \lambda_n - \omega^2} \cos \omega t. \]

Again, we have \( B_n = 0 \) and

\[ C_n(0) = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx, \]

such that

\[ A_n = C_n(0) - \frac{\gamma_n}{c^2 \lambda_n - \omega^2}. \]

If \( \omega = \sqrt{\lambda_m} c \) for some \( n = m \), then resonance occurs and we obtain a different coefficient for \( n = m \):

\[ C_m(t) = A_m \cos \sqrt{\lambda_m} ct + B_m \sin \sqrt{\lambda_m} ct + \frac{\gamma_m t}{2 \sqrt{\lambda_m} c} \sin \sqrt{\lambda_m} ct, \]

which follows from the method of undetermined coefficients and grows in time.
Problem 3. [Haberman 8.5.3] Consider a vibrating string with time-dependent forcing

\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t} + g(x) \cos \omega t, \]

\[ u(0, t) = 0, \quad u(x, 0) = f(x), \]

\[ u(L, t) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = 0. \]

(a) Solve the initial value problem if \( \beta \) is moderately small \((0 < \beta < 2c\pi/L)\).

(b) Compare this solution to Exercise 8.5.2(b).

(a) We proceed as in the previous problem. The homogeneous problem is given by

\[ \frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2} - \beta \frac{\partial w}{\partial t}, \]

\[ w(0, t) = 0, \quad w(L, t) = 0, \]

which for \(0 < \beta < 2c\pi/L\), again has eigenfunctions

\[ \lambda_n = \left( \frac{n\pi}{L} \right)^2, \quad \phi_n(x) = \sin \left( \frac{n\pi x}{L} \right), \quad n = 1, 2, 3, \ldots. \]

We therefore again take the eigenfunction expansion

\[ u(x, t) = \sum_{n=1}^{\infty} C_n(t) \sin \left( \frac{n\pi x}{L} \right). \]

The homogeneous boundary conditions also mean we can differentiate the sine series term by term, so the PDE gives

\[ C_n''(t) + \beta C_n'(t) + \lambda_n c^2 C_n(t) = q_n(t), \]  

(1)

where

\[ Q(x, t) = g(x) \cos \omega t, \]

\[ q_n(t) = \frac{2}{L} \int_0^L Q(x, t) \sin \left( \frac{n\pi x}{L} \right) dx = \frac{2 \cos \omega t}{L} \int_0^L g(x) \sin \left( \frac{n\pi x}{L} \right) dx \equiv \gamma_n \cos \omega t. \]

The solution to the ODE is

\[ C_n(t) = e^{-\beta t/2} \left( A_n \cos \mu_n t + B_n \sin \mu_n t \right) + \frac{\gamma_n \left[ \left( \lambda_n c^2 - \omega^2 \right) \cos \omega t + \beta \omega \sin \omega t \right]}{c^4 \lambda_n^2 + (\beta^2 - 2\lambda_n c^2) \omega^2 + \omega^4}, \]  

(2)

where by the assumption on \( \beta \),

\[ \mu_n = \frac{1}{2} \sqrt{\beta^2 - 4\lambda_n c^2} \in \mathbb{R}. \]

(b) Since the ODE in part (a) now involves the \( \beta \) damping term, the right-hand side \( \cos \omega t \) will never resonate with the homogeneous ODE on the left hand side of (1) (in other words, the roots of the homogeneous equation are now complex and can therefore never correspond to the real-valued forcing frequency \( \omega \)). Therefore, the particular solution will not give growth in \( t \). The interpretation is therefore the same as that for resonance in second order ODEs: the damping term means that there will still be a resonant response of the system but it will not grow in time (i.e. the amplitude of the particular solution in (2) will be higher for certain values of \( \omega \) but will not become infinite in this case).
Problem 4. Solve Poisson’s equation in a circular disk with a circularly symmetric source (we assume axial symmetry of the solution since the source is symmetric):

\[ \nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) = Q(r), \]

subject to the boundary condition

\[ u_r(a) = 0, \]

using an expansion in the eigenfunctions of the homogeneous problem

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \lambda u = 0, \quad u_r(a) = 0. \]

What condition must \( Q(r) \) satisfy for a solution to exist?

The homogeneous problem is

\[ \nabla^2 w = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial w}{\partial r} \right) + \lambda w = 0, \]

subject to the boundary condition

\[ u_r(a) = 0, \]

which we know has eigenfunctions

\[ \phi_n(r) = J_0(\sqrt{\lambda_n} r), \]

where

\[ J_0(\sqrt{\lambda_n} a) = 0, \quad n = 1, 2, 3 \ldots, \]

to satisfy the boundary condition. We therefore let

\[ u(r) = \sum_{n=1}^{\infty} C_n J_0(\sqrt{\lambda_n} r), \]

such that the Poisson PDE gives

\[ \nabla^2 u(r) = \sum_{n=1}^{\infty} (-\lambda_n) C_n J_0(\sqrt{\lambda_n} r) = Q(r). \]

Thus by orthogonality,

\[ -\lambda_n C_n = q_n, \]

where

\[ q_n = \frac{1}{\int_0^a \left[ J_0(\sqrt{\lambda_n} r) \right]^2 r dr} \int_0^a Q(r) J_0(\sqrt{\lambda_n} r) r dr. \]

The condition on \( Q(r) \) for a solution to exist is known as a solvability condition and is found by integrating:

\[ \int_R Q(r) dV = \int_R \nabla^2 u dV = \int_R \nabla \cdot (\nabla u) dV = \int_{\partial R} \nabla u \cdot ndS = \int_0^a \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) r dr = a \tilde{c}_r u(a) = 0. \]

Thus, the total heat inputted and removed by \( Q(r) \) over the whole domain must integrate to zero.