Problem 1. [Haberman 7.5.5 (a) and (b)] See exercise 7.5.4. Consider the three-dimensional eigenvalue problem given in
\[ \nabla \cdot (p\nabla \phi) + \lambda \sigma(x, y, z) \phi = 0 \text{ with } \phi = 0 \text{ on the boundary.} \]
(a) Prove that the eigenfunctions belonging to different eigenvalues are orthogonal (with what weight?).
(b) Prove that all the eigenvalues are real.

Problem 2. [Haberman 7.5.8] Suppose that in a three-dimensional region \( R \),
\[ \nabla^2 \phi = f(x, y, z), \]
with \( f \) given and \( \nabla \phi \cdot \hat{n} = 0 \) on the boundary.
(a) Show mathematically that (if there is a solution)
\[ \int \int \int_R f \, dx \, dy \, dz = 0. \]
(b) Briefly explain physically (using the heat flow model) why condition (a) must hold for a solution. What happens in a heat flow problem if
\[ \int \int \int_R f \, dx \, dy \, dz > 0? \]

Problem 3. [Haberman 7.6.3] Redo exercise 7.6.1 (first, show that \( \lambda \geq 0 \); then, determine whether \( \lambda = 0 \) is an eigenvalue and if so, determine the corresponding eigenfunction) if the differential equation is
\[ \nabla \cdot (p\nabla \phi) + \lambda \sigma(x, y, z) \phi = 0 \]
with the following boundary conditions. Assume \( p > 0 \).
(a) \( \phi = 0 \) on the boundary
(b) \( \nabla \phi \cdot \hat{n} = 0 \) on the boundary

Problem 4. [Haberman 7.7.1] Solve as simply as possible:
\[ \frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \]
with \( u(a, \theta, t) = 0, u(r, \theta, 0) = 0 \) and \( \frac{\partial u}{\partial t}(r, \theta, 0) = \alpha(r) \sin 3\theta. \)
Problem 5. [Haberman 7.9.2 (a)] Solve Laplace’s equation inside a semicircular cylinder, subject to the boundary conditions

(a)

\[ u(r, \theta, 0) = 0, \quad u(r, \theta, H) = \alpha(r, \theta), \quad u(r, 0, z) = 0, \]
\[ u(r, \pi, z) = 0, \quad u(\alpha, \theta, z) = 0 \]