Homework 4

**Problem 1.** [Haberman 5.3.3] Consider the non-Sturm-Liouville differential equation

\[
\frac{d^2 \phi}{dx^2} + \alpha(x) \frac{d\phi}{dx} + [\lambda \beta(x) + \gamma(x)] \phi = 0.
\]

Multiply this equation by \( H(x) \). Determine \( H(x) \) such that the equation may be reduced to the standard Sturm-Liouville form:

\[
\frac{d}{dx} \left[ p(x) \frac{d\phi}{dx} \right] + [\lambda \sigma(x) + q(x)] \phi = 0.
\]

Given \( \alpha(x) \), \( \beta(x) \), and \( \gamma(x) \), what are \( p(x) \), \( \sigma(x) \), and \( q(x) \)?

**Problem 2.** [Haberman 5.3.9] Consider the eigenvalue problem

\[
x^2 \frac{d^2 \phi}{dx^2} + x \frac{d\phi}{dx} + \lambda \phi = 0, \quad \text{with} \quad \phi(1) = 0 \quad \text{and} \quad \phi(b) = 0. \tag{1}
\]

(a) Show that multiplying by \( 1/x \) puts this in the Sturm-Liouville form. (This multiplicative factor is derived in Exercise 5.3.3.)

(b) Show that \( \lambda \geq 0 \).

(c) Since (1) is an equidimensional (Euler) equation, determine all positive eigenvalues. Is \( \lambda = 0 \) an eigenvalue? Show there is an infinite number of eigenvalues with a smallest, but no largest.

(d) The eigenfunctions are orthogonal with what weight according to Sturm-Liouville theory? Verify the orthogonality using properties of integrals.

(e) Show that the \( n \)th eigenfunction has \( n - 1 \) zeros.

**Problem 3.** [Haberman 5.4.1] Consider

\[
c \rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( K_0 \frac{\partial u}{\partial x} \right) + \alpha u,
\]

where \( c, \rho, K_0, \alpha \) are functions of \( x \), subject to

\[
\begin{align*}
  u(0, t) &= 0, \\
  u(L, t) &= 0, \\
  u(x, 0) &= f(x).
\end{align*}
\]

Assume that the appropriate eigenfunctions are known.

(a) Show that the eigenvalues are positive if \( \alpha < 0 \) (see Sec. 5.2.1).
(b) Solve the initial value problem.

(c) Briefly discuss $\lim_{t \to x} u(x, t)$.

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**Problem 4.** [Haberman 5.5.9] For the eigenvalue problem

$$\frac{d^4 \phi}{dx^4} + \lambda e^x \phi = 0,$$

subject to the boundary conditions

$$\phi(0) = 0, \quad \phi'(0) = 0, \quad \phi(1) = 0, \quad \phi''(1) = 0,$$

show that the eigenvalues are less than or equal to zero ($\lambda \leq 0$). (Don’t worry; in a physical context that is exactly what is expected.) Is $\lambda = 0$ an eigenvalue?

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**Problem 5.** [Haberman 5.5.14] If $L$ is the following first-order linear differential operator

$$L = p(x) \frac{d}{dx},$$

then determine the adjoint operator $L^*$ such that

$$\int_a^b \left[ u L^*(v) - v L(u) \right] dx = B(x) \bigg|_a^b.$$ 

What is $B(x)$? [Hint: Consider $\int_a^b v L(u) dx$ and integrate by parts.]