**Problem 1.** [Haberman 3.3.2] For the following functions, sketch the Fourier sine series of \( f(x) \) and determine its Fourier coefficients:

(a) \( f(x) = \cos \frac{\pi x}{L} \)  \[\text{[Verify formula (3.3.13)]}\]

(b) \( f(x) = \left\{ \begin{array}{ll} 0 & : x < L/2 \\ 1 & : x > L/2 \end{array} \right. \)

(a) We use the trigonometric identity

\[
2 \sin(\theta) \cos(\phi) = \sin(\theta + \phi) + \sin(\theta - \phi).
\]

The Fourier sine coefficients are given by

\[
B_n = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{n\pi x}{L} \right) dx = \frac{2}{L} \int_0^L \cos \left( \frac{\pi x}{L} \right) \sin \left( \frac{n\pi x}{L} \right) dx
\]

\[
= \frac{1}{L} \left[ \int_0^L \sin \left( \frac{(n+1)\pi x}{L} \right) dx + \int_0^L \sin \left( \frac{(n-1)\pi x}{L} \right) dx \right]
\]

\[
= -\frac{1}{\pi} \left[ \frac{1}{n+1} \cos \left( \frac{(n+1)\pi}{L} \right) + \frac{1}{n-1} \cos \left( \frac{(n-1)\pi}{L} \right) \right]_0^L
\]

\[
= -\frac{1}{\pi} \left[ \frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} \right]
\]

\[
= \frac{1}{\pi} \left[ \frac{(-1)^n + 1}{n+1} + \frac{(-1)^n + 1}{n-1} \right]
\]

\[
= \frac{1}{\pi} \left[ \frac{1}{n} \frac{(n-1)(n-1) + 1}{n(n-1)} \right]
\]

\[
= \left\{ \begin{array}{ll} 0 & n \text{ odd} \\ \frac{4n}{\pi(n^2-1)} & n \text{ even} \end{array} \right. 
\]

In the figure below, we have set \( L = 1 \) because I felt like it.
(b) Since \( f(x) \) is zero (and therefore so is the integral) on the interval from 0 to \( L/2 \), we use integration by parts to obtain the Fourier sine coefficients:

\[
B_n = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx = \frac{2}{L} \int_{L/2}^L x \sin \left( \frac{n\pi x}{L} \right) \, dx \\
= \left( - \frac{2x}{n\pi} \cos \left( \frac{n\pi x}{L} \right) \right) \bigg|_{L/2}^L + \frac{2}{n\pi} \int_{L/2}^L \cos \left( \frac{n\pi x}{L} \right) \, dx \\
= \left( - \frac{2x}{n\pi} \cos \left( \frac{n\pi x}{L} \right) \right) \bigg|_{L/2}^L + \frac{2L}{(n\pi)^2} \left( \sin \left( \frac{n\pi x}{L} \right) \right) \bigg|_{L/2}^L \\
= -\frac{2L}{n\pi} \cos \left( \frac{n\pi}{2} \right) + \frac{L}{n\pi} \cos \left( \frac{n\pi}{2} \right) - \frac{2L}{(n\pi)^2} \sin \left( \frac{n\pi}{2} \right).
\]

This value will change depending on whether \( n = 4k \), \( n = 4k + 1 \), \( n = 4k + 2 \), or \( n = 4k + 3 \), where \( k \) is an integer greater than zero. Using the unit circle,

\[
A_n = \begin{cases} 
-\frac{L}{n\pi} & : n = 4k \\
\frac{2L(n\pi - 1)}{(n\pi)^2} & : n = 4k + 1 \\
-\frac{3L}{n\pi} & : n = 4k + 2 \\
-\frac{2L(n\pi - 1)}{(n\pi)^2} & : n = 4k + 3
\end{cases}
\]

In the figure below, we have set \( L = 1 \) because I felt like it.

---

**Problem 2.** [Haberman 3.3.5] For the following functions, sketch the Fourier cosine series of \( f(x) \) and determine its Fourier coefficients:

\( f(x) = x^2 \)  \hspace{1cm} (c) \hspace{1cm} f(x) = \begin{cases} 
0 & : x < L/2 \\
x & : x > L/2
\end{cases} \)
(a) We integrate by parts twice to obtain

\[ A_n = \frac{2}{L} \int_0^L f(x) \cos \left( \frac{n\pi x}{L} \right) \, dx = \frac{2}{L} \int_0^L x^2 \cos \left( \frac{n\pi x}{L} \right) \, dx \]

\[ = \left( \frac{2}{n\pi} x^2 \sin \left( \frac{n\pi x}{L} \right) \right)_0^L - \frac{4}{n\pi} \int_0^L x \sin \left( \frac{n\pi x}{L} \right) \, dx \]

\[ = 0 + \left( \frac{4}{(n\pi)^2} \cos \left( \frac{n\pi x}{L} \right) \right)_0^L - \frac{4}{n\pi} \int_0^L \cos \left( \frac{n\pi x}{L} \right) \, dx \]

In the figure below, we have set \( L = 1 \) because I felt like it.

(c) Since \( f(x) \) is zero (and therefore so is the integral) on the interval from 0 to \( L/2 \), we use integration by parts to obtain the Fourier sine coefficients:

\[ A_n = \frac{2}{L} \int_0^{L/2} f(x) \cos \left( \frac{n\pi x}{L} \right) \, dx = \frac{2}{L} \int_{L/2}^L x \cos \left( \frac{n\pi x}{L} \right) \, dx \]

\[ = \left( \frac{2x}{n\pi} \sin \left( \frac{n\pi x}{L} \right) \right)_0^{L/2} - \frac{2}{n\pi} \int_{L/2}^L \sin \left( \frac{n\pi x}{L} \right) \, dx \]

\[ = \left( \frac{2x}{n\pi} \sin \left( \frac{n\pi x}{L} \right) \right)_0^{L/2} + \frac{2L}{(n\pi)^2} \left( \cos \left( \frac{n\pi x}{L} \right) \right)_0^{L/2} \]

\[ = -\frac{L}{n\pi} \sin \left( \frac{n\pi}{2} \right) + \frac{2L}{(n\pi)^2} \cos \left( \frac{n\pi}{2} \right) - \frac{2L}{(n\pi)^2} \cos \left( \frac{n\pi}{2} \right). \]

This value will change depending on whether \( n = 4k \), \( n = 4k + 1 \), \( n = 4k + 2 \), or \( n = 4k + 3 \), where \( k \) is an integer greater than zero. Using the unit circle, for \( n > 0 \),

\[ A_n = \begin{cases} 
0 & : n = 4k \\
-\frac{L(2+n\pi)}{n^2\pi} & : n = 4k + 1 \\
\frac{2^2L}{n^2\pi} & : n = 4k + 2 \\
\frac{L(-2+n\pi)}{(n\pi)^2} & : n = 4k + 3
\end{cases} \]
while for $n = 0,$

$$A_0 = \frac{1}{L} \int_0^L f(x)dx = \frac{1}{L} \int_{L/2}^L xdx = \frac{1}{L} \left( \frac{x^2}{2} \right)_L^{L/2} = \frac{1}{L} \left( \frac{L^2}{2} - \frac{L^2}{8} \right) = \frac{3L}{8}.$$  

In the figure below, we have set $L = 1$ because I felt like it.

\begin{center}
\includegraphics[width=\textwidth]{figure.png}
\end{center}

---

**Problem 3.** Which of the Fourier series in problems 1 and 2 can be differentiated term by term?

For Fourier sine series, the hypotheses which allow differentiation term by term are either

(a) The function $f'(x)$ is piecewise smooth, the function $f(x)$ is continuous, and $f(0) = f(L) = 0.$

(b) The function $f'(x)$ is piecewise smooth and the Fourier sine series of $f(x)$ is continuous.

Recall that continuity of $f$ is enough to imply that the Fourier series is continuous. In each of the examples above in problems 1 and 2, $f'(x)$ is piecewise smooth. In problem 1 (a), the function is continuous but it is not zero at the endpoints. Furthermore, the Fourier sine series is not continuous at $x = 0,$ where it takes on the value of 0 but approaches ±1 in the limit. Thus, we cannot differentiate term by term. In problem 1 (c), the function is not continuous, nor is its Fourier sine series, which again takes on the mean value at the jumps at $x = L/2$ and $x = L.$ Thus, we can’t differentiate either example in problem 1.

For Fourier cosine series, the hypotheses which allow differentiation term by term are either

(c) The function $f'(x)$ is piecewise smooth and $f(x)$ is continuous.

(d) The function $f'(x)$ is piecewise smooth and the Fourier cosine series of $f(x)$ is continuous.

Again, these functions each have a piecewise smooth $f'(x).$ In problem 2 (a), the function is continuous and therefore we can differentiate term by term. In problem 2 (c), the function has jump discontinuities, and its Fourier cosine series will converge in the mean at these points. Therefore, neither the function or its series are continuous, so we cannot differentiate term by term.

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**Problem 4.** [Haberman 3.4.9] Consider the heat equation with a known source $q(x,t)$:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + q(x,t) \quad \text{with} \quad u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0.$$
Assume that $q(x, t)$ (for each $t > 0$) is a piecewise smooth function of $x$. Also assume that $u$ and $\partial u/\partial x$ are continuous functions of $x$ (for $t > 0$) and $\partial^2 u/\partial x^2$ and $\partial u/\partial t$ are piecewise smooth. Thus

$$u(x, t) = \sum_{n=1}^{\infty} b_n(t) \sin \frac{n\pi x}{L}.$$  

Justify spatial term-by-term differentiation. What ordinary differential equation does $b_n(t)$ satisfy? Do not solve this differential equation.

Since $u$ and $\partial u/\partial x$ are continuous functions of $x$ (for $t > 0$) and $\partial^2 u/\partial x^2$ and $\partial u/\partial t$ are piecewise smooth, we can clearly differentiate the Fourier series for $u$ and $\partial u/\partial x$ term by term with respect to $x$. Note that while we didn’t prove it in class, problem 3.4.7 states that if $\partial u/\partial t$ is piecewise smooth, then $u(x, t)$ can be differentiated term by term with respect to the parameter $t$ as well. Finally, $q(x, t)$ is piecewise smooth, which implies that we can express it as a Fourier series.

We take term by term derivatives of the Fourier series for $u(x, t)$ to obtain

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} b_n'(t) \sin \frac{n\pi x}{L},$$  

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} \left( \frac{n\pi}{L} \right) b_n(t) \cos \frac{n\pi x}{L},$$  

$$-\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} \left( \frac{n\pi}{L} \right)^2 b_n(t) \sin \frac{n\pi x}{L}.$$  

The heat equation with source $q(x, t)$ gives

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} b_n'(t) \sin \frac{n\pi x}{L} = k \frac{\partial^2 u}{\partial x^2} + q(x, t) = -k \sum_{n=1}^{\infty} \left( \frac{n\pi}{L} \right)^2 b_n(t) \sin \frac{n\pi x}{L} + q(x, t)$$  

$$= -k \sum_{n=1}^{\infty} \left( \frac{n\pi}{L} \right)^2 b_n(t) \sin \frac{n\pi x}{L} + \sum_{n=1}^{\infty} q_n(t) \sin \frac{n\pi x}{L},$$  

where

$$q_n(t) = \frac{2}{L} \int_0^L q(x, t) \sin \frac{n\pi x}{L} dx.$$  

Recall the orthogonality of sine functions when $m = n$. We multiply each side of the above equation by $\sin \left( \frac{m\pi x}{L} \right)$, where $m$ is an integer, and integrate from $x = 0$ to $x = L$ to obtain

$$\int_0^L \sum_{n=1}^{\infty} b_n'(t) \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \sum_{n=1}^{\infty} b_n'(t) \int_0^L \left( \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \right) dx$$  

$$= \frac{L}{2} b_m'(t)$$  

$$= \int_0^L \left( -k \sum_{n=1}^{\infty} \left( \frac{n\pi}{L} \right)^2 b_n(t) + q_n(t) \right) \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx$$  

$$= -\frac{Lk}{2} \left( \frac{m\pi}{L} \right)^2 b_n(t) + \frac{L}{2} q_n(t).$$  

We therefore obtain the ODEs

$$b_n'(t) + k \left( \frac{n\pi}{L} \right)^2 b_n(t) = q_n(t) = \frac{2}{L} \int_0^L q(x, t) \sin \frac{n\pi x}{L} dx.$$
Problem 5. [Haberman 4.4.3] Consider a slightly damped vibrating string that satisfies
\[ \rho \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}. \]

(a) Briefly explain why \( \beta > 0. \)

(b) Determine the solution (by separation of variables) that satisfies the boundary conditions
\[ u(0, t) = 0 \quad \text{and} \quad u(L, t) = 0, \]
and the initial conditions
\[ u(x, 0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x, 0) = g(x). \]
You can assume that this frictional coefficient is relatively small \( (\beta^2 < 4\pi^2 \rho_0 T_0 / L^2) . \)

(a) If we consider the PDE, the acceleration \( u_{tt} \) is equated to the term \(-\beta u_t\). The negative sign implies that a positive velocity \( u_t \) will create a negative component of the acceleration \( u_{tt} \), which implies that the acceleration resists a velocity in the positive direction, or damp it. The same can be said of a negative velocity. This dampening effect becomes clearer in the ODE’s of the next part.

(b) We separate variables \( u(x, t) = T(t)\phi(x) \) to obtain
\[ \rho_0 T''(t)\phi(x) = T_0 T''(t)\phi''(x) - \beta T'(t)\phi(x), \]
or
\[ \frac{\rho_0 T'' + \beta T'}{T_0 T} = \frac{\phi''}{\phi} = -\lambda, \]
where \( \lambda \) is a constant. This equation gives the separated equations
\[ \rho_0 T'' + \beta T' + T_0 \lambda T = 0, \quad \phi'' + \lambda \phi = 0. \]
The boundary conditions \( u(0, t) = 0 \) and \( u(L, t) = 0 \) give \( \phi(0) = \phi(L) = 0. \) These with the second equation above give the first boundary value (eigenvalue) problem which we have seen, giving the eigenvalues and eigenfunctions
\[ \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad \phi_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots. \]
We now solve the equation for \( T \) (for each \( n \)), which we recognize as a damped harmonic oscillator. Solutions will decay in time as per this ODE. This is a second order linear ODE with constant coefficients, which has exponential solutions of the form \( T = e^{\kappa t}. \) If we seek solutions \( T = e^{\kappa t} \), we obtain the polynomial
\[ \rho_0 \kappa^2 + \beta \kappa + \lambda_n T_0 = 0, \]
which has the roots
\[ \kappa = \frac{-\beta \pm \sqrt{\beta^2 - 4\rho_0 \lambda_n T_0}}{2\rho_0}. \]
Note that $\lambda_n = (n\pi/L)^2$, such that
\[
\beta^2 - 4 \rho_0 T_0 L^2 \geq \beta^2 - 4 \rho_0 (n\pi/L)^2 T_0 = \beta^2 - 4 \rho_0 \lambda_n T_0, \quad n \geq 1.
\]
The problem states that we may assume that $\beta^2 < 4 \rho_0 T_0 L^2$, which implies that
\[
0 > \beta^2 - 4 \rho_0 T_0 L^2 \geq 4 \rho_0 (n\pi/L)^2 T_0 = \beta^2 - 4 \rho_0 \lambda_n T_0, \quad n \geq 1.
\]
This implies that
\[
-\mu_n^2 \equiv \beta^2 - 4 \rho_0 \lambda_n T_0 < 0
\]
for $\mu_n$ real-valued, which implies
\[
\kappa = \frac{-\beta \pm \sqrt{\beta^2 - 4 \rho_0 \lambda_n T_0}}{2 \rho_0} = \frac{-\beta \pm \sqrt{-\mu_n^2}}{2 \rho_0} = \frac{-\beta \pm \mu_n i}{2 \rho_0} \equiv \zeta_n \pm \eta_n i.
\]
Note that the real part of this root $\kappa$ is $\zeta_n = -\beta/2 \rho_0$, while the imaginary part is $\eta_n = \mu_n/2 \rho_0$. Solutions to second order linear ODE with constant coefficients may be re-expressed (rather than as strict exponentials) as decaying trigonometric functions of the form
\[
T_n = e^{-\zeta_n t} [A_n \cos(\eta_n t) + B_n \sin(\eta_n t)].
\]
We combine this with $\phi_n(x)$ via superposition to obtain
\[
u(x, t) = \sum_{n=1}^{\infty} e^{-(\beta/2 \rho_0) t} \left[ A_n \cos\left(\frac{\mu_n t}{2 \rho_0}\right) + B_n \sin\left(\frac{\mu_n t}{2 \rho_0}\right) \right] \sin\left(\frac{n \pi x}{L}\right),
\]
where $\mu_n = \sqrt{4 \rho_0 \lambda_n T_0 - \beta^2}$. Note that this Fourier series is equivalent to the one for the standard wave equation when $t = 0$, which implies that we obtain the same coefficients
\[
A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n \pi x}{L}\right) \, dx, \quad n \geq 1.
\]
However, we must be careful in differentiating in time by using the product rule:
\[
\partial_t u(x, t) = \sum_{n=1}^{\infty} e^{-(\beta/2 \rho_0) t} \left[ \left( -\frac{\beta}{2 \rho_0} \right) \cos\left(\frac{\mu_n t}{2 \rho_0}\right) - \left( \frac{\mu_n}{2 \rho_0} \right) \sin\left(\frac{\mu_n t}{2 \rho_0}\right) \right] A_n \sin\left(\frac{n \pi x}{L}\right)
+ e^{-(\beta/2 \rho_0) t} \left[ \left( -\frac{\beta}{2 \rho_0} \right) \sin\left(\frac{\mu_n t}{2 \rho_0}\right) + \left( \frac{\mu_n}{2 \rho_0} \right) \cos\left(\frac{\mu_n t}{2 \rho_0}\right) \right] B_n \sin\left(\frac{n \pi x}{L}\right)
\]
so
\[
g(x) = \partial_t u(x, 0) = \sum_{n=1}^{\infty} \left[ \left( -\frac{\beta}{2 \rho_0} \right) A_n + \left( \frac{\mu_n}{2 \rho_0} \right) B_n \right] \sin\left(\frac{n \pi x}{L}\right) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n \pi x}{L}\right),
\]
which gives the Fourier coefficients
\[
D_n = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n \pi x}{L}\right) \, dx, \quad n \geq 1,
\]
and therefore
\[
B_n = \frac{\beta}{\mu_n} A_n + \frac{4 \rho_0}{L \mu_n} \left( \frac{\mu_n}{2 \rho_0} \right) \int_0^L g(x) \sin\left(\frac{n \pi x}{L}\right) \, dx, \quad n \geq 1.
\]
We finally note that the addition of the $-\beta u_t$ term in the original PDE resulted in the exponential decay rate $-(\beta/2\rho_0)t$ in the solution $u$. It also changed the temporal oscillation frequency to $\mu_n t/2\rho_0$.

**Problem 6.** [Haberman 4.4.9] From (4.4.1), derive conservation of energy for a vibrating string,

$$\frac{dE}{dt} = c^2 \frac{\partial u}{\partial t} \frac{\partial u}{\partial t} |_0^L,$$

where the total energy $E$ is the sum of the kinetic energy, defined by $\int_0^L \frac{1}{2} \left( \frac{\partial u}{\partial t} \right)^2 \, dx$, and the potential energy, defined by $\int_0^L \frac{c^2}{2} (\frac{\partial u}{\partial x})^2 \, dx$.

Multiply the wave equation

$$u_{tt} = c^2 u_{xx},$$

by the function $u_t$ to obtain (with the chain rule)

$$u_t u_{tt} = \frac{1}{2} \frac{\partial}{\partial t} (u_t)^2 = c^2 u_{xx} u_t.$$

We integrate in space and switch the order of integration to obtain

$$\frac{1}{2} \int_0^L \frac{\partial}{\partial t} (u_t)^2 \, dx = \frac{1}{2} \frac{d}{dt} \int_0^L u_t^2 \, dx = c^2 \int_0^L (u_{xx} u_t) \, dx$$

$$= c^2 \left[ u_x u_t |_0^L - \int_0^L (u_x u_{xx}) \, dx \right]. \quad (1)$$

Note that

$$u_x u_{xt} = \frac{1}{2} \frac{\partial}{\partial t} (u_x^2),$$

which we combine with (2) to obtain

$$\frac{1}{2} \frac{d}{dt} \int_0^L u_t^2 \, dx = c^2 \left[ u_x u_t |_0^L - \frac{1}{2} \int_0^L \left( \frac{\partial}{\partial t} \frac{u_t^2}{2} (u_x^2) \right) \, dx \right]$$

$$= c^2 \left[ u_x u_t |_0^L - \frac{1}{2} \frac{d}{dt} \int_0^L u_x^2 \, dx \right],$$

or

$$\frac{dE}{dt} = \frac{1}{2} \left( \frac{d}{dt} \int_0^L [u_t^2 + c^2 u_x^2] \, dx \right) = c^2 \left( u_x u_t |_0^L \right).$$

where

$$E = \frac{1}{2} \int_0^L [u_t^2 + c^2 u_x^2].$$
Problem 7. Consider the (linear) Klein-Gordon equation

\[ \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + \alpha u. \]  

(3)

Find the wave number \( k(\omega) \) such that the plane wave

\[ u(x, t) = Ae^{i(kx - \omega t)} = Ae^{ik[x-(\omega/k)t]}, \]

is a solution to (3), where the amplitude \( A \) is an arbitrary constant. What is the traveling wave (phase) velocity \( \omega/k(\omega) \)? How does this differ from the phase velocity for the standard wave equation? What is the time-independent (steady state) plane wave solution? What is the high frequency limit of the wave speed \( \omega/k(\omega) \) as \( \omega \to \infty \)? This distinction is known as wave dispersion, which is covered in detail in chapter 13 of the textbook.

We insert the plane wave \( u(x, t) = Ae^{i(kx - \omega t)} \) into (3) to obtain

\[-i\omega)^2 Ae^{i(kx - \omega t)} = (i\omega)^2 E + \alpha Ae^{i(kx - \omega t)} = \left[ c^2(-ik)^2 + \alpha \right] Ae^{i(kx - \omega t)}, \]

and therefore

\[ \omega^2 = c^2 k^2 - \alpha, \]

or

\[ k(\omega) = \pm \frac{1}{c} \sqrt{\omega + \alpha}. \]

The wave speed is therefore

\[ \frac{\omega}{k(\omega)} = \pm \frac{c \omega}{\sqrt{\omega + \alpha}}. \]

This is different from the standard wave equation wave speed

\[ \frac{\omega}{k(\omega)} = \pm c. \]

This differs from the current equation (3) because in the standard wave equation, the wave speed is equal to the constant \( c \), which is independent of the frequency \( \omega \) or wave vector \( k(\omega) \). Dispersive wave PDEs refer to partial differential equations for which the wave speed propagation depends on the frequency (or wavelength), as is the case with the linear PDE (3); if an initial condition consists of several frequencies, then some frequencies will spread faster than others. That is, the wave will disperse.