Homework 2 Solutions

Problem 1. [Haberman 2.3.1] For the following partial differential equations, what ordinary differential equations are implied by the method of separation of variables?

(b) \[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} - v_0 \frac{\partial u}{\partial x}. \]
(d) \[ \frac{\partial u}{\partial t} = k \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right). \]

(b) We separate variables with \( u(x,t) = \phi(x)T(t) \) and take derivatives to obtain

\[ \phi' = k\phi'' - v_0\phi'. \]

Dividing by \( k\phi \) gives

\[ \frac{T'}{kT} = \phi' - \frac{v_0}{k}\phi' = -\lambda, \]

from which we extract the separated equations,

\[ T' + k\lambda T = 0, \quad \phi'' - \frac{v_0}{k}\phi' + \lambda\phi = 0. \]

(d) We separate variables with \( u(x,t) = \phi(r)T(t) \) and take derivatives to obtain

\[ \phi' = k \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) T. \]

Dividing by \( k\phi \) gives

\[ \frac{T'}{kT} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) = -\lambda, \]

from which we extract the separated equations,

\[ T' + k\lambda T = 0, \quad \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) + \lambda r^2 \phi = 0. \]

Problem 2. [Haberman 2.4.2] Solve

\[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \]

with

\[ \frac{\partial u}{\partial x}(0,t) = 0, \quad u(L,t) = 0, \quad u(x,0) = f(x). \]

For this problem, you may assume that no solutions of the heat equation exponentially grow in time. You may also guess appropriate orthogonality conditions for the eigenfunctions.
We already know that the separated equations for the heat equation are
\[ T' + k\lambda T = 0, \quad X'' + \lambda X = 0. \]

Solutions to the first equation are exponentials, which will grow in time if \( \lambda > 0 \), which we may rule out according to the assumptions of the problem. Turning to the \( x \) equation, the boundary conditions on the PDE give the boundary conditions \( \phi'(0) = \phi(L) = 0 \). For \( \lambda = 0 \), we obtain the solution
\[ \phi(x) = C_1 x + C_2, \quad (1) \]
from which we have \( \phi'(0) = C_1 = 0 \) and \( \phi(L) = C_2 = 0 \), such that there are no nontrivial solutions in this case.

For the \( \lambda > 0 \) case, we obtain
\[ \phi(x) = C_1 \sin(\sqrt{\lambda}x) + C_2 \cos(\sqrt{\lambda}x), \]
and the first condition gives \( \phi'(0) = C_1 \cos(0) - C_2 \sin(0) = C_1 = 0 \), while the second condition gives \( \phi(L) = C_2 \cos(\lambda L) = 0 \).

To avoid the trivial solution, we need \( \cos(\lambda L) = 0 \). Cosine is zero at odd multiples of \( \pi/2 \), which gives
\[ \sqrt{\lambda}L = n\pi - \pi/2 = \frac{(2n - 1)\pi}{2}, \quad n = 1, 2, \ldots . \]

Therefore,
\[ \lambda_n = \left[ \frac{(2n - 1)\pi}{2L} \right]^2, \quad \phi_n = \cos\left( \frac{(2n - 1)\pi x}{2L} \right), \quad n = 1, 2, \ldots . \]

These eigenfunctions are orthogonal, which we may assume from the problem hypotheses (indeed, one can prove this). From the \( T \) equation, we now have
\[ T_n = C_n e^{-k\lambda_n t} = C_n e^{-k\left[ \frac{(2n - 1)\pi x}{2L} \right]^2 t}. \]

Taking a superposition of the separated solutions, we have
\[ u(x, t) = \sum_{n=1}^{\infty} C_n e^{-k\left[ \frac{(2n - 1)\pi x}{2L} \right]^2 t} \cos\left( \frac{(2n - 1)\pi x}{2L} \right). \]

The initial condition gives
\[ u(x, 0) = f(x) = \sum_{n=1}^{\infty} C_n \cos\left( \frac{(2n - 1)\pi x}{2L} \right). \]

The assumed orthogonality conditions (just as in the standard cases for sines and cosines) are
\[ \frac{2}{L} \int_0^L \cos\left( \frac{(2m - 1)\pi x}{2L} \right) \cos\left( \frac{(2n - 1)\pi x}{2L} \right) \, dx = \delta_{mn}, \]
from which we derive
\[ C_n = \frac{2}{L} \int_0^L f(x) \cos\left( \frac{(2n - 1)\pi x}{2L} \right) \, dx. \]
Problem 3. [Haberman 2.5.1] Solve Laplace’s equation inside a rectangle $0 \leq x \leq L$, $0 \leq y \leq H$, with the following boundary conditions [Hint: Separate variables. If there are two homogeneous boundary conditions in $y$, let $u(x, y) = h(x)\phi(y)$, and if there are two homogeneous boundary conditions in $x$, let $u(x, y) = \phi(x)h(y)$]:

(b) $\frac{\partial^2 u}{\partial x^2}(0, y) = g(y), \quad \frac{\partial^2 u}{\partial y^2}(L, y) = 0, \quad u(x, 0) = 0, \quad u(x, H) = 0.$

(f) $u(0, y) = f(y), \quad u(L, y) = 0, \quad \frac{\partial u}{\partial y}(x, 0) = 0, \quad \frac{\partial u}{\partial y}(x, H) = 0.$

(b) The separated equations for Laplace’s equation on a rectangular domain (from class, given $u(x, t) = h(x)\phi(y)$) are

$$h'' - \lambda h = 0, \quad \phi'' + \lambda \phi = 0.$$  

Note that the boundary conditions are homogeneous in the $y$ direction and therefore are on $\phi$, which is why we choose this sign convention for $\lambda$. These conditions

$$u(x, 0) = 0, \quad u(x, H) = 0,$$

give the standard Dirichlet conditions $\phi(0) = \phi(H) = 0$. We have solved this eigenvalue problem many times in class (think back to the first heat equation example) to obtain

$$\lambda_n = \left(\frac{n\pi}{H}\right)^2, \quad \phi_n(y) = \sin\left(\frac{n\pi y}{H}\right).$$

No nontrivial solutions exist for $\lambda \leq 0$. The $h$ equation now gives (since the sign on $\lambda$ is opposite that on the $\phi$ equation)

$$h(x) = C_1 \sinh\left(\sqrt{\lambda_n}(x - L)\right) + C_2 \cosh\left(\sqrt{\lambda_n}(x - L)\right) = C_1 \sinh\left(\frac{n\pi(x - L)}{H}\right) + C_2 \cosh\left(\frac{n\pi(x - L)}{H}\right),$$

and the boundary condition

$$\frac{\partial u}{\partial x}(L, y) = 0,$$

gives $h'(L) = 0$, and therefore

$$h'(L) = C_1 \cosh(0) + C_2 \sinh(0) = C_1 = 0, \quad \implies \quad h_n(x) = C_n \cosh\left(\frac{n\pi(x - L)}{H}\right).$$

The superposition of the separated solutions gives

$$u(x, y) = \sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi(x - L)}{H}\right) \sin\left(\frac{n\pi y}{H}\right).$$

We now apply the final boundary condition

$$\frac{\partial u}{\partial x}(0, y) = g(y),$$

to obtain

$$\frac{\partial u}{\partial x}(0, y) = g(y) = \sum_{n=1}^{\infty} C_n \left(\frac{n\pi}{H}\right) \sinh\left(\frac{n\pi(-L)}{H}\right) \sin\left(\frac{n\pi y}{H}\right) = -\sum_{n=1}^{\infty} C_n \left(\frac{n\pi}{H}\right) \sinh\left(\frac{n\pi L}{H}\right) \sin\left(\frac{n\pi y}{H}\right).$$
From orthogonality of the sine functions, we obtain

\[-C_n \left( \frac{n\pi}{H} \right) \sinh \left( \frac{n\pi L}{H} \right) = \frac{2}{H} \int_0^H g(y) \sin \left( \frac{n\pi y}{H} \right) \, dy,\]

or

\[C_n = \frac{-2}{n\pi \sinh \left( \frac{n\pi L}{H} \right)} \int_0^H g(y) \sin \left( \frac{n\pi y}{H} \right) \, dy.\]

(f) The separated equations for Laplace’s equation on a rectangular domain (see part (b) ) are

\[h'' - \lambda h = 0, \quad \phi'' + \lambda \phi = 0.\]

The conditions

\[\frac{\partial u}{\partial y}(x, 0) = 0, \quad \frac{\partial u}{\partial y}(x, H) = 0\]

give \(\phi'(0) = \phi'(H) = 0\). We have solved this eigenvalue problem many times in class (think back to the second Neumann heat equation example) to obtain

\[\lambda_n = \left( \frac{n\pi}{H} \right)^2, \quad \phi_n(y) = \cos \left( \frac{n\pi y}{H} \right),\]

No nontrivial solutions exist for \(\lambda < 0\). The \(\lambda = 0\) case gives the constant solution

\[\phi_0 = 1.\]

The \(h\) equation now gives (since the sign on \(\lambda\) is opposite that on the \(\phi\) equation)

\[h(x) = C_1 \sinh \left( \frac{n\pi (x - L)}{H} \right) + C_2 \cosh \left( \frac{n\pi (x - L)}{H} \right),\]

for \(\lambda > 0\) and

\[h(x) = C_1 x + C_2,\]

for \(\lambda = 0\). The boundary condition

\[u(L, y) = 0,\]

gives \(h(L) = 0\), and therefore for \(\lambda > 0\),

\[h(L) = C_1 \sinh(0) + C_2 \cosh(0) = C_2 = 0 \quad \implies \quad h_n(x) = C_n \sinh \left( \frac{n\pi (x - L)}{H} \right).\]

and for \(\lambda = 0\),

\[h(L) = C_1 L + C_2 = 0 \quad \implies \quad C_2 = -C_1 L \quad \implies \quad h_0(x) = C_0 (L - x).\]

The superposition of the separated solutions gives

\[u(x, y) = C_0 (L - x) + \sum_{n=1}^{\infty} C_n \sinh \left( \frac{n\pi (x - L)}{H} \right) \cos \left( \frac{n\pi y}{H} \right).\]

We now apply the final boundary condition

\[u(0, y) = f(y),\]
to obtain

\[ u(0, y) = f(y) = C_0 L + \sum_{n=1}^{\infty} C_n \sinh \left( \frac{n\pi(-L)}{H} \right) \cos \left( \frac{n\pi y}{H} \right) = C_0 L - \sum_{n=1}^{\infty} C_n \sinh \left( \frac{n\pi L}{H} \right) \cos \left( \frac{n\pi y}{H} \right). \]

From orthogonality of the sine functions, we obtain

\[ C_0 L = \frac{1}{H} \int_{0}^{H} f(y) dy, \]

\[ -C_n \sinh \left( \frac{n\pi L}{H} \right) = \frac{2}{H} \int_{0}^{H} f(y) \cos \left( \frac{n\pi y}{H} \right) dy, \quad n > 0 \]

or

\[ C_n = \frac{-2}{H \sinh \left( \frac{n\pi L}{H} \right)} \int_{0}^{H} f(y) \cos \left( \frac{n\pi y}{H} \right) dy. \]

**Problem 4.** [Haberman 2.5.6] Solve Laplace’s equation inside a semicircle of radius \( a \) (0 < \( r < a \), 0 < \( \theta < \pi \)) subject to the boundary conditions [Hint: In polar coordinates,

\[ \nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0, \]

it is known that if \( u(r, \theta) = \phi(\theta)G(r) \), then \( \frac{r}{\partial} \left( r \frac{\partial G}{\partial r} \right) = -\frac{1}{\phi} \frac{\partial^2 \phi}{\partial \theta^2} \).

(b) the diameter is insulated and \( u(a, \theta) = g(\theta) \).

As in class, the separated equations for Laplace’s equation in polar coordinates (with \( u(r, \theta) = G(r)\phi(\theta) \)) are

\[ \phi'' + \lambda \phi = 0, \quad r \frac{\partial}{\partial r} \left( r \frac{\partial G}{\partial r} \right) - \lambda G = 0. \]

In polar coordinates, the radial \( r \) direction and the angular \( \theta \) direction are orthogonal. The radial component points outward from the origin, and the angular component points in the direction of rotation. The normal vector across the diameter boundary is orthogonal to the \( r \) direction (i.e. downwards), and must be the \( \theta \) direction. Therefore, the insulated boundary condition on the diameter of the semicircle means that the normal derivative of the function \( u(x, y) \) must be zero, which implies

\[ \frac{\partial u}{\partial \theta}(r, 0) = \frac{\partial u}{\partial \theta}(r, \pi) = 0. \]

For these to hold for any \( r \), we obtain the boundary conditions on \( \phi(\theta) \):

\[ \phi'(0) = \phi'(\pi) = 0. \]

Combined with the equation \( \phi'' + \lambda \phi = 0 \), we obtain the eigenvalues and eigenfunctions

\[ \lambda_n = n^2, \quad \phi_n(\theta) = \cos (n\theta). \]

for \( \lambda > 0 \) and \( \phi_0(\theta) = 1 \) for \( \lambda = 0 \). See the solution for problem 3(f). The equation for \( G \) is an Euler equation, for which we extracted the solutions

\[ G_0(r) = C_0 + B_0 \log r, \]

\[ G_n(r) = C_n r^n + B_n r^{-n}, \quad n > 0. \]
As in the case of the full disk, we need the boundedness condition at the origin $r = 0$, giving $|u(r, \theta)| < \infty$ and therefore $|G_n(r)| < \infty$, which implies

$$G_0(r) = C_0, \quad G_n(r) = C_nr^n. \quad n > 0.$$  

From superposition of the separated solutions, we obtain

$$u(r, \theta) = C_0 + \sum_{n=1}^{\infty} C_nr^n \cos(n\theta).$$

The last boundary condition $u(a, \theta) = g(\theta)$ gives, by orthogonality,

$$C_0 = \frac{1}{\pi} \int_0^\pi g(\theta), \quad a^n C_n = \frac{2}{\pi} \int_0^\pi g(\theta) \cos(n\theta) d\theta,$$

or

$$C_0 = \frac{1}{\pi} \int_0^\pi g(\theta), \quad C_n = \frac{2}{\pi a^n} \int_0^\pi g(\theta) \cos(n\theta) d\theta.$$

**Problem 5.** [Haberman 2.5.8] Solve Laplace’s equation inside a circular annulus ($a < r < b$) subject to the boundary conditions

$$\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0,$$

it is known that if $u(r, \theta) = \phi(\theta)G(r)$, then $r \frac{d}{dr} \left( r \frac{dG}{dr} \right) = -\frac{1}{\phi} \frac{d^2 \phi}{d\theta^2}$.

(b) $\frac{\partial u}{\partial r}(a, \theta) = 0, \quad u(b, \theta) = g(\theta)$.

As in class, the separated equations for Laplace’s equation in polar coordinates (with $u(r, \theta) = G(r)\phi(\theta)$) are

$$\phi'' + \lambda \phi = 0, \quad r \frac{\partial}{\partial r} \left( r \frac{\partial G}{\partial r} \right) - \lambda G = 0.$$

Since an annulus is a connected circular domain, as in the disk example, we have periodic boundary conditions on $u$ given by

$$u(r, \pi) = u(r, -\pi) = 0, \quad \frac{\partial u}{\partial \theta}(r, \pi) = \frac{\partial u}{\partial \theta}(r, -\pi) = 0.$$

(see the disk example from class for details). These conditions translate to $\phi(\pi) = \phi(-\pi) = 0$ and $\phi'(\pi) = \phi'(-\pi) = 0$, which all give

$$\phi_0(\theta) = 1, \quad \lambda_0 = 0, \quad \phi_n(\theta) = C_1 \cos(n\theta) + C_2 \sin(n\theta), \quad \lambda_n = n^2, \quad n = 1, 2, 3, \ldots.$$
The Euler equation for \( G(r) \) again gives
\[
G'(r) = A_n + B_n \log r,
\]
\[
G_n(r) = A_n r^n + B_n r^{-n}, \quad n > 0.
\]

We therefore use superposition to obtain
\[
u(r, \theta) = A_0 + B_0 \log r + \sum_{n=1}^{\infty} \left[ A_n r^n + B_n r^{-n} \right] \cos(n\theta) + \sum_{n=1}^{\infty} \left[ C_n r^n + D_n r^{-n} \right] \sin(n\theta).
\]

The inner boundary condition \( \partial_r u(a, \theta) = 0 \) gives
\[
\partial_r u(a, \theta) = 0 = \frac{B_n}{a} + \sum_{n=1}^{\infty} \left[ nC_n a^{n-1} - nB_n a^{-n-1} \right] \cos(n\theta)
\]
\[+ \sum_{n=1}^{\infty} \left[ nC_n a^{n-1} - nD_n a^{-n-1} \right] \sin(n\theta).
\]

For this to be true for all \( \theta \), we need \( B_n = 0 \) and
\[
nA_n a^{n-1} - nB_n a^{-n-1} = 0 \implies B_n = A_n a^{2n},
\]
\[
nC_n a^{n-1} - nD_n a^{-n-1} = 0 \implies D_n = C_n a^{2n}.
\]

Thus,
\[
u(r, \theta) = A_n + \sum_{n=1}^{\infty} A_n \left[ r^n + a^{2n} r^{-n} \right] \cos(n\theta) + \sum_{n=1}^{\infty} C_n \left[ r^n + a^{2n} r^{-n} \right] \sin(n\theta).
\]

Finally, we use orthogonality and the last boundary condition \( u(b, \theta) = g(\theta) \) to obtain
\[
A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta) d\theta,
\]
\[
\left[ b^n + a^{2n} b^{-n} \right] A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \cos(n\theta) d\theta,
\]
\[
\left[ b^n + a^{2n} b^{-n} \right] B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \sin(n\theta) d\theta,
\]
or
\[
A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta) d\theta,
\]
\[
A_n = \frac{1}{\pi \left[ b^n + a^{2n} b^{-n} \right]} \int_{-\pi}^{\pi} g(\theta) \cos(n\theta) d\theta,
\]
\[
B_n = \frac{1}{\pi \left[ b^n + a^{2n} b^{-n} \right]} \int_{-\pi}^{\pi} g(\theta) \sin(n\theta) d\theta.
\]