Consider an ordinary differential equation of the form
\[ \frac{dx(t)}{dt} = \frac{b(t)}{a(x(t))}, \quad x(t_0) = x_0. \] (1)

Rewrite this as
\[ a(x(t)) \frac{dx(t)}{dt} = b(t). \] (2)

Define the improper integrals
\[ A(z) = \int a(z) \, dz, \quad B(z) = \int b(z) \, dz, \] (3)

such that (2) is equivalent to
\[ \left. \frac{dA(z)}{dz} \right|_{z=x(t)} \left( \frac{dx(t)}{dt} \right) = \left. \frac{dB(z)}{dz} \right|_{z=t}. \] (4)

The chain rule implies that
\[ \frac{dA(x(t))}{dt} = \left( \frac{dA(z)}{dz} \right)_{z=x} \left( \frac{dx(t)}{dt} \right), \] (5)

so (4) implies
\[ \frac{dA(x(t))}{dt} - \frac{dB(t)}{dt} = \frac{d}{dt} (A(x(t)) - B(t)) = 0 \implies A(x(t)) - B(t) = C. \] (6)

for some $C$ constant. Next, since $x(t_0) = x_0$ is an initial condition, it satisfies (2) and therefore the equivalent equation (6):
\[ A(x(t_0)) - B(t_0) = A(x_0) - B(t_0) = C. \] (7)

Combining (6) and (7), we have
\[ A(x(t)) - A(x_0) = B(t) - B(t_0). \] (8)

But by the fundamental theorem of calculus, we must have
\[ A(x(t)) - A(x_0) = \int_{x_0}^{x(t)} a(z) \, dz, \quad \int_{t_0}^{t} b(z) \, dz = B(t) - B(t_0), \] (9)

such that (8) gives
\[ \int_{x_0}^{x(t)} a(z) \, dz = \int_{t_0}^{t} b(z) \, dz. \] (10)