Homework #5

Problem 1

We have \( x = K(4 - y - x^2) \)
\( y = 1/(x-1) \)
and see that fixed points lie at \((0,0)\), \((1,3)\), \((3,0)\), and \((-2,0)\).

Linearizing we get
\[
A = \begin{pmatrix} 4-y-3x^2 & -x \\ y & x-1 \end{pmatrix}
\]

so at \((0,0)\), \( A_1 = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \lambda = -1, \mu = \) saddle point since \( \lambda_1 < 0 < \lambda_2 \).

at \((2,0)\), \( A_2 = \begin{pmatrix} -8 & -2 \\ 0 & 1 \end{pmatrix} \Rightarrow \lambda = -8, 1 \Rightarrow \) saddle point since \( \lambda_1 < 0 < \lambda_2 \).

at \((-2,0)\), \( A_3 = \begin{pmatrix} -8 & 2 \\ 0 & -3 \end{pmatrix} \Rightarrow \lambda = -8, -3 \Rightarrow \) stable node with \( \lambda_1 < \lambda_2 < 0 \).

at \((1,3)\), \( A_4 = \begin{pmatrix} -2 & -1 \\ 3 & 0 \end{pmatrix} \Rightarrow \lambda^2 + 2\lambda + 3 = 0 \Rightarrow \lambda = 1 \pm 2i \Rightarrow  \) potential center or sink / source.

Since \((2,0)\) and \((0,0)\) are saddle points, we know
\( I_{c_1} \) and \( I_{c_2} \) are \(-1\) and thus no such \( C_1 \) or \( C_2 \) can be deformed into a closed orbit since such an orbit would also have this index.

Though the stable node at \((-2,0)\) has \( I_{c_3} = 1 \), we know \( C_3 \) could not be deformed into a trajectory about \((-2,0)\) which is closed since we could deform \( C_3 \) close enough to the fixed point so that the local behavior indicated by linearization holds. Clearly no closed periodic trajectory could exist here since this local behavior is that of a stable node.
We also note that the system has nullclines on $x = 0$ and $y = 0$. Along $x = 0$, $\dot{x} = 0$ and $\dot{y} = -y$ while along $y = 0$, $\dot{x} = (1 - x^2)$. Thus, a closed curve could exist on a deformation of $C_4$. However, $C_4$ crosses these nullclines and violates uniqueness if it could be a closed trajectory since $x = 0$ and $y = 0$ must be actual trajectories; this is because $x = 0$ on $x = 0$ and $y = 0$ on $y = 0$. So a trajectory on these lines will continue in more along them.

Similarly, no closed curve trajectory could coincide $(1,3)$ or any other fixed point since it would again cross $x = 0$ or $y = 0$.

Thus, the only possible closed orbit would be around $(1,3)$ and no other fixed points, indicated by $C_5$.

We rule this out be the assumption that a trajectory connects $(2,0) + (1,3)$ a closed orbit in the first quadrant would intersect with this trajectory and violate uniqueness.
Problem 2

We have \( V = ax^2 + by^2 \) such that
\[
V = 2ax \dot{x} + 2by \dot{y} = 2ax(\dot{y} - x^3) + 2by(-\dot{x} - y^3)
= 2axy - 2ax^4 - 2byx - 2by^4
= -2(ax^4 + by^4) + 2xy(a - b)
\]
So when \( a = b, a > 0 \) we have
\[
\dot{V} = -2a(x^4 + y^4) \leq 0,
\]
while \( V = a(x^2 + y^2) \geq 0, \)
but if \( x \neq 0 \) or \( y \neq 0, \) \( \dot{V} < 0 \) and \( V > 0. \)

We also note that
\[
\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y - x^3 \\ -x - y^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x = y^2 \quad \Rightarrow \quad x = 0, y = 0
\]
So \((x^*, y^*) = (0, 0)\) is the only fixed point.

Thus,
\[
V(x, y) > 0 \quad \text{for all } (x, y) \neq (x^*, y^*) = (0, 0),
\]
\[
V(x, y) < 0 \quad \text{for all } (x, y) \neq (x^*, y^*) = (0, 0)
\]
and \( V(x^*, y^*) = V(0, 0) = 0. \)

So \( V \) is a Liapunov function and according to pg. 201 [5] \((x^*, y^*)\) is globally asymptotically stable, i.e. \((x(t), y(t)) \to (0, 0)\),
and the system has no closed orbits.

(Note that \( V \) is clearly real-valued and continuously differentiable since it is a polynomial.)
Problem 3 (7, 3, 3)

\[
\begin{align*}
\dot{x} &= x - y - x^3 \\
\dot{y} &= x + y - y^3
\end{align*}
\]

First observe that \((x^*, y^*) = (0, 0)\) is a fixed pt

\[
DF (x^*, y^*) = \begin{pmatrix}
1 - 3x^2 & -1 \\
1 & 1 - 3y^2
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix}
\]

\[
\Rightarrow (\lambda - 1)^2 + 1 = \lambda^2 - 2\lambda + 2 = 0
\]

\[
\Rightarrow \lambda = 2 \pm \sqrt{4 - 8} = 1 \pm i
\]

This is an unstable spiral which means locally, i.e. on a circle sufficiently small about \((0, 0)\), trajectories will point outward.
Next, note that for $x + y$ large
\[ \dot{x} \approx -x^3 \]
\[ \dot{y} \approx -y^3 \]

i.e. on large enough curve surrounding origin, trajectories point back toward zero (since $x^3$ and $y^3$ are odd funs).

Let $X = 100$, $-10 \leq Y \leq 10$.

then
\[ \dot{x} = 10 - 1000 - y \leq 20 - 1000 < 0 \]

Same can be done for
(i) $X = -10$, $-10 \leq Y \leq 10$
\[ \Rightarrow \dot{x} > 0 \]

(ii) $Y = 10$, $-10 \leq X \leq 10$
\[ \Rightarrow \dot{y} < 0 \]

(iii) $Y = -10$, $-10 \leq X \leq 10$
\[ \Rightarrow \dot{y} > 0 \]
Finally, are there any other fixed pts in set B?

\[
\begin{align*}
x &= x - y - x^3 = 0 \\
y &= x + y - y^3 = 0 \\
\end{align*}
\]

\[\Rightarrow y = x^3 - x\]

\[\Rightarrow x + y - y^3 = x + x^3 - x - (x^3 - x)^3\]

\[= x^9 + 3x^5 - 3x^7\]

\[= x^5(x^4 - 3x^2 + 3) = 0\]

\[\Rightarrow x = 0; \quad x^2 = \frac{3 \pm \sqrt{9 - 12}}{2} = \frac{1}{2} (3 \pm \sqrt{3}i)\]

\[\Rightarrow \text{not real!}\]

So we apply the Poincaré–Bendixson theorem to the invariant set \(R\) containing no fixed pts

\[\Rightarrow \text{exists periodic orbit.}\]
Problem 4

7.3.4

(a) We have

\[
\begin{align*}
X &= X (1 - x^2 - y^2) - \frac{1}{2} y (1 + x) \\
y &= y (1 - x^2 - y^2) + 2 x (1 + x)
\end{align*}
\]

and linearize about the origin such that

\[
A = \begin{pmatrix}
\frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} \\
\frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y}
\end{pmatrix} = \begin{pmatrix}
1 - 12 x^2 - y^2 - \frac{1}{2} y & -2 x y - \frac{1}{2} - \frac{1}{2} x \\
-8 x y + 2 + 4 x & 1 - 4 x^2 - 3 y^2
\end{pmatrix}
\]

\(= \begin{pmatrix} 1 & -1/2 \\ 2 & 1 \end{pmatrix} \Rightarrow (\lambda^2 - 1)^2 + 1 = \lambda^2 - 2 \lambda + 2 = 0 \Rightarrow \lambda = 1 \pm i \) as in 7.3.4.

\[
\begin{pmatrix} 1 & -1/2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = (1 + i) \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \Rightarrow -2 i V_1 = V_2 \Rightarrow V = \begin{pmatrix} 1 \\ -2i \end{pmatrix}
\]

\[
\begin{pmatrix} 1 & -1/2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = (1 - i) \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \Rightarrow 2 i W_1 = W_2 \Rightarrow W = \begin{pmatrix} 1 \\ -2i \end{pmatrix}
\]

So locally:

\[
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{t} \left( c_1 e^{it} V + c_2 e^{-it} W \right)
\]

\[
e^{t} \left( \rho \cos(t) + \frac{q}{2} \sin(t) \right) \quad \text{for} \quad \rho = c_1 + c_2, \quad q = i(c_1 - c_2)
\]

As in 7.3.3, the vector of sines and cosines indicates elliptical motion while the \( e^t \) will increase the solution as \( t \to \infty \), so again we have an unstable spiral.

Since linearization's predictions are true for nodes, saddles, and spirals, we have that there is actually an unstable spiral at the origin.
(b) We take \( V = (1-4x^2-y^2)^2 \) and find that
\[
\dot{V} = 2 \left( 1-4x^2-y^2 \right) \frac{dl}{dt} (1-4x^2-y^2)
\]
\[
= 2 \left( 1-4x^2-y^2 \right) (-8x \dot{x} - 2y \dot{y})
\]
\[
= 2 \left( 1-4x^2-y^2 \right) (-8x^2 (1-4x^2-y^2) + 4yx(1+4x^2-y^2))
\]
\[
= -4 \left( 1-4x^2-y^2 \right)^2 (4x^2+y^2) \leq 0
\]
Since \( (1-4x^2-y^2)^2 \geq 0 \), \((4x^2+y^2) \geq 0\).

We now note that since \( \dot{V} < 0 \) whenever \((1-4x^2-y^2) \neq 0\)
and \((4x^2+y^2) \neq 0\), then the value of the scalar field \( V \) must be decreasing whenever these conditions are met.
So \( V \) will not be decreasing when \((4x^2+y^2) = 0 \Leftrightarrow (0,0)\)
or when \((1-4x^2-y^2) = 0 \Leftrightarrow 4x^2+y^2 = 1\), where \( \dot{V} = 0 \).

We know however that at \((0,0)\), the fixed point is unstable,
so trajectories will not approach \((0,0)\) as \( V \) decreases along
them (since \( \dot{V} < 0 \)). Instead, then, trajectories must approach
the ellipse
\[
4x^2+y^2 = 1.
\]

We know they won't just spiral away to infinity, because if
they did so, \( \dot{V} < 0 \) there would be contradicted by the
fact that as \((x,y) \to (00,00)\),
\[ V = (1-4x^2-y^2)^2 \to +\infty \]
and that we would at some point have \( \dot{V} > 0 \) (contradiction).
Problem 5

9.3.10

We have \( \dot{\mathbf{X}} = A \mathbf{X} - r^2 \mathbf{X} \) with fixed points at
\[
A \mathbf{X} - r^2 \mathbf{X} = 0 = (A - r^2 I)\mathbf{X} = 0
\]

So
\[
A \mathbf{X} = \|\mathbf{X}\|^2 \mathbf{X},
\]
which is only true when \( \mathbf{X} = 0 \) or when \( \|\mathbf{X}\|^2 \) is an eigenvalue multiple of \( A \) and \( \mathbf{X} \) happens to be an eigenvector. We know the latter can never happen, however, since \( \|\mathbf{X}\|^2 \) is real while the eigenvalues and eigenvalues of \( A \) will be complex valued since \( \lambda_i = \alpha \pm i\beta \). So the only fixed point is \( \mathbf{X} = 0 \).

At the origin, \( \mathbf{X} = \|\mathbf{X}\| \mathbf{X} = 0 \), so
\[
\dot{\mathbf{X}} = A \mathbf{X}
\]
indicating that the fixed point is a spiral since its eigenvalues are of the form \( \alpha \pm i\beta \) (this result was shown in the previous problems of the set that had spirals).

So \( \mathbf{X} = e^{\alpha t} (\ldots ) \) where the parenthesized term is a vector involving sines and cosines, representing elliptical behavior. Thus, it is clear that at the origin, \( \alpha < 0 \Rightarrow \) the fixed point is attractive; \( \alpha > 0 \Rightarrow \) the fixed point is a repeller in forward time.

We also have \( A \) can be diagonalized to the canonical form
\[
A = T A' T^{-1} \quad \text{where} \quad A' = \begin{pmatrix} \alpha & B \\ -B & \alpha \end{pmatrix}
\]
so we have
\[
\dot{\mathbf{X}} = T A' T^{-1} \mathbf{X} + \|\mathbf{X}\|^2 \mathbf{X}
\]

\[
T^{-1} \dot{\mathbf{X}} = A' T^{-1} \mathbf{X} + \|\mathbf{X}\|^2 T^{-1} \mathbf{X}
\]

We note that \( T^{-1} \mathbf{X} = \frac{d}{dE} (T^{-1} \mathbf{X}) \) since \( T^{-1} \) is a canonical matrix.

So then
\[
(T^{-1} \mathbf{X})^T \frac{d}{dE} (T^{-1} \mathbf{X}) = (T^{-1} \mathbf{X})^T A' (T^{-1} \mathbf{X}) + \|\mathbf{X}\|^2 (T^{-1} \mathbf{X})^T (T^{-1} \mathbf{X})
\]
We know \((T^{-1} \vec{x})^T (T^{-1} \vec{x}) = \|T^{-1} \vec{x}\|^2\)
and that
\((T^{-1} \vec{x})^T \frac{d}{dt} (T^{-1} \vec{x}) = \frac{1}{2} \frac{d}{dt} \|T^{-1} \vec{x}\|^2\)

since \(\frac{1}{2} \frac{d}{dt} (a^2 + b^2) = \frac{1}{2} \left( \frac{d}{dt} a + \frac{d}{dt} b \right) = (a, b)^T \frac{d}{dt} (a, b)\)

and we write
\[(T^{-1} \vec{x})^T A' (T^{-1} \vec{x}) = (T^{-1} \vec{x})^T \begin{pmatrix} (\alpha, 0) + (0, \beta) \\ (0, \alpha) \end{pmatrix} (T^{-1} \vec{x}) = \alpha \|T^{-1} \vec{x}\|^2\]

Since
\[
(a, b) \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = (-\beta b, \beta a) \begin{pmatrix} a \\ b \end{pmatrix} = -\beta ba + \beta ab = 0
\]

and
\[
(a, b) \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = (a \alpha, b \alpha) \begin{pmatrix} a \\ b \end{pmatrix} = \alpha (a, b)^T (a, b) = \alpha (a^2 + b^2) = \| (a, b) \|^2
\]

so
\[
\frac{1}{2} \frac{d}{dt} \|T^{-1} \vec{x}\|^2 = \alpha \|T^{-1} \vec{x}\|^2 - \|\vec{x}\|^2 \frac{d}{dt} \|T^{-1} \vec{x}\|^2
\]

Thus, we see that if \(\alpha\) is positive (ad since \(T^{-1}\) is a constant matrix) the vector field will point away from the origin for \(\vec{x}\) small when \(\alpha \|T^{-1} \vec{x}\|^2 > \|\vec{x}\|^2 \|T^{-1} \vec{x}\|^2\)

since \(\|T^{-1} \vec{x}\|^2\) is clearly proportional to the size of \(\vec{x}\), \(\|\vec{x}\|^2 \).

But for \(\vec{x}\) large and \(\alpha\) constant, we will have
\(\alpha \|T^{-1} \vec{x}\|^2 < \|\vec{x}\|^2 \|T^{-1} \vec{x}\|^2\)

such that the field will move inward towards the origin other passing this threshold. (Origin of \(Y = T^{-1} X = 0 <\Rightarrow \vec{x} = 0\))

Thus, as shown, if \(\alpha\) is negative the field will always point towards the origin of no trapping region exists and all trajectories move towards the origin.
However, for $\alpha > 0$, we should note that at the origin, trajectories locally move away from the origin, while for large distances (some $R$, dependent on $\alpha$) away from the origin, trajectories will move toward the origin so a trapping region must exist.

So in the disk defined by $R_0$ (infinitesimal) and $R_L$, we have a trapping region. Bernoulli's indicatrix indicates that there must be a closed orbit in the region since it is absent of fixed points.

(Also depend on $A\mathbf{x} = e^\mathbf{x}$ being continuously differentiable, which must be true since it only deals with powers of $\mathbf{x}$ and $Ax$ will have no singularities or discontinuities.)