6.2.1

The claim for the uniqueness/existence theorem was that trajectories do not intersect in the open connected set \( D \subset \mathbb{R}^n \) for some interval \((-\epsilon, \epsilon)\) about \( x_0 \) since solutions on this interval are unique.

Though we might be able to extend this interval by taking an \( x_0 \) later in time, we will still be only applying the theorem in finite time. As such, unless the trajectories begin at a fixed point, trajectories which appear to intersect at a fixed point actually only approach that point as \( t \to \pm \infty \). Thus, the trajectories never reach the fixed point or any interval \((-\epsilon, \epsilon)\) given by the theorem and never actually intersect at the fixed points so much as approach them in infinite time.

6.2.2

(a) We want \( f \) to be continuous and have partial derivatives that are continuous on \( x^2 + y^2 < 1 \), open and connected, which we call \( D \).

First, we see that the domain \( x^2 + y^2 < 1 \) must be open and connected since it defines an open disk in \( \mathbb{R}^2 \), i.e., any point in \( D \) can be surrounded by a smaller neighborhood which is still in \( D \) (allowed by the fact that the disk's circumference isn't contained in \( D \)).

We also see \( D \) is connected since any \((x, y) \in D \) can be connected by a line contained in \( D \) to any other \((x', y') \in D \) since \( D \) is a circle and will be homeomorphic properly (Chords or Secants).

We now note that
\[
\dot{f} = \begin{pmatrix} y \\ -x + (1-x^2-y^2)y \end{pmatrix}
\]
is continuous since it is the composition of two polynomial functions which are themselves continuous everywhere in \( \mathbb{R}^2 \) (as is the nature of polynomials).
Similarly, \( \frac{dx_1}{dx} = 0 \), \( \frac{dy_1}{dy} = 1 \), \( \frac{dx_2}{dx} = 1 - 2xy \), and \( \frac{dy_2}{dy} = 1 - x^2 - 3y^2 \).

are all also polynomials and thus are continuous for all \( \mathbb{R}^2 \).

So \( f \) and its partials are continuous on \( D \) open / connected, ensuring existence and uniqueness.

(b) We see that if \( y(t) = \sin t \), \( y(t) = \cos t \),
\( x(t) = \cos t \), \( y(t) = -\sin t \),
and
\[
\begin{align*}
f_1(x, y) &= y = \cos t \\
f_2(x, y) &= -x + (1 - (x^2 + y^2))y \\
&= -\sin t + (1 - (\sin^2 t + \cos^2 t))\cos t \\
&= -\sin t + (1 - 1)\cos t = \sin t.
\end{align*}
\]

So \( \left( \begin{array}{c}
x(t) \\
y(t)
\end{array} \right) = \left( \begin{array}{c}
f_1 \\
f_2
\end{array} \right) = \tilde{f}(x, y) \)

and thus these are exact solutions.

(c) We note that \( x(t) = \sin t \), \( y(t) = \cos t \) defines a circle of radius 1 since \( x^2(t) + y^2(t) = 1 \).
Clearly the other solution where \( \left( \begin{array}{c}
x(0) \\
y(0)
\end{array} \right) = \left( \begin{array}{c}
\frac{1}{2} \\
0
\end{array} \right) \) has an initial point inside the circle since \((\frac{1}{2})^2 + 0^2 = (\frac{1}{4}) < 1 \).

The solutions from (b) resides in \( D \) where existence / uniqueness hold for \( t < \infty \) (and \( t > -\infty \)), so no other soln can intersect it in finite time.

As such, \( x = \frac{1}{2}, y = 0 \) is not on this trajectory as shown above and its solution's trajectory cannot cross that which \( x^2 + y^2 < 1 \) in finite time, implying its trajectory must remain within this circle (also in \( D \)) where \( x(t)^2 + y(t)^2 < 1 \) for \(-\infty < t < \infty \).
6.3.2 \[ \dot{x} = \sin y, \quad \dot{y} = x - x^3 \]
\[ \dot{x} = 0 = \sin y \quad \Rightarrow \quad y = n \pi, \ n \in \mathbb{Z} \]
\[ \dot{y} = 0 = k \left( 1 - x^2 \right) \quad \Rightarrow \quad x = 0, -1, 1 \]

We have the Jacobian
\[
\begin{pmatrix}
\frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\
\frac{\partial y}{\partial x} & \frac{\partial y}{\partial y}
\end{pmatrix} =
\begin{pmatrix}
0 & \cos y \\
1 - 3x^2 & 0
\end{pmatrix}
\]

We note that the \((x^k, y^k)\) found will provide 4 cases for bifurcation, but then since \(\cos(n\pi)\) will be \(\pm 1\) and since \(1 - 3(1)^2 = 1 - 3(-1)^2\), so we have

**Case (i)**

For \(y^k = 2k\pi, \ k \in \mathbb{Z}\)
\[ x^k = \pm 1 \]

with \(\det (A - \lambda I) = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda \end{vmatrix} = \lambda^2 + 2 \Rightarrow \lambda = \pm \sqrt{2} i \)

\[
\begin{pmatrix}
0 & 1 \\
-2 & 0
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix} = \sqrt{2}
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix} \Rightarrow v_2 = \sqrt{2} v_1 \quad \Rightarrow \quad \begin{pmatrix}
v_1 \\
v_2
\end{pmatrix} = \begin{pmatrix}
1 \\
\sqrt{2}
\end{pmatrix}
\]

So
\[
A = T^{-1} A T = \begin{pmatrix}
1 & 0 \\
0 & \sqrt{2}
\end{pmatrix} \begin{pmatrix}
0 & 1 \\
-2 & 0
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & \sqrt{2}
\end{pmatrix} = \begin{pmatrix}
0 & \sqrt{2} \\
-\sqrt{2} & 0
\end{pmatrix}
\]

and in turn
\[
\begin{pmatrix}
0 & \sqrt{2} \\
-\sqrt{2} & 0
\end{pmatrix}
\begin{pmatrix}
0 \\
1
\end{pmatrix} = \left( \sin \frac{\pi t}{2}, \cos \frac{\pi t}{2} \right)
\]

So these fixed points are centers, surrounded by ellipses locally.

**Case (ii)**

For \(x^k = 2k\pi, \ k \in \mathbb{Z}\)
\[ x^k = \pm 1 \]

with \(\det (A - \lambda I) = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda \end{vmatrix} = \lambda^2 + 2 \Rightarrow \lambda = \pm \sqrt{2} i \)
\[
\begin{pmatrix}
0 & 1 \\
-2 & 0
\end{pmatrix}
\begin{pmatrix}
V_1 \\
V_2
\end{pmatrix} = \sqrt{2}
\begin{pmatrix}
V_1 \\
V_2
\end{pmatrix} \Rightarrow \vec{v} = \begin{pmatrix}
1 \\
\sqrt{2}
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 \\
-2 & 0
\end{pmatrix}
\begin{pmatrix}
W_1 \\
W_2
\end{pmatrix} = -\sqrt{2}
\begin{pmatrix}
W_1 \\
W_2
\end{pmatrix} \Rightarrow \vec{w} = \begin{pmatrix}
1 \\
-\sqrt{2}
\end{pmatrix}
\]

\[
A^T = T^{-1}AT = \begin{pmatrix}
1/2 & -\sqrt{2}/4 \\
1/2 & \sqrt{2}/4
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
-2 & 0
\end{pmatrix}
\begin{pmatrix}
1/2 & 1/2 \\
-\sqrt{2}/4 & \sqrt{2}/4
\end{pmatrix} = \begin{pmatrix}
1/2 & 0 \\
0 & 1/2
\end{pmatrix}
\]

\[
e^{At} = T\begin{pmatrix}
1/2 & 1/2 \\
1/2 & -1/2
\end{pmatrix}
\begin{pmatrix}
e^t & 0 \\
0 & e^{-t}
\end{pmatrix}
\begin{pmatrix}
1/2 & 1/2 \\
-1/2 & 1/2
\end{pmatrix}
\begin{pmatrix}
1/2 & 0 \\
0 & 1/2
\end{pmatrix}
\begin{pmatrix}
\cosh(-t) & (\sqrt{2}/2) \sinh(-t) \\
(-\sqrt{2}/2) \sinh(-t) & \cosh(-t)
\end{pmatrix}
\]

So these fixed pts are saddles as indicated by the one positive + one negative real eigen values.

\[\text{Case (iii)}\]

For \(y^* = (2k\pi)\) for \(k \in \mathbb{Z} : A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \)

\(X^* = 0\)

and

\[
\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 - 1 \Rightarrow \lambda = \pm 1
\]

\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \Rightarrow V_1 = V_2 \Rightarrow \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \Rightarrow W_2 = -W_1 \Rightarrow \vec{w} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]

So

\[
A^T = T^{-1}AT = \begin{pmatrix}
1/2 & 1/2 \\
1/2 & -1/2
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
1/2 & 1/2 \\
1/2 & -1/2
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\]

\[
e^{At} = T\begin{pmatrix}
1/2 & 1/2 \\
1/2 & -1/2
\end{pmatrix}
\begin{pmatrix}
e^t & 0 \\
0 & e^{-t}
\end{pmatrix}
\begin{pmatrix}
1/2 & 1/2 \\
1/2 & -1/2
\end{pmatrix} = \begin{pmatrix}
\cosh(t) & \sinh(t) \\
\sinh(t) & \cosh(t)
\end{pmatrix}
\]

So these fixed pts are saddles as indicated by the one positive + one negative real eigen values.
Case (iv)

For \( \gamma^* = (2k+1)\pi \) for \( k \in \mathbb{Z} \) : 
\( A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \)

\( \kappa^* = 0 \)

with \( \det(\lambda I - A) = \begin{vmatrix} \lambda & 1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 + 1 \Rightarrow \lambda = \pm i \)

so \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = i \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix} \)

\( A^* = T^{-1}AT - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \)

And hence \( A^* = T e^{A^* \tau} T^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos \tau & \sin \tau \\ -\sin \tau & \cos \tau \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos \tau & -\sin \tau \\ \sin \tau & \cos \tau \end{pmatrix} \)

So these fixed pts are centers, surrounded by circles locally.
Fixed points + types appear where they do because of the reasoning primarily given. Flow shapes at the saddles are given by the eigenvector shapes; similarly, flow directions (into or out of saddle) depend on whether eigenvalues of the associated vector are positive or negative (since $e^{-\lambda t} \to 0$ as $t \to \infty$) or decay to fixed point while $e^{\lambda t} \to \infty$ as $t \to \infty$. Flow away from fixed point for $\lambda > 0$ in these cases.

Flow direction about centers follows from the saddle flow directions but is also be surmised by simply $(x, y)$ near (but not on) the center points:

- at $(\pi/2, 1)$, $(\dot{x}, \dot{y}) = (\sin(\pi/2), 0) < 0$ for $E > 0$.

Similarly, outside flow directions found as follows:

- at $(\pi/2, 0)$, $(\dot{x}, \dot{y}) = (\sin(\pi/2), 0) = (1, 0)$
- at $(\pi/2, 0)$, $(\dot{x}, \dot{y}) = (\sin(\pi/2), 0) = (-1, 0)$

We then take some function $H$:

\[ \frac{dH}{dy} = \frac{dx}{dt} \cdot \sin y \Rightarrow H = -\cos y + C \]

\[ \frac{dH}{dx} = \frac{dy}{dt} = x - x^3 \Rightarrow \frac{dC}{dx} = x^3 - x \Rightarrow C = \frac{x^4}{4} - \frac{x^2}{2} \]

It follows by the chain rule that

\[ \frac{dH}{dt} = \frac{dH}{dx} \frac{dx}{dt} + \frac{dH}{dy} \frac{dy}{dt} = -\frac{dx}{dt} + \frac{dy}{dt} \frac{dy}{dt} = 0. \]
6.4.1 \[ \dot{x} = x(3 - x - y) = 0 \implies (x^*, y^*) = (0, 0), (3, 0), (0, 2) \]
\[ \dot{y} = y(2 - x - y) = 0 \]

Since eqn 1 suggests \( x = 0 \) or \( x = 3 - y \)

Eqn 2 suggests \( y = 0 \) or \( y = 2 - x \)

So we linearize

\[ A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 3 - 2x - y & -x \\ -y & 2 - 2y - x \end{pmatrix} \]

(i) At \((0, 0)\), \( A_1 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \) \implies \( (\lambda - 3)\chi_2 = 0 \) \implies \( \lambda = 0, 3 \)

\( 0 < \chi_1 < \chi_2 \) \: unstable fixed point

\[ \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \]

\[ \implies 3v_1 = 2v_1 \]

\[ \therefore v_1 = 0 \]

\[ \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 3 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \]

\[ \implies 3w_1 = 3w_1 \]

\[ \therefore w_1 = 0 \]

\[ \Rightarrow \vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

\[ \Rightarrow \vec{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

(ii) At \((3, 0)\), \( A_2 = \begin{pmatrix} -3 & -3 \\ 0 & -1 \end{pmatrix} \) \implies \( (\lambda + 3)(\lambda + 1) = 0 \) \implies \( \lambda = -1, -3 \)

\( \chi_1 < \chi_2 < 0 \) \: Stable fixed point

\[ \begin{pmatrix} -3 & -3 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -3 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \]

\[ \implies -3v_1 - 3v_2 = -3v_1 \]

\[ \therefore v_2 = 0 \]

\[ \begin{pmatrix} -3 & -3 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = -3 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \]

\[ \implies -3w_1 - 3w_2 = -3w_1 \]

\[ \therefore w_2 = 0 \]

\[ \Rightarrow \vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

\[ \Rightarrow \vec{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

(iii) At \((0, 2)\), \( A_3 = \begin{pmatrix} 1 & 0 \\ -2 & -2 \end{pmatrix} \) \implies \( (\lambda - 1)(\lambda + 2) = 0 \) \implies \( \lambda = -2, 1 \)

\( \chi_1 < 0 < \chi_2 \) \: saddle point

\[ \begin{pmatrix} 1 & 0 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \]

\[ \implies v_1 = -2v_1 \]

\[ \therefore v_1 = 0 \]

\[ \begin{pmatrix} 1 & 0 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \]

\[ \implies -2w_1 - 2w_2 = w_2 \]

\[ \therefore w_2 = -2/3 \]

\[ \Rightarrow \vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

\[ \Rightarrow \vec{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]
The nullclines will be wherever $x = x(3-x-y) = 0$ or $\dot{y} = y(2-x-y)$

i.e. $x = 0$, $y = 0$, $x = 3 - x$, and $y = 2 - x$.

Sketching these we see

![Graph showing nullclines](image)

Note: Rabbit/Sheep Problem
does not relate to behavior for $y < 0$ or $x < 0$.

Fixed points where $\dot{x} = 0$ nullclines meet $\dot{y} = 0$ nullclines

On $x = 0$, $\dot{y} = y(2-y)$; positive for $0 < y < 2$, negative for $y < 0, y > 2$.

On $y = 0$, $\dot{x} = x(x(3-x))$; positive for $0 < x < 3$, negative for $x < 0, x > 3$.

On $y = 2-x$, $\dot{x} = x$; positive for $x > 0$, negative for $x < 0$.

On $y = 3-x$, $\dot{y} = -y$; positive for $y < 0$, negative for $y > 0$.

We know that the stability analysis from the previous page is correct since the analysis indicates these three fixed points are not borderline cases, so we can deduce the phase portrait from local behavior and the eigen values/vectors.

![Phase portrait](image)

**Basin of Attraction for**

(3,0) stable:

$V(x,y)$ s.t. $x > 0$. 

$C(-\frac{1}{2\sqrt{3}})$
6.5.2

(a) \( \dddot{x} = x - x^2 \)
\[
\frac{\partial}{\partial t} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ x - x^2 \end{pmatrix} \Rightarrow y^* = 0
\]
\( x^* = 0, 1 \) since \( x - x^2 = x(1-x) \)

Linearizing we get:

(i) At \((x^*, y^*) = (0,0)\), \( A_1 = \begin{pmatrix} \frac{\partial x}{\partial x} & \frac{\partial y}{\partial x} \\ \frac{\partial x}{\partial y} & \frac{\partial y}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \)

\[
\Rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \lambda = \pm 1
\]

\[
\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

\( \lambda, < 0 \Rightarrow \lambda = \pm i \) \quad Saddle point

(ii) At \((x^*, y^*) = (1,0)\),
\[
A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i
\]

\[
\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

We see that for the system, we have \( E \) as follows:
\[
\frac{dx}{dt} = \frac{\partial E}{\partial y} = y \Rightarrow E = \frac{1}{2} y^2 + W(x)
\]
\[
\frac{dy}{dt} = -\frac{\partial E}{\partial x} = x - x^2 \Rightarrow W(x) = \frac{1}{3} x^3 - \frac{1}{2} x^2
\]

\[
\frac{\partial E}{\partial x} = \frac{\partial E}{\partial x} \frac{dx}{dt} + \frac{\partial E}{\partial y} \frac{dy}{dt} = \left( -\frac{dy}{dt} \right) \frac{dx}{dt} + \left( \frac{dy}{dt} \right) \frac{dx}{dt} = 0
\]

Is conserved quantity \( E = \frac{1}{2} y^2 + \frac{1}{3} x^3 - \frac{1}{2} x^2 = C \) constant.

We also see
\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x^2 - y^2 \\ y \end{pmatrix} \bigg|_{(1,0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

and \( D^2 E(0,1) = \begin{pmatrix} E_{xx} & E_{xy} \\ E_{yx} & E_{yy} \end{pmatrix} = \begin{pmatrix} \frac{1}{8} x^3 - 2x & 0 \\ 0 & 1 \end{pmatrix} \bigg|_{(0,1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \)

which is positive definite since
\[ (x, y) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = (x, y) \begin{pmatrix} x \\ x^2 + y^2 \geq 0 \end{pmatrix} \] for \((x, y) \neq 0\).

So for smooth \( f(x, y) = \begin{pmatrix} -y \\ x-x^2 \end{pmatrix} \) we have by Thm 6.5.1 that all trajectories about \((0,0)\) must be closed curves \(\Rightarrow (1,0)\) is a center.

(b) We see from finding the fixed pts in part (a) that the system has nullclines at \(y = 0, x = 0, x = 1\).
Along \(y = 0\), \(\dot{x} = 0 + \dot{y} = x(1-x)\) will be positive for \(0 < x < 1\) and negative for \(x > 1\) or \(x < 0\). On \(x = 0 + x = 1\), \(\dot{y} = 0\) and \(\dot{x} = y\) will be positive for \(y > 0\), negative for \(y < 0\).

At \((0,0)\) trajectories are locally along \((1) + (-1)\) according to the eigen vectors from linearization.

The shape of trajectory \(T\) will be explained in part (c).

(c) We know trajectories will be determined by paths of equal energy, i.e. a constant \(C\), where
\[ E = C = \frac{1}{2} y^2 + W(x) \] for \(W(x) = \frac{x^3}{3} - \frac{x^2}{2}\). 
\[ W(x) \text{ has } \frac{dW}{dx} = x^2 - x = x(x-1) \text{ and } \frac{d^2W}{dx^2} = 2x - 1 \]

So at \( x = 1 \), it has a max and at \( x = 0 \) it has a min.

So

\[ W(1) = -\frac{1}{6} \]

At \( x = 1 \),

So when \( y = 0 \), \( W(x) = C \). As we move the value \( C \) up the plot above, we have one intersection for \( C < -\frac{1}{6} \), two for \( -\frac{1}{6} < C < 0 \), and none for \( C > 0 \).

The two intersections indicate that the curves are periodic i.e. closed, so the closed curve which separates closed from non-closed orbits is either the one at \( C = -\frac{1}{6} \) or that at \( C = 0 \).

We see the one at \( C = 0 \) is the true curve since the one at \( C = -\frac{1}{6} \) corresponds to the fixed point at \( x = 1 + \text{some curve hitting } x = -\frac{1}{2} \) (see \( W(-\frac{1}{2}) = -\frac{1}{6} \) as well) which indicates a simple trajectory since the fixed point isn't on one.

That is, a trajectory for this system could be at the fixed point or any point of lower potential \( W(x) \) for \( x < -\frac{1}{6} \).

So we know our trajectory is at \( C = 0 \) where

\[ 0 = \frac{1}{2} y^2 + W(x) \Rightarrow y = \pm \sqrt{-2W(x)} = \pm \sqrt{2 \left( \frac{x^3}{3} + \frac{x^2}{2} \right)} \]

Since \( y \propto x^{3/2} \) near for large \( x \), it follows that \( y \) increases faster than \( x \) in this trajectory, explaining why this curve leads upwards and downwards in the phase portrait.