Homework 2

**Problem 1.** [Strogatz, 2.6.2] (No periodic solutions to $\dot{x} = f(x)$) Here’s an analytic proof that periodic solutions are impossible for a vector field on the line. Suppose on the contrary that $x(t)$ is a nontrivial periodic solution, i.e. $x(t) = x(t + T)$ for some $T > 0$, and $x(t) \neq x(t + s)$ for all $0 < s < T$. Derive a contradiction by considering

$$\int_0^T f(x) \frac{dx}{dt} dt.$$

**Problem 2.** [Strogatz, 2.7.3] Plot the potential function $V(x)$ for the following vector field and identify all the equilibrium points and their stability:

$$\dot{x} = \sin(x).$$

**Problem 3.** [Strogatz, 3.1.2] For the following system, sketch all the qualitatively different vector fields that occur as $r$ is varied. Show that a saddle-node bifurcation occurs at a critical value of $r$, to be determined. Finally, sketch the bifurcation diagram of fixed points $x^*$ versus $r$:

$$\dot{x} = r - \cosh x.$$

**Problem 4.** [Strogatz, 3.4.2] For the following system, sketch all the qualitatively different vector fields that occur as $r$ is varied. Show that a pitchfork bifurcation occurs at a critical value of $r$, to be determined. Finally, sketch the bifurcation diagram of fixed points $x^*$ versus $r$:

$$\dot{x} = rx - \sinh x.$$

In the following systems, find the values of $r$ at which bifurcations occur, and classify those as saddle-node, transcritical, or pitchfork. Finally, sketch the bifurcation diagram of fixed points $x^*$ vs. $r$.

**Problem 5.** [Strogatz, 3.4.9] $\dot{x} = x + \tanh(rx)$.

**Problem 6.** [Strogatz, 3.4.10]

$$\dot{x} = rx + \frac{x^3}{1 + x^2}.$$