Homework 1

**Problem 1.** [Strogatz, 2.3.1] (Exact solution of logistic equation) There are two ways to solve the logistic equation

\[
\dot{N} = rN \left( 1 - \frac{N}{K} \right),
\]

analytically for an arbitrary initial condition \(N_0\).

a) Separate variables and integrate, using partial fractions.

b) Make the change of variables \(x = 1/N\). Then derive and solve the resulting differential equation for \(x\).

**Problem 2.** [Strogatz, 2.3.3] (Tumor growth) The growth of cancerous tumor can be modeled by the Gompertz law

\[
\dot{N} = -aN \ln(bN),
\]

where \(N(t)\) is proportional to the number of cells in the tumor, and \(a, b > 0\) are parameters.

a) Interpret \(a\) and \(b\) biologically.

b) Sketch the vector field and then graph \(N(t)\) for various initial values. The predictions of this simple model agree surprisingly well with data on tumor growth, as long as \(N\) is not too small; see Aroesty et al. (1973) and Newton (1980) for examples.

Use linear stability analysis to classify the fixed points of the following systems. If \(f'(x^*) = 0\), use a graphical argument to decide the stability.

**Problem 3.** [Strogatz, 2.4.1] \(\dot{x} = x(1 - x)\).

**Problem 4.** [Strogatz, 2.4.7] \(\dot{x} = ax - x^3\), where \(a\) can be positive, negative, or zero. Discuss all three cases.

**Problem 5.** [Strogatz, 2.5.6] (The leaky bucket) The following example (Hubbard and West 1991, p. 159) shows that in some physical situations, non-uniqueness is natural and obvious, not pathological.

Consider a water bucket with a hole in the bottom. If you see an empty bucket with a puddle beneath it, can you figure out when the bucket was full? No, of course not! It could have finished emptying a minute ago, ten minutes ago, or whatever. The solution to the corresponding differential equation must be non-unique when integrate backwards in time.

Here’s a crude model of the situation. Let \(h(t)\) =height of the water remaining in the bucket at time \(t\); \(a = \)are of the hole; \(A = \)cross-sectional area of the bucket (assumed constant); \(v(t) = \)velocity of the water passing through the hole.
a) Show that $av(t) = A\dot{h}(t)$. What physical law are you invoking?

b) To derive an additional equation, use conservation of energy. First, find the change in potential energy in the system, assuming that the height of the water in the bucket decreases by an amount $\Delta h$ and that the water has density $\rho$. Then find the kinetic energy transported out of the bucket by the escaping water. Finally, assuming all the potential energy is converted into kinetic energy, derive the equation $v^2 = 2gh$.

c) Combining (b) and (c), show $\dot{h} = -C\sqrt{h}$, where $C = \sqrt{2\gamma \left(\frac{A}{\rho}\right)}$.

d) Given $h(0) = 0$ (bucket empty at $t = 0$), show that the solution for $h(t)$ is non-unique in backwards time, i.e. for $t < 0$.

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**Problem 6. [ODE Review (Separable Equations)]**

a) Show that choosing a new dependent variable $u = x/t$ transforms the differential equation

\[ \dot{x} = f \left( \frac{x}{t} \right), \]

into a separable equation.

b) Solve the initial value problem

\[ \dot{x} = \frac{x}{t} - \frac{t^2}{x^2}, \quad x(1) = 1, \]

In what interval is this solution valid (including backwards in time)?

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**Problem 7. [ODE Review (Separable Equations)]** Solve the initial-value problem

\[ \dot{x} = \frac{x \cos(t)}{1 + 2x^2}, \quad x(0) = 1, \quad (1) \]

as far as you can. Furthermore, conclude that the solution $\phi(t)$ of (1) can be extended to the whole interval $-\infty < t < \infty$ by completing the following outline:

(i) By finding the maximum and minimum values of $x/(1 + 2x^2)$, show that

\[ \left| \frac{x}{1 + x^2} \right| \leq \frac{1}{2\sqrt{2}}, \]

for all $x$, and hence that the absolute value of the right-hand side of the differential equation in (1) is bounded by $\frac{1}{2\sqrt{2}}$ for all $t$ and $x$.

(ii) Use the result of part (i) to show that

\[ |\phi(t) - 1| \leq \frac{|t|}{2\sqrt{2}}, \]

for all $t$. Conclude that $\phi(t)$ can indeed be extended to the whole interval $-\infty < t < \infty$.

HINT: First formally integrate (1) as it stands to derive an integral equation for $\phi(t)$. Note that it is clear that $\phi(t)$ can be extended to $-\infty < t < \infty$ precisely when it never blows up.