Final Exam

Make sure to show and explain your work. The exam is due on Wednesday, 12/16 by 5pm in Amos Eaton, room 301.

Problem 1: Find the fixed points of the following system, classify the bifurcation with respect to the parameter \( r \), and draw a bifurcation diagram:

\[
\dot{x} = 1 - (1 + r)x + rx^2
\]

Problem 2: Draw the phase portrait of the following two-dimensional system and classify the fixed point.

\[
\dot{x} = Ax, \quad x \in \mathbb{R}^2, \\
A = \begin{pmatrix} -3 & -1 \\ 1 & -1 \end{pmatrix}
\]

Problem 3: Solve the following systems for \( x(t) \):

a) \[
\begin{align*}
\dot{x} - 5\dot{x} + 6x &= 2t \, e^{3t} \\
x(0) &= 0, \quad \dot{x}(0) = 0
\end{align*}
\]

b) \[
\begin{align*}
\dot{x} - 5\dot{x} + 6x &= 5(t - 1) \sin(t) \\
x(0) &= 0, \quad \dot{x}(0) = 0
\end{align*}
\]

c) \[
\begin{align*}
\dot{x} - 5\dot{x} + 6x &= 2t \, e^{3t} + 5(t - 1) \sin(t) \\
x(0) &= 0, \quad \dot{x}(0) = 0
\end{align*}
\]

Problem 4: Let \( 0 < \epsilon \ll 1 \). Use the method of multiple scales (two-timing) to find the periodic solution (limit cycle) of the following system to leading (\( \mathcal{O}(1) \)) order:

\[
\ddot{x} + 2x - \epsilon(x^2 - 2)\dot{x} = 0
\]

Is the cycle stable or unstable?

Problem 5: Find the fixed points of the following one-dimensional map and classify their stability. Show that orbits which begin inside a certain closed interval remain in that interval for all time. Show that solutions which begin outside this interval become unbounded.

\[
x_{n+1} = -2x_n + x_n^3
\]