Suggested Homework 8

Section 10.7.

Find the Maclaurin series and find the interval on which the expansion is valid.

3. $f(x) = \frac{1}{1 - 2x}$
4. $f(x) = \frac{x}{1 - x^4}$
5. $f(x) = \cos(3x)$
6. $f(x) = \sin(2x)$
9. $f(x) = \ln(1 - x^2)$
11. $f(x) = \tan^{-1}(x^2)$
12. $f(x) = x^2e^{x^2}$
13. $f(x) = e^{x^2}$

3. We can use the geometric series formula

$$\frac{1}{1 - r} = \sum_{n=0}^{\infty} r^n$$

which is valid for $|r| < 1$. The series on the right diverges otherwise. For $r = 2x$, we get the Maclaurin series

$$f(x) = \frac{1}{1 - 2x} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n,$$

which converges for $|r| = |2x| < 1$, or $|x| < 1/2$. Since this is a geometric series, we know that this series diverges at the endpoints, which we can check by plugging in $x = 1/2$ and $x = -1/2$ to get the divergent series

$$\sum_{n=0}^{\infty} 1 \quad \text{and} \quad \sum_{n=0}^{\infty} (-1)^n,$$

respectively. The interval of convergence of the Maclaurin series for $f(x)$ is therefore $x \in (-1/2, 1/2)$.

4. Similar to above, we use the geometric series formula with $r = x^4$ to get

$$\frac{1}{1 - x^4} = \sum_{n=0}^{\infty} x^{4n},$$

convergent for $|r| = |x|^4 < 1$ or $|x| < 1$ and divergent otherwise. Multiplying by $x$ gives

$$f(x) = \frac{x}{1 - x^4} = \sum_{n=0}^{\infty} x^{4n+1}.$$
which converges for $|x| < 1$ and diverges otherwise. One can easily check again that this series diverges for $x = \pm 1$ since these points give the respective divergent series
\[
\sum_{n=0}^{4n+1} = \sum_{n=0}^{1}
\]
and
\[
\sum_{n=0}^{(-1)}.
\]
The interval of convergence for the Maclaurin series for $f(x)$ is therefore $x \in (-1, 1)$.

5. As shown in class, the Maclaurin series for $\cos u$ is
\[
\cos u = \sum_{k=0}^{\infty} \frac{(-1)^k u^{2k}}{(2k)!}
\]
which is convergent everywhere, so letting $u = 2x$ gives
\[
\cos 3x = \sum_{k=0}^{\infty} \frac{(-1)^k (3x)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-1)^k 3^{2k} x^{2k}}{(2k)!}
\]
which is convergent for all $u = 3x$ and therefore for all $x$.

6. As shown in class, the Maclaurin series for $\sin u$ is
\[
\sin u = \sum_{k=0}^{\infty} \frac{(-1)^k u^{2k+1}}{(2k+1)!}
\]
which is convergent everywhere, so letting $u = 2x$ gives
\[
\sin 2x = \sum_{k=0}^{\infty} \frac{(-1)^k (2x)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1} x^{2k+1}}{(2k+1)!}
\]
which is convergent for all $u = 2x$ and therefore for all $x$.

9. The Maclaurin series for $\ln(1 - u)$ was found in class by integrating the geometric series for $1/(1 - u)$ term by term to obtain
\[
\ln(1 - u) = -\int \frac{du}{1 - u} = -\sum_{n=1}^{\infty} \frac{u^n}{n}
\]
The geometric series for $1/(1 - u)$ converges for $|u| < 1$ and diverges otherwise. We found that the series above also converges for $|u| < 1$ (which can be checked by the ratio test),
as well as at $u = -1$ by the alternating series test. Thus the interval of convergence was $u \in [-1, 0)$. We now let $u = x^2$ to obtain

$$\ln(1 - x^2) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\sum_{n=1}^{\infty} \frac{x^{2n}}{n}$$

The ratio test again gives convergence for $u = |x| < 1$ or $|x| < 1$, but we must check the endpoints. In this case, $x = 1$ gives the divergent harmonic series

$$-\sum_{n=1}^{\infty} \frac{1^{2n}}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$$

Furthermore, $x = -1$ also gives the same divergent series:

$$-\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$$

so the interval of convergence for the Maclaurin series of $\ln(1 - x^2)$ is $x \in (-1, 1)$.

11. The Maclaurin series for $\tan^{-1} u$ is given in the book to be

$$\tan^{-1} u = \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n+1}}{2n+1}$$

for $|u| \leq 1$. This is shown by integrating the geometric series $1/(1 - r)$ for $r = u^2$:

$$\tan^{-1} u = \int \frac{du}{1 + u^2} = \int \sum_{n=0}^{\infty} (-u^2)^n = \int \left[ \sum_{n=0}^{\infty} (-1)^n u^{2n} \right] du = \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n+1}}{2n+1}.$$ 

The endpoints at $u = 1$ and $u = -1$ both give a convergent alternating series (by the alternating series test)

$$\sum_{n=0}^{\infty} \frac{(-1)^n 1^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

and

$$\sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}.$$ 

We now substitute $u = x^2$ to obtain

$$\tan^{-1} x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{2n+1}.$$
For $x = \pm 1$, we obtain the convergent alternating series
\[
\sum_{n=0}^{\infty} \frac{(-1)^n (\pm 1)^{4n+2}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}
\]
The interval of convergence for the Maclaurin series for $\tan^{-1} x^2$ is therefore $x \in [-1, 1]$.

12. The Maclaurin series for $e^u$ convergent everywhere is
\[
e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}
\]
so the Maclaurin series for $e^{x^2}$ convergent everywhere is
\[
e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}
\]
Multiplying this by $x^2$ gives the Maclaurin series
\[
x^2 e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n+2}}{n!}
\]
which will also converge for all $x$ (check this with the ratio test!).

13. The Maclaurin series for $e^x$ convergent everywhere is
\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}
\]
so the Maclaurin series for $e^{x-2}$ convergent everywhere is
\[
e^{x-2} = e^{-2} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{e^{2n} n!}
\]

Find the terms through degree four of the Maclaurin series of $f(x)$. Use multiplication and substitution as necessary.

19. $f(x) = e^x \sin x$.

The Maclaurin series for $e^x$ and $\sin x$ respectively through degree four are
\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + ...
\]
and
\[
\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + ...
\]
so we multiply to get

\[
e^x \sin x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots \right) \left(x - \frac{x^3}{6} + \ldots \right)
\]

\[
= x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} - \frac{x^3}{6} - \frac{x^4}{12} - \frac{x^6}{36} - \frac{x^7}{6 \cdot 24} + \ldots
\]

\[
= x + x^2 + \frac{x^3}{3} + \ldots
\]

where we kept only up to the fourth order terms.

Find the Taylor series centered at \(c\) and the interval on which the expansion is valid.

29. \(f(x) = \frac{1}{x}, \ c = 1\)

31. \(f(x) = \frac{1}{1-x}, \ c = 5\)

29. We must write this as a geometric series centered about \(c = 1\) so we have

\[
f(x) = \frac{1}{1 - (1-x)} = \sum_{n=0}^{\infty} (1-x)^n = \sum_{n=0}^{\infty} (-1)^n(x-1)^n,
\]

which is a convergent geometric series when \(|1-x| = |x-1| < 1\), or \(0 < x < 2\) and diverges at the end points (and outside of them) since it is a geometric series.

31. We must write this as a geometric series centered about \(c = 5\) so using \(1-x = -[4+(x-5)]\), we have

\[
f(x) = -\left(\frac{1}{4+(x-5)}\right) = -\frac{1}{4} \left(\frac{1}{1 - (5-x)/4}\right) = -\frac{1}{4} \sum_{n=0}^{\infty} \frac{(5-x)^n}{4^n} = -\frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n(x-5)^n}{4^n},
\]

which is convergent when \(|x-5|/4 < 1\). That is, \(|x-5| < 4\) or \(1 < x < 9\), and divergent at and away from the endpoints by virtue of being a geometric series.

42. Differentiate the Maclaurin series for \(\frac{1}{1-x}\) twice to find the Maclaurin series of \(\frac{1}{(1-x)^3}\).

For \(f(x) = \frac{1}{1-x}\), we have

\[
f'(x) = \frac{1}{(1-x)^2}, \quad f''(x) = -\frac{2}{(1-x)^3}
\]

We have the geometric series

\[
f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for} \quad |x| < 1,
\]

which we differentiate twice to get

\[
f''(x) = -\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)x^n.
\]
Thus

\[- \frac{1}{2} f''(x) = \frac{1}{(1-x)^3} = - \frac{1}{2} \sum_{n=0}^{\infty} (n+2)(n+1)x^n\]

This will converge for \(|x| < 1\) as expected, which can be confirmed using the ratio test. Plugging in \(x = \pm 1\) gives a divergent series

\[- \frac{1}{2} \sum_{n=0}^{\infty} (n+2)(n+1)(\pm 1)^n\]

due to the \(n\)th term test.

Express the integral as an infinite series.

53. \(\int_0^x \frac{1 - \cos t}{t} dt\) for all \(x\), 55. \(\int_0^x \ln(1 + t^2) dt\) for \(|x| < 1\)

53. The Maclaurin series for \(\frac{1 - \cos t}{t}\), using the Maclaurin series for \(\cos t\) (valid for all \(t\)), is

\[1 - \cos t = \frac{1}{t} \left( 1 - \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} t^{2n-1}}{(2n)!}\]

We have

\[\int_0^x t^{2n-1} dt = \frac{t^{2n}}{2n} \bigg|_0^x = \frac{x^{2n}}{2n}\]

so

\[\int_0^x \frac{1 - \cos t}{t} dt = \int_0^x \sum_{n=1}^{\infty} \frac{(-1)^{n+1} t^{2n-1}}{(2n)!} dt = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n)(2n)!}\]

The ratio test will show that this is convergent for all \(x\), as was the series for \(\cos t\). We can check with

\[\rho = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{|x|^{2n+2}(2n)(2n)!}{|x|^{2n}(2n+2)(2n+2)!} = \lim_{n \to \infty} \frac{(2n)(2n)!}{(2n+2)(2n+2)!} = \lim_{n \to \infty} \frac{(2n)}{(2n+2)(2n+1)} = 0 < 1\]

55. The Maclaurin series for \(\ln(1 + t^2)\) is given by

\[\ln(1 + t^2) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} t^{2n}}{n}\]

with interval of convergence \(t \in (-1, 1)\) (see problem 9 for a similar result). We integrate

\[\int_0^x t^{2n} dt = \frac{t^{2n+1}}{2n+1} \bigg|_0^x = \frac{x^{2n+1}}{2n+1}\]
so
\[
\int_0^x \ln(1 + t^2) dt = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{2n+1}}{(2n + 1)n}
\]

The ratio test gives
\[
\rho = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{x^{2n+3}(2n + 1)n}{x^{2n+1}(2n + 3)(n + 1)}
\]
\[
= |x|^2 \lim_{n \to \infty} \frac{(2n + 1)n}{(2n + 3)(n + 1)} = |x|^2 < 1,
\]
which gives convergence when $|x| < 1$ and divergence when $|x| > 1$. At the endpoints $x = \pm 1$,
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^{2n+1}}{(2n + 1)n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n + 1)n},
\]
and
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(1)^{2n+1}}{(2n + 1)n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n + 1)n},
\]
are both absolutely convergent and therefore convergent using a limit comparison test with $b_n = 1/n^2$ (in fact, they also both pass the alternating series test). Thus, the interval of convergence for the final series is $x \in [-1, 1]$.

57. Which function has Maclaurin series $\sum_{n=0}^{\infty} (-1)^n 2^n x^n$?

Since all terms in the series are raised to the $n$th power, this appears to be a geometric series with $r = -2x$, so
\[
\sum_{n=0}^{\infty} (-1)^n 2^n x^n = \frac{1}{1 - (-2x)} = \frac{1}{1 + 2x},
\]
which converges for $x \in (-1/2, 1/2)$ and diverges outside of this interval.

59. Using Maclaurin series, determine to exactly what value the following series converges.
\[
\sum_{n=0}^{\infty} (-1)^n \frac{(\pi)^{2n}}{(2n)!}
\]

We rewrite
\[
\sum_{n=0}^{\infty} (-1)^n \frac{(\pi)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n(\pi^2)^n}{(2n)!}.
\]
Recall the Maclaurin series for $\sin x$ given by

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!},$$

such that plugging in $x = \pi$ gives

$$\cos \pi = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!} = -1.$$

60. Using Maclaurin series, determine to exactly what value the following series converges.

$$\sum_{n=0}^{\infty} \frac{(\ln 5)^n}{n!}$$

Recall the Maclaurin series for $e^x$ given by

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

so plugging in $x = \ln 5$ gives

$$e^{\ln 5} = \sum_{n=0}^{\infty} \frac{(\ln 5)^n}{n!} = 5.$$