Suggested Homework 7

Section 10.6.

3. Show that the power series (a)-(c) have the same radius of convergence. Then show that 
(a) diverges at both endpoints, (b) converges at one endpoint but diverges at the other, and 
(c) diverges at both endpoints.

\[
\text{(a) } \sum_{n=1}^{\infty} \frac{x^n}{3^n} \quad \text{(b) } \sum_{n=1}^{\infty} \frac{x^n}{n3^n} \quad \text{(c) } \sum_{n=1}^{\infty} \frac{x^n}{n^23^n}
\]

We use the ratio test for \(a_n = \frac{x^n}{3^n}, \ b_n = \frac{x^n}{(n3^n)}, \) and \(c_n = \frac{x^n}{(n^23^n)}\) to get

\[
\rho_a = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}3^n}{x^n3^{n+1}} \right| = \frac{|x|}{3},
\]

\[
\rho_b = \lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}n3^n}{x^n(n+1)3^{n+1}} \right| = \frac{|x|}{3} \left( \frac{n}{n+1} \right),
\]

\[
\rho_c = \lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}n^23^n}{x^n(n+1)^23^{n+1}} \right| = \frac{|x|}{3} \left( \frac{n}{n+1} \right)^2.
\]

The ratio test tells us that each of the series corresponding to these sequences will converge if \(\rho_a = \rho_b = \rho_c < 1\) and will diverge if \(\rho_a = \rho_b = \rho_c > 1\), which is when

\[
\frac{|x|}{3} < 1 \quad \implies \quad |x| < 3
\]

and

\[
\frac{|x|}{3} > 1 \quad \implies \quad |x| > 3
\]

respectively. So the radius of convergence is 3.

Series (a) diverges at the endpoints because when \(x = 3\), the series is

\[
\sum_{n=1}^{\infty} \frac{3^n}{3^n} = \sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + ...
\]

which approaches \(\infty\) and when \(x = -3\), the series is

\[
\sum_{n=1}^{\infty} \frac{(-3)^n}{3^n} = \sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + 1 - 1 - ...
\]
which oscillates between 1 and \(-1\) and never converges.

Series (b) diverges when \(x = 3\) because it is given by

\[
\sum_{n=1}^{\infty} \frac{3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n}
\]

which is the harmonic series and diverges by the \(p\)-test. It converges when \(x = -3\) because it is given by

\[
\sum_{n=1}^{\infty} \frac{(-3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},
\]

which is the alternating harmonic series and, as we have shown in class, converges by the alternating series test.

Series (c) converges when \(x = 3\) because it is given by

\[
\sum_{n=1}^{\infty} \frac{3^n}{n^23^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}
\]

which converges by the \(p\)-test. It converges when \(x = -3\) because it is given by

\[
\sum_{n=1}^{\infty} \frac{(-3)^n}{n^23^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2},
\]

which can also be shown to converge by the alternating series test (but also converges absolutely since \(\sum_{n=1}^{\infty} \frac{1}{n^2}\) converges by the \(p\)-test).

5. Show that \(\sum_{n=0}^{\infty} n^n x^n\) diverges for all \(x \neq 0\).

The ratio test gives

\[
\rho = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{|(n+1)^n x^{n+1}|}{|n^n x^n|} = \lim_{n \to \infty} \frac{(n+1)^n |x|^{n+1}}{n^n |x|^n} = |x| \lim_{n \to \infty} \frac{(n+1)^n}{n^n}.
\]

The limit is given by

\[
\lim_{n \to \infty} \frac{(n+1)^n}{n^n} = \lim_{n \to \infty} \left(\frac{n+1}{n}\right)^n (n+1),
\]

where the first fraction term will approach 1 in the limit of \(n\) large while the second term gets larger and larger. Only when \(x = 0\) will \(\rho = 0 < 1\) for convergence.
Find the interval of convergence.

9. \( \sum_{n=0}^{\infty} nx^n \)
10. \( \sum_{n=1}^{\infty} \frac{2^n}{n} x^n \)
13. \( \sum_{n=4}^{\infty} \frac{x^n}{n^5} \)
15. \( \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} \)
16. \( \sum_{n=0}^{\infty} \frac{8^n}{n!} x^n \)
25. \( \sum_{n=1}^{\infty} n(x-3)^n \)
26. \( \sum_{n=1}^{\infty} \frac{(-5)^n}{n^2} (x-3)^n \)
29. \( \sum_{n=1}^{\infty} \frac{2^n}{3n} (x+3)^n \)
31. \( \sum_{n=0}^{\infty} \frac{(-5)^n}{n!} (x+10)^n \)
32. \( \sum_{n=10}^{\infty} n!(x+5)^n \)

9. The ratio test gives

\[
\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)x^{n+1}}{nx^n} \right| = \lim_{n \to \infty} \frac{(n+1)|x|^{n+1}}{n|x|^n} = |x| \lim_{n \to \infty} \frac{n+1}{n} = |x|,
\]

We have convergence for \( \rho = |x| < 1 \) when \( |x| < 1 \) and divergence for \( |x| > 1 \). Checking the endpoints, we find for \( x = 1 \) that

\[
\sum_{n=0}^{\infty} nx^n = \sum_{n=0}^{\infty} n = 1 + 2 + 3 + 4 + ...
\]

which is infinite and for \( x = -1 \),

\[
\sum_{n=0}^{\infty} nx^n = \sum_{n=0}^{\infty} n(-1)^n = 1 - 2 + 3 - 4 + ...
\]

which will keep adding larger values and will never converge (the nth term test confirms that both of these diverge). So the interval of convergence is \((-1, 1)\) with no endpoints included.

10. The ratio test gives

\[
\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{2^{n+1}x^{n+1}(n+1)}{2^n x^n n} \right| = \lim_{n \to \infty} \frac{2^{n+1}|x|^{n+1}(n+1)}{2^n |x|^n n} = 2|x| \lim_{n \to \infty} \frac{n+1}{n} = 2|x|,
\]

so we get convergence for \( \rho = 2|x| < 1 \) when \( |x| < 1/2 \) and divergence for \( |x| > 1/2 \). Checking the endpoints, we find for \( x = 1/2 \) that

\[
\sum_{n=0}^{\infty} \frac{2^n}{n} x^n = \sum_{n=0}^{\infty} \frac{2^n}{n} (1/2)^n = \sum_{n=0}^{\infty} \frac{1}{n}
\]
which is the divergent harmonic series. For \( x = -1/2 \), we have
\[
\sum_{n=0}^{\infty} \frac{2^n}{n} x^n = \sum_{n=0}^{\infty} \frac{2^n}{n} (-1/2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n}
\]
which is again the convergent alternating harmonic series. The interval of convergence is therefore \([-1/2, 1/2)\).

13. The ratio test gives
\[
\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{|x^{n+1} n^5|}{|x^n (n+1)^5|} = |x| \lim_{n \to \infty} \frac{n^5}{(n+1)^5} = |x|,
\]
We have convergence for \( \rho = |x| < 1 \) when \( |x| < 1 \) and divergence for \( |x| > 1 \). Checking the endpoints, we find for \( x = 1 \) that
\[
\sum_{n=4}^{\infty} \frac{x^n}{n^5} = \sum_{n=4}^{\infty} \frac{1}{n^5}
\]
which converges by the \( p \)-test, and for \( x = -1 \),
\[
\sum_{n=4}^{\infty} \frac{x^n}{n^5} = \sum_{n=4}^{\infty} \frac{(-1)^n}{n^5}
\]
which is a convergent alternating series (by the alternating series test, or by absolute convergence since \( \sum_{n=4}^{\infty} \frac{1}{n^5} \) converges by the \( p \)-test). The interval of convergence is therefore \([-1, 1]\).

15. The ratio test gives
\[
\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{|x^{n+1} (n!)^2|}{|x^n ((n+1)!)^2|} = |x| \lim_{n \to \infty} \frac{|x|^{n+1} (n!)^2}{|x^n ((n+1)!)^2|} = |x| \lim_{n \to \infty} \left( \frac{1}{n+1} \right)^2 = |x|.
\]
We have convergence for \( \rho = |x| < 1 \) when \( |x| < 1 \) and divergence for \( |x| > 1 \). Checking the endpoints, we find for \( x = 1 \) that
\[
\sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} = \sum_{n=0}^{\infty} \frac{1}{(n!)^2},
\]
which converges by the direct comparison test since \( n! \geq n \) implies that
\[
0 \leq \frac{1}{(n!)^2} \leq \frac{1}{n^2}
\]
and \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges by the \( p \)-test. For \( x = -1 \),
\[
\sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2},
\]
which is absolutely convergent by the fact that the series for $x = 1$ was also convergent.

16. The ratio test gives

$$\rho = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{|8^{n+1}x^{n+1}n!|}{|8^n x^n (n+1)!|} = 8|x| \lim_{n \to \infty} \frac{1}{n+1} = |x| \lim_{n \to \infty} \frac{1}{n+1} = 0,$$

so this series converges for all $x$.

25. The ratio test gives

$$\rho = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{|(n+1)(x-3)^{n+1}|}{|n(x-3)^n|} = \lim_{n \to \infty} \frac{(n+1)|x-3|^{n+1}}{n|x|^n} = |x-3| \lim_{n \to \infty} \frac{n+1}{n} = |x-3|.$$

We have convergence for $|x-3| < 1$ when $2 < x < 4$ and divergence for $|x-3| > 1$, that is $x < 2$ or $x > 4$. Checking the endpoints, we find for $x = 2$ that

$$\sum_{n=1}^{\infty} n(x-3)^n = \sum_{n=1}^{\infty} n(-1)^n$$

which diverges by the $n$th term test. For $x = 4$, we find

$$\sum_{n=1}^{\infty} n(x-3)^n = \sum_{n=1}^{\infty} n$$

which also diverges by the $n$th term test.

26. The ratio test gives

$$\rho = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{|(-5)^{n+1}(x-3)^{n+1}n^2|}{|(-5)^n(x-3)^n(n+1)^2|} = \lim_{n \to \infty} \frac{5^{n+1}|x-3|^{n+1}n^2}{5^n|x-3|^n(n+1)^2} = 5|x-3| \lim_{n \to \infty} \frac{n^2}{(n+1)^2} = 5|x-3|.$$

We have convergence for $|x-3| < 1$ when $-1/5 < x - 3 < 1/5$, i.e. $14/5 < x < 16/5$, and divergence for $5|x-3| > 1$, that is $x < 14/5$ or $x > 16/5$. Checking the endpoints, we find for $x = 14/5$ that

$$\sum_{n=1}^{\infty} \frac{(-5)^n(x-3)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-5x+15)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which converges by the $p$-test, and for $x = 16/5$,

$$\sum_{n=1}^{\infty} \frac{(-5)^n(x-3)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-5x+15)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2},$$
which is absolutely convergent by the previous $p$-test, so the interval of convergence is $[14/5, 16/5]$.

29. The ratio test gives

$$
\rho = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{|2^{n+1}(x + 3)^{n+1}3n|}{|2^n(x + 3)^n3(n + 1)|} = 2|x + 3| \lim_{n \to \infty} \frac{3n}{3(n + 1)} = 2|x + 3|
$$

We have convergence for $\rho = 2|x + 3| < 1$ when $-1/2 < x + 3 < 1/2$, i.e. $-7/2 < x < -5/2$, and divergence for $2|x + 3| > 1$, that is $x < -7/2$ or $x > -5/2$. Checking the endpoints, we find for $x = -7/2$ that

$$
\sum_{n=1}^{\infty} \frac{2^n}{3n} (x + 3)^n = \sum_{n=1}^{\infty} \frac{(2x + 6)^n}{3n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{3n} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n}
$$

which is the convergent alternating harmonic series. For $x = -5/2$, we have

$$
\sum_{n=1}^{\infty} \frac{2^n}{3n} (x + 3)^n = \sum_{n=1}^{\infty} \frac{(2x + 6)^n}{3n} = \sum_{n=1}^{\infty} \frac{1}{3n} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n}
$$

which is the divergent harmonic series. Therefore, the interval of convergence is $[-7/2, -5/2)$.

31. The ratio test gives

$$
\rho = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{|(-5)^{n+1}(x + 10)^{n+1}n!|}{|(-5)^n(x + 10)^n(n + 1)!|}

= \lim_{n \to \infty} \frac{5^{n+1}|x + 10|^{n+1}n!}{5^n|x + 10|^n(n + 1)!} = 5|x + 10| \lim_{n \to \infty} \frac{1}{n + 1} = 0,
$$

so this series converges for all $x$.

32. The ratio test gives

$$
\rho = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{|(n + 1)!(x + 5)^{n+1}|}{|n!(x + 5)^n|} = |x + 5| \lim_{n \to \infty} \frac{n + 1}{n}
$$

which will never converge anywhere unless $x = -5$ in which case the limit is zero. Thus, $x = 0$ is the only point of convergence.

Use equation (2)

$$
\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n \quad \text{for} \quad |x| < 1
$$

to expand the function in a power series with center $c = 0$ and determine the interval of convergence.

35. $f(x) = \frac{1}{1 - 3x}$

37. $f(x) = \frac{1}{3 - x}$

38. $f(x) = \frac{1}{4 + 3x}$

39. $f(x) = \frac{1}{1 + x^2}$
35. We substitute \( x \to 3x \) to get
\[
f(x) = \frac{1}{1-3x} = \sum_{n=0}^{\infty} (3x)^n = \sum_{n=0}^{\infty} 3^n x^n
\]
which converges for \(|3x| < 1\) or \(|x| < 1/3\) and diverges otherwise since it is a geometric series. Thus, the interval is \((-1/3, 1/3)\).

37. We write
\[
f(x) = \frac{1}{3-x} = \frac{1}{3} \left( \frac{1}{1-(x/3)} \right) = \frac{1}{3} \sum_{n=0}^{\infty} \frac{(x/3)^n}{3^n}
\]
which converges for \(|x/3| < 1\) or \(|x| < 3\) and diverges otherwise since it is a geometric series. Thus the interval is \((-3, 3)\).

38. We write
\[
f(x) = \frac{1}{4+3x} = \frac{1}{4} \left( \frac{1}{1-(3x/4)} \right) = \frac{1}{4} \sum_{n=0}^{\infty} \frac{(3x/4)^n}{3^n},
\]
which converges for \(|3x/4| < 1\) or \(|x| < 4/3\) and diverges otherwise since it is a geometric series. Thus the interval is \((-4/3, 4/3)\).

39. We write
\[
f(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n}
\]
which converges for \(|x^2| = x^2 < 1\) or \(|x| < 1\) and diverges otherwise since it is a geometric series. Thus the interval is \((-1, 1)\).

41. Use the equalities
\[
\frac{1}{1-x} = \frac{1}{3 - (x - 4)} = \frac{-1/3}{1 + \left(\frac{x-4}{3}\right)}
\]
to show that for \(|x-4| < 3\),
\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-4)^n}{3^{n+1}}
\]
We write
\[
\frac{1}{1-x} = \frac{1}{-3 - (x - 4)} = \frac{-1/3}{1 - \left(-\frac{x-4}{3}\right)} = \frac{-1}{3} \sum_{n=0}^{\infty} \left(-\frac{x-4}{3}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-4)^n}{3^{n+1}},
\]
which is valid for $|(x - 4)/3| < 1$ or $|x - 4| < 3$ by the fact that this is a geometric series. Therefore, the interval is $1 < x < 7$.

42. Use the method of Exercise 41 to expand $1/(1 - x)$ in power series with centers $c = 2$ and $c = -2$. Determine the interval of convergence.

We have for $c = 2$,
\[
\frac{1}{1 - x} = \frac{1}{-1 - (x - 2)} = -\frac{1}{1 + (x - 2)} = -\sum_{n=0}^{\infty} (-1)^n (x - 2)^n = -\sum_{n=0}^{\infty} (2 - x)^n,
\]
which is valid for $|x - 2| < 1$, or $1 < x < 3$, by the fact that this is a geometric series.

We also have for $c = -2$,
\[
\frac{1}{1 - x} = \frac{1}{3 - (x + 2)} = \frac{1}{3} \left( \frac{1}{1 - (x + 2)/3} \right) = \frac{1}{3} \sum_{n=0}^{\infty} \frac{(x + 2)^n}{3^n}
\]
which is valid for $|x + 2| < 1$, or $-3 < x < -1$, by the fact that this is a geometric series.

43. Use the method of Exercise 41 to expand $1/(4 - x)$ in a power series with center $c = 5$. Determine the interval of convergence.

We have
\[
\frac{1}{4 - x} = \frac{1}{-1 - (x - 5)} = -\frac{1}{1 + (x - 5)} = -\sum_{n=0}^{\infty} (-1)^n (x - 5)^n = -\sum_{n=0}^{\infty} (5 - x)^n,
\]
which is valid for $|x - 5| < 1$, or $4 < x < 6$, by the fact that this is a geometric series.