Suggested Homework 2

Section 7.1.
Evaluate the integral using the Integration by Parts formula with the given choice of $u$ and $dv$.

6. $\int \tan^{-1} x \, dx \quad u = \tan^{-1} x, dv = dx$

6. We have $du = \frac{1}{1+x^2} dx$ and $v = \int dv = x$, so integrating by parts gives

$$\int \tan^{-1} x \, dx = uv - \int u dv = x \tan^{-1} x - \int \frac{xdx}{1+x^2}.$$  

The substitution rule for $w = 1 + x^2$ and $dw = 2xdx$ gives

$$\int \frac{x dx}{1+x^2} = \frac{1}{2} \int \frac{dw}{w} = -\frac{1}{2} \ln |w| + C = -\frac{1}{2} \ln |x^2 + 1| + C,$$

so

$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \frac{1}{2} \ln |x^2 + 1| + C.$$  

Note we don’t really need the absolute value since $x^2 + 1 > 0$.

Evaluate using Integration by Parts.

9. $\int xe^{5x+2} dx$,  

17. $\int e^{-5x} \sin x \, dx$

21. $\int x^2 \ln x \, dx$  

24. $\int x(\ln x)^2 dx$

25. $\int \cos^{-1} x \, dx$

9. Let $u = x$ with $du = dx$ and $dv = e^{5x+2}dx$. Using the substitution rule for $w = 5x + 2$ and $dw = 5dx$,

$$v = \int dv = \int e^{5x+2} dx = \frac{1}{5} \int e^w dw = \frac{1}{5} e^w = \frac{1}{5} e^{5x+2},$$

where we leave out the constant of integration until later. Integration by parts then gives

$$\int xe^{5x+2} dx = \frac{x}{5} e^{5x+2} - \frac{1}{5} \int e^{5x+2} dx = \frac{x}{5} e^{5x+2} - \frac{1}{25} e^{5x+2} + C.$$
17. Let \( u = \sin x \) with \( du = \cos x \, dx \) and \( dv = e^{-5x} \). Note that the substitution rule gives \( v = \int e^{-5x} \, dx = (-1/5)e^{-5x} \), so integrating by parts gives

\[
\int e^{-5x} \sin x \, dx = -\frac{e^{-5x}}{5} \sin x + \frac{1}{5} \int e^{-5x} \cos x \, dx.
\]

The same approach on the new integral gives

\[
\int e^{-5x} \cos x \, dx = \frac{e^{-5x}}{5} \cos x + \frac{1}{5} \int e^{5x} \sin x \, dx.
\]

Combining the above equations, we get

\[
\int e^{-5x} \sin x \, dx = -\frac{e^{-5x}}{5} \sin x - \frac{e^{-5x}}{25} \cos x + \frac{1}{25} \int e^{-5x} \sin x \, dx,
\]

so solving for \( \int e^{-5x} \sin x \, dx \) by subtracting \( (1/25) \int e^{-5x} \sin x \, dx \) from both sides gives

\[
\left( 1 + \frac{1}{25} \right) \int e^{-5x} \sin x \, dx = -\frac{e^{-5x}}{5} \sin x - \frac{e^{-5x}}{25} \cos x + C.
\]

Therefore, dividing by \( (1 + 1/25) = 26/25 \) gives

\[
\int e^{5x} \sin x \, dx = -\frac{1}{26} \left( 5e^{-5x} \sin x - e^{-5x} \cos x \right) + C_2,
\]

where \( C_2 = (25/26)C \) is an arbitrary constant.

21. We let \( u = \ln x \) with \( du = (1/x) \, dx \) and \( dv = x^2 \) with \( v = x^3/3 \) to obtain via IBP,

\[
\int x^2 \ln x \, dx = \frac{x^3 \ln x}{3} - \int \frac{x^2}{3} \, dx = \frac{x^3 \ln x}{3} - \frac{x^3}{9} + C.
\]

24. Let \( u = (\ln x)^2 \) such that by the chain rule, \( du = 2 \ln x/x \, dx \), and \( dv = x \, dx \) such that \( v = x^2/2 \) and

\[
\int x(\ln x)^2 \, dx = \frac{x^2(\ln x)^2}{2} - \int x \ln x \, dx.
\]

The second integral we solved in class with integration by parts to get

\[
\int x \ln x \, dx = \frac{x^2 \ln x}{2} - \frac{1}{2} \int x \, dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C,
\]

so

\[
\int x(\ln x)^2 \, dx = \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4} + C_2,
\]

where \( C_2 = -C \) is an arbitrary constant.
25. We let \( u = \cos^{-1} x \) with \( du = (-1/\sqrt{1-x^2})dx \) and \( dw = dx \) with \( w = x \) such that
\[
\int \cos^{-1} x \, dx = x \cos^{-1} x + \int \frac{dx}{\sqrt{1-x^2}}.
\]
The substitution rule with \( w = 1 - x^2 \) and \( -dw = 2xdx \) gives
\[
\int \frac{xdx}{\sqrt{1-x^2}} = -\frac{1}{2} \int \frac{dw}{w^{1/2}} = -\frac{1}{2} (2w^{1/2} + C) = -w^{1/2} + C_2 = -\sqrt{1-x^2} + C_2.
\]
where \( C_2 = (-1/2)C \). Thus,
\[
\int \cos^{-1} x \, dx = x \cos^{-1} x - \sqrt{1-x^2} + C_2.
\]
Evaluate using substitution and then Integration by Parts.

36. \( \int x^3 e^{x^2} \, dx \)

36. Let \( w = x^2 \) with \( dw = 2xdx \) to get
\[
\int x^3 e^{x^2} \, dx = \frac{1}{2} \int we^w \, dw.
\]
We found this integral in class using integration by parts to be
\[
\int we^w \, dw = we^w - e^w + C = x^2 e^{x^2} - e^{x^2} + C,
\]
so
\[
\int x^3 e^{x^2} \, dx = \frac{1}{2} \left( x^2 e^{x^2} - e^{x^2} + C \right).
\]
Evaluate using Integration by Parts, substitution, or both if necessary.

41. \( \int \cos x \ln(\sin x) \, dx \)

45. \( \int \frac{\ln(\ln x) \ln x \, dx}{x} \)

46. \( \int \sin(\ln x) \, dx \)

52. \( \int_0^1 \frac{x^3}{\sqrt{9 + x^2}} \, dx \)

41. Use substitution with \( w = \sin x \) and \( dw = \cos x \, dx \) to obtain
\[
\int \cos x \ln(\sin x) \, dx = \int \ln w \, dw,
\]
which we computed in class to be given by
\[
\int \cos x \ln(\sin x) \, dx = \int \ln w \, dw = w \ln |w| - w + C = \sin x \ln |\sin x| - \sin x + C.
\]
45. Use substitution with \( w = \ln x \) and \( dw = (1/x)dx \) to obtain
\[
\int \frac{\ln(x) \ln x}{x} \, dx = \int w \ln w \, dw.
\]
Now let \( u = \ln w \) with \( du = (1/w)dw \) and \( dv = wdw \) with \( v = (w^2/2) \) such that integration by parts gives
\[
\int w \ln w \, dw = \frac{w^2 \ln w}{2} - \frac{1}{2} \int w \, dw = \frac{w^2 \ln w}{2} - \frac{w^2}{4} + C.
\]
so
\[
\int \frac{\ln(x) \ln x}{x} \, dx = \int w \ln w \, dw = \frac{w^2 \ln w}{2} - \frac{w^2}{4} + C = \frac{(\ln x)^2 \ln(\ln x)}{2} - \frac{(\ln x)^2}{4} + C,
\]
(that was a tough one).

46. Use substitution with \( w = \ln x \) and \( dw = (1/x)dx \), or \( dx = xdw = e^w \, dw \) such that
\[
\int \sin(\ln x) \, dx = \int e^w \sin w \, dw = \frac{1}{2} e^w (\sin w - \cos w) + C
\]
\[
= \frac{1}{2} e^{\ln x} [\sin(\ln x) - \cos(\ln x)] + C = \frac{1}{2} x [\sin(\ln x) - \cos(\ln x)] + C,
\]
where we found the second integral by integrating by parts as we did in class using the recursive trick.

52. We can use substitution with \( w = 9 + x^2 \) with \( dw = 2xdx \) and \( x^2 = w - 9 \) to obtain
\[
\int_0^1 \frac{x^3}{\sqrt{9+x^2}} \, dx = \frac{1}{2} \int_0^1 (w - 9) \frac{dw}{w^{1/2}} = \frac{1}{2} \int_{w(0)}^{w(1)} w^{1/2} - 9w^{-1/2} \, dw = \left( \frac{1}{3} w^{1/2} - \frac{9}{2} w^{3/2} \right)_{w(0)}^{w(1)}
\]
\[
= \left( 9(9 + x^2)^{1/2} - \frac{3}{2} (9 + x^2)^{3/2} \right)_{x^2=0}^{x^2=1} = 9(10)^{1/2} - \frac{3}{2} (10)^{3/2} - 9(9)^{1/2} + \frac{1}{3} (9)^{3/2}
\]
\[
= 18 - \frac{17}{3} \sqrt{10}.
\]
We could have also used IBP with \( u = x^2 \) and \( du = 2xdx \) along with \( dv = xdx/\sqrt{9+x^2} \) and \( v = \sqrt{9+x^2} \) (use substitution to get \( v \) ) to obtain
\[
\int_0^1 \frac{x^3}{\sqrt{9+x^2}} \, dx = x^2 \sqrt{9+x^2} \bigg|_0^1 - 2 \int_0^1 x \sqrt{9+x^2} \, dx = \left( x^2 \sqrt{9+x^2} - \frac{2}{3} (9+x^2)^{3/2} \right) \bigg|_0^1
\]
\[
= \sqrt{10} - \frac{2}{3} 10^{3/2} + \frac{2}{3} \cdot 9^{3/2} = 18 - \left( 1 - \frac{20}{3} \right) \sqrt{10} = 18 - \frac{17}{3} \sqrt{10}.
\]
Note that we got the same answer, which has to be the case!
56. If we let \( u = \tan^{-1} x \) with \( du = dx/(1 + x^2) \) and \( dv = 1 \) with \( v = 1 \), then IBP gives

\[
\int_0^1 \tan^{-1} x \, dx = x \tan^{-1} x
\]

**Section 7.2.**

Use the method for odd powers to evaluate the integral.

2. \( \int \sin^5 x \, dx \)

5. \( \int \sin^3 t \cos^3 t \, dt \)

Evaluate the integral using methods employed in Examples 4 and 5.

9. \( \int \cos^4 y \, dy \)

Evaluate the integral.

23. \( \int \cos^5 x \sin x \, dx \)

30. \( \int \sin^2 x \cos^6 x \, dx \)

35. \( \int \tan x \sec^2 x \, dx \)

39. \( \int \tan^6 x \sec^4 x \, dx \)

2. This is a sin to an odd power so we leave one sin out and let \( \sin^4 x = (1 - \cos^2 x)^2 = 1 - 2 \cos^2 x + \cos^4 x \). Then

\[
\int \sin^5 x \, dx = \int (1 - 2 \cos^2 x + \cos^4 x) \sin x \, dx = \int \sin x \, dx + 2 \int u^2 - u^4 \, du
\]

\[
= - \cos x + \frac{2u^3}{3} - \frac{u^5}{5} + C = - \cos x + \frac{2 \cos^3 x}{3} - \frac{\cos^5 x}{5} + C.
\]

where we used the usual substitution with \( u = \cos x \) and \( du = -\sin x \, dx \).

5. Leave one sin alone and write

\[
\int \sin^3 t \cos^3 t \, dt = \int \sin t(1 - \cos^2 t) \cos^3 t \, dt = \int \sin t(\cos^3 t - \cos^5 t) \, dt = \int u^5 - u^3 \, du
\]

\[
= \frac{u^6}{6} - \frac{u^4}{4} + C = \frac{\cos^6 t}{6} - \frac{\cos^4 t}{4} + C.
\]

9. We have

\[
\int \cos^4 x \, dx = \int \left( \frac{1 + \cos 2x}{2} \right)^2 \, dx = \frac{1}{4} \int (1 + 2 \cos 2x + \cos 2x) \, dx
\]

\[
= \frac{1}{4} \int (1 + 2 \cos 2x) \, dx + \frac{1}{4} \int \frac{1 + \cos 4x}{2} \, dx
\]

\[
= \frac{3x}{8} + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C.
\]
23. Since cosine appears to an odd power, let \( \cos^4 x = (1 - \sin^2 x)^2 \) such that

\[
\int \cos^5 x \sin x \, dx = \int (\sin x - 2 \sin^3 x + \sin^5 x) \cos x \, dx = \int u - 2u^3 + u^5 \, du
\]

\[
= \frac{u^2}{2} - \frac{u^4}{2} + \frac{u^6}{6} + C = \frac{\sin^2 x}{2} - \frac{\sin^4 x}{2} + \frac{\sin^6 x}{6} + C.
\]

where we used \( u = \sin x \) and \( du = \cos x \, dx \).

30. Use the power reduction formulas to get

\[
\int \sin^2 x \cos^6 x \, dx = \int \left(\frac{1 - \cos 2x}{2}\right) \left(\frac{1 + \cos 2x}{2}\right)^3 \, dx
\]

\[
= \int \left(\frac{1}{16} + \frac{1}{8} \cos 2x - \frac{1}{8} \cos^3 2x - \frac{1}{16} \cos^4 2x\right) \, dx
\]

\[
= \frac{x}{16} + \frac{1}{16} \sin 2x - \frac{1}{8} \int \cos^3 2x \, dx - \frac{1}{16} \int \cos^4 2x \, dx.
\]

Next, we have

\[
\int \cos^3 2x \, dx = \int (1 - \sin^2 2x) \cos 2x \, dx = \frac{1}{2} \int (1 - u^2) \, du
\]

\[
= \frac{u}{2} - \frac{u^3}{6} + C_1 = \frac{1}{2} \sin 2x - \frac{\sin^3 2x}{6} + C_1.
\]

where we used \( u = \sin 2x \) and \( du = 2 \cos 2x \, dx \), and

\[
\int \cos^4 2x \, dx = \int \left(\frac{1 + \cos 4x}{2}\right)^2 \, dx = \int \frac{1}{4} + \frac{1}{2} \cos 4x + \frac{1}{4} \cos^2 4x \, dx
\]

\[
= \frac{3x}{8} + \frac{1}{8} \sin 4x + \frac{1}{64} \sin 8x + C_2,
\]

where we used

\[
\int \cos^2 4x \, dx = \int \left(\frac{1 + \cos 8x}{2}\right) \, dx = \frac{x}{2} + \frac{1}{16} \sin 8x + C_2.
\]

Combining everything, we have

\[
\int \sin^2 x \cos^6 x \, dx
\]

\[
= \frac{5x}{128} + \frac{1}{64} \sin 2x - \frac{1}{128} \sin 4x - \frac{1}{192} \sin 6x - \frac{1}{1024} \sin 8x + C
\]

where \( C \) is a combined constant.

35. If we let \( u = \sec x \) with \( du = \sec x \tan x \, dx \), we obtain

\[
\int \tan x \sec^2 x \, dx = \int \sec x (\tan x \sec x) \, dx = \int u \, du = \frac{u^2}{2} + C = \frac{\sec^2 x}{2} + C.
\]
39. Let $\sec^2 x = 1 + \tan^2 x$ such that

$$
\int \tan^6 x \sec^4 x \, dx = \int \tan^6 x (1 + \tan^2 x) \sec^2 x \, dx = \int (\tan^6 x + \tan^8 x) \sec^2 x \, dx
$$

$$
= \int u^6 + u^8 \, du = \frac{u^7}{7} + \frac{u^9}{9} + C = \frac{\tan^7 x}{7} + \frac{\tan^9 x}{9} + C.
$$

Section 7.3.
Evaluate using trigonometric substitution. Refer to the table of trigonometric integrals as necessary.

17. $\int \frac{dx}{x\sqrt{x^2 + 16}}$  
19. $\int \frac{dx}{\sqrt{x^2 - 9}}$  
22. $\int \frac{dx}{\sqrt{9 - x^2}}$  
28. $\int \frac{dx}{(x^2 - 4)^2}$

Evaluate the integral by completing the square and using trigonometric substitution.

38. $\int \frac{dx}{\sqrt{2 + x - x^2}}$

17. Let $x = 4 \tan \theta$ such that $\sqrt{x^2 + 16} = 4 \sec \theta$ and $dx = 4 \sec^2 \theta \, d\theta$. Then,

$$
\int \frac{dx}{x\sqrt{x^2 + 16}} = \int \frac{4 \sec \theta \, d\theta}{16 \tan \theta} = \frac{1}{4} \int \csc \theta \, d\theta = -\frac{1}{4} \ln |\csc \theta + \cot \theta| + C
$$

$$
= -\frac{1}{4} \ln \left| \csc \left( \sin^{-1} \frac{x}{4} \right) + \cot \left( \sin^{-1} \frac{x}{4} \right) \right| + C.
$$

Note that

$$
\csc \left( \sin^{-1} \frac{x}{4} \right) = \frac{1}{\sin \left( \sin^{-1} \frac{x}{4} \right)} = \frac{1}{\frac{x}{4}} = \frac{4}{x},
$$

and

$$
\cot \left( \sin^{-1} \frac{x}{4} \right) = \frac{1}{\tan \left( \sin^{-1} \frac{x}{4} \right)} = \frac{1}{\sqrt{x^2 + 16/x}} = \frac{x}{\sqrt{x^2 + 16}},
$$

(draw out the triangle for this one) so

$$
\int \frac{dx}{x\sqrt{x^2 + 16}} = -\frac{1}{4} \ln \left| \frac{4 + \sqrt{x^2 + 16}}{x} \right| + C
$$

19. Let $x = 3 \sec \theta$ with $dx = 3 \sec \theta \tan \theta \, d\theta$ and $\sqrt{x^2 - 9} = 3 \tan \theta$ and

$$
\int \frac{dx}{\sqrt{x^2 - 9}} = \int \frac{3 \sec \theta \tan \theta \, d\theta}{3 \tan \theta} = \int \sec \theta \, d\theta = \ln |\sec \theta + \tan \theta| + C.
$$
Next, we have

$$\sec \theta = \sec \sec^{-1} \frac{x}{3} = \frac{x}{3}$$

and

$$\tan \theta = \tan \sec^{-1} \frac{x}{3} = \frac{\sqrt{x^2 - 9}}{3},$$

(draw out the triangle again) so

$$\int \frac{dx}{\sqrt{x^2 - 9}} = \ln \left| \frac{x + \sqrt{x^2 - 9}}{3} \right| + C.$$

22. Let $x = 3 \sin \theta$ with $dx = 3 \cos \theta d\theta$ and $\sqrt{9 - x^2} = 3 \cos \theta$ such that

$$\int x^3 \sqrt{9 - x^2} dx = 243 \int \sin^3 \theta \cos^2 \theta d\theta.$$

Now we use $\sin^2 \theta = 1 - \cos^2 \theta$ such that

$$\int \sin^3 \theta \cos^2 \theta d\theta = \int (\cos^2 \theta - \cos^4 \theta) \sin \theta d\theta = \int u^2 - u^4 du = \frac{u^3}{3} - \frac{u^5}{5} + C$$

Next,

$$\cos \theta = \cos \sin^{-1} \frac{x}{3} = \frac{\sqrt{9 - x^2}}{3},$$

so

$$\int x^3 \sqrt{9 - x^2} dx = \frac{243}{3} \left( \frac{\sqrt{9 - x^2}}{3} \right)^3 - \frac{243}{5} \left( \frac{\sqrt{9 - x^2}}{3} \right)^5 + C.$$

28. We let $x = 2 \sin \theta$ with $dx = 2 \cos \theta d\theta$ and $(x^2 - 4)^2 = (4 - x^2)^2 = 16 \cos^4 \theta$

$$\int \frac{dx}{(x^2 - 4)^2} = \int \frac{dx}{(4 - x^2)^2} = \int \frac{d\theta}{8 \cos^3 \theta} = \frac{1}{8} \int \sec^3 \theta d\theta$$

$$= \frac{1}{16} \sec \theta \tan \theta + \frac{1}{16} \ln |\sec \theta + \tan \theta| + C.$$

(we can use integration by parts to evaluate this integral, as we saw in class). Note that

$$\tan \theta = \tan \sin^{-1} \frac{x}{2} = \frac{x}{\sqrt{4 - x^2}},$$
and
\[
\sec \theta = \sec \sin^{-1} \frac{x}{2} = \frac{1}{\cos \sin^{-1}(x/2)} = \frac{2}{\sqrt{4 - x^2}},
\]
so
\[
\int \frac{dx}{(x^2 - 4)^2} = \frac{x}{8(4 - x^2)} + \frac{1}{16} \ln \left| \frac{x + 2}{\sqrt{4 - x^2}} \right| + C.
\]

38. We complete the square with \(-x^2 + x + 2 = -(x - \frac{1}{2})^2 + \frac{9}{4}\), so let \(u = x - 1/2\) with \(du = dx\) and \(a^2 = 9/4\).

\[
\int \frac{dx}{\sqrt{2 + x - x^2}} = \int \frac{du}{\sqrt{a^2 - u^2}}.
\]

As in the first example in the book section 7.3, we let \(u = a \sin \theta\) to obtain
\[
\int \frac{dx}{\sqrt{2 + x - x^2}} = \int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + C = \sin^{-1} \left( \frac{2x - 1}{3} \right) + C.
\]