

Outline of the Steepest Descent Algorithm

To use the SDM to solve the linear system $\mathbf{Ax} = \mathbf{b}$, where \mathbf{A} is an $n \times n$ symmetric positive definite matrix, do the following:

- 1) pick \mathbf{x}_1 and set $\mathbf{r}_1 = \mathbf{b} - \mathbf{Ax}_1$
- 2) for $k = 1, 2, 3, \dots$

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{r}_k$$

$$\text{where } \alpha_k = \frac{\mathbf{r}_k^T \mathbf{r}_k}{\mathbf{r}_k^T \mathbf{q}_k} \text{ for } \mathbf{q}_k = \mathbf{Ar}_k$$

$$\mathbf{r}_{k+1} = \mathbf{r}_k - \alpha_k \mathbf{q}_k$$

This algorithm requires the storage of four vectors (\mathbf{x} , \mathbf{r} , \mathbf{d} , \mathbf{q}) as well as the nonzero components of \mathbf{A} . The stopping criterion for the algorithm can require that one of the following is "small":

$$\|\mathbf{x}_k - \mathbf{x}_{k-1}\|, \quad \frac{\|\mathbf{x}_k - \mathbf{x}_{k-1}\|}{\|\mathbf{x}_k\|}, \quad \frac{\|\mathbf{x}_k - \mathbf{x}_{k-1}\|_A}{\|\mathbf{x}_k\|_A}$$
$$\|\mathbf{r}_k\|, \quad \frac{\|\mathbf{r}_k\|}{\|\mathbf{r}_1\|}, \quad \|\mathbf{x}_k - \mathbf{x}_{k-1}\|_A.$$

Note that the first three add $O(n)$ operations to each iteration (unlike the last three which add only a couple of operations to each iteration...assuming the Euclidean norm is used with the residuals).