Outline of the Conjugate Gradient Algorithm

To use the CGM to solve the linear system $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is an nxn symmetric positive definite matrix, do the following:

1) pick $\mathbf{x}_1$ and set $\mathbf{r}_1 = \mathbf{b} - \mathbf{A}\mathbf{x}_1$

2) for k = 1, 2, 3, ...

$$\mathbf{d}_k = \mathbf{r}_k + \beta_k \mathbf{d}_{k-1}$$

where $\beta_1 = 0$ otherwise $\beta_k = \dfrac{\mathbf{r}_k \cdot \mathbf{r}_k}{\mathbf{r}_{k-1} \cdot \mathbf{r}_{k-1}}$

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k$$

where $\alpha_k = \dfrac{\mathbf{r}_k \cdot \mathbf{r}_k}{\mathbf{d}_k \cdot \mathbf{q}_k}$ for $\mathbf{q}_k = \mathbf{A}\mathbf{d}_k$

$$\mathbf{r}_{k+1} = \mathbf{r}_k - \alpha_k \mathbf{q}_k$$

This algorithm requires the storage of four vectors ($\mathbf{x}$, $\mathbf{r}$, $\mathbf{d}$, $\mathbf{q}$) as well as the nonzero components of $\mathbf{A}$. The stopping criterion for the algorithm can require that one of the following is "small":

$$\|\mathbf{x}_k - \mathbf{x}_{k-1}\| , \quad \frac{\|\mathbf{x}_k - \mathbf{x}_{k-1}\|}{\|\mathbf{x}_k\|} , \quad \frac{\|\mathbf{x}_k - \mathbf{x}_{k-1}\|_A}{\|\mathbf{x}_k\|_A}$$

$$\|\mathbf{r}_k\| , \quad \frac{\|\mathbf{r}_k\|}{\|\mathbf{r}_1\|} , \quad \|\mathbf{x}_k - \mathbf{x}_{k-1}\|_A .$$

Note that the first three add O(n) operations to each iteration (unlike the last three which add only a couple of operations to each iteration...assuming the Euclidean norm is used with the residuals).