1) \( u_x + v_y = 0 \)

\[ u_t + uu_x + vv_y = -\frac{1}{\rho} p_x + \sqrt{u_{xx} + u_{yy}} \]

\[ v_t + uv_x + vv_y = -\frac{1}{\rho} p_y + \sqrt{v_{xx} + v_{yy}} \]

a) \( u_x = v_x = 0 \) \( \Rightarrow \) \( v_y = 0 \) from continuity eqn.

Also, \( v = 0 \) from B.C.s at \( y = 0, h \)

\[ y \text{- mom. } \Rightarrow \quad p_y = 0 \quad \Rightarrow \quad p = p(x, t) \text{ at wall} \]

\[ x \text{- mom. } \Rightarrow \quad u_t = -\frac{1}{\rho} p_x + \sqrt{v_{yy}} \]

Since \( u_t \) and \( v_{yy} \) depend on \( (y, t) \) at most implies that \( p_x \) is a fct. of \( t \) at most.

b) (i) Steady with \( p_x = 0 \) \( \Rightarrow \) \( v_{yy} = 0 \)

\[ \text{Soln } \Rightarrow \quad u = \mathcal{U} \left( \frac{y}{h} \right) \]

(ii) Unsteady with \( p_x = G = \text{const.} \)

\[ u_t = v v_{yy} - \frac{1}{\rho} G \]
with \( u = 0 \) at \( y = 0, h \)

\[ u = 0 \] at \( t = 0 \)

**Soln.**

\[ u(y, t) = \hat{u}(y) + \tilde{u}(y, t) \]

*steady* \begin{align*}
\text{deviation from} \\
\text{steady}
\end{align*}

**Steady soln**

\[ \hat{u} = -\frac{G}{2\mu} y (h - y), \quad \mu = \rho \nu \]

**Deviation from steady soln**

\[ \tilde{u}_t = \nu \tilde{u}_{yy}, \quad 0 < y < h, \quad t > 0 \]

\[ \tilde{u} = 0 \] at \( y = 0, h \)

\[ \tilde{u} = -\hat{u} \] at \( t = 0 \)

**Separation of variables** \( \Rightarrow \)

\[ \tilde{u} = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{h}\right)^2 \nu t} \sin \left(\frac{n\pi y}{h}\right) \]

\[ A_n = \frac{2}{h} \int_0^h (-\hat{u}(y)) \sin \left(\frac{n\pi y}{h}\right) dy \]

\[ = \frac{2Gh^2(1 - \cos(n\pi))}{\mu} \left(\frac{1}{n^2 \pi^2}\right) \]
c) \[ u_t = v u_{yy} \Rightarrow u = t^\alpha f(y), \quad y = \frac{\gamma}{\sqrt{vt}} \]

\[ \Rightarrow \alpha f - \frac{1}{2} \gamma f' = f'' \]

i) \[ u = V \text{ at } \gamma = 0 \]
\[ \alpha = 0, \quad f(0) = V \]
\[ u = 0 \text{ at } t = 0, \quad f(\infty) = 0 \]

\[ \text{soln} \Rightarrow f(y) = V \sqrt{1 - \text{erf}\left(\frac{y}{\sqrt{2}}\right)} \]

ii) \[ u_y = \frac{\gamma}{\mu} \text{ at } \gamma = 0 \]
\[ \alpha = \frac{1}{2}, \quad f'(-1) = \frac{\gamma}{\mu} \sqrt{\nu} \]
\[ u = 0 \text{ at } t = 0, \quad f(\infty) = 0 \]

\[ \text{soln} \Rightarrow f(y) = \frac{\gamma}{\mu} \sqrt{\nu} \left[ y (1 - \text{erf}\left(\frac{y}{\sqrt{2}}\right)) + \frac{2}{\sqrt{\pi}} e^{-\gamma^2/4} \right] \]

2) \[ \frac{\partial}{\partial t} \left( \rho v \right) + \frac{\partial}{\partial x} \left( \rho v^2 + \rho a^2 \right) = 0 \]
a) Quasi-linear form: \[ \frac{\partial \xi}{\partial t} + A \frac{\partial \eta}{\partial x} = 0 \]
\[ \xi = \int p \, dq \quad A = \begin{bmatrix} v \\ \frac{a^2}{p} \\ v \end{bmatrix} \]

b) Characteristic form
\[ \frac{d}{dt} (pa^2) = pa \quad \frac{dv}{dt} = v + a \]
\[ v \text{ same as for full Euler} \]
\[ \text{since } pa^2 = q \text{ for isothermal} \]

3) Have
\[ u_{xx} + u_{yy} = \left( x \frac{\partial^2}{\partial x^2} + y \frac{\partial^2}{\partial y^2} \right) (xu_x + yu_y) \]

Standard form
\[ A u_{xx} + 2B u_{xy} + C u_{yy} = 0 \]
\[ \Rightarrow A = 1 - x^2, \quad B = -xy, \quad C = 1 - y^2 \]
\[ B^2 - AC = x^2 + y^2 - 1 > 0 \]

Hyperbolic region is outside the unit circle.
5) Characteristic \( A\partial_y \partial_x + C \partial_x \partial_y = 0 \)

use polar coords \( x = r \cos \theta, \quad y = r \sin \theta \)

\[ r^2 (r^2 - 1) \partial \theta^2 = \partial r^2 \]

Integrate

\[ \frac{1}{r} \cos (\theta - \theta_0) = 1, \quad \theta_0 = \text{const} \]

family of lines tangent to the unit circle

4)

\[ \frac{\partial}{\partial t} \left[ \frac{1}{h} \right] + \frac{\partial}{\partial x} \left[ \frac{hv}{hv^2 + \frac{1}{2}gh^2} \right] = 0 \]

a) Receding barrier

char. form \( \Rightarrow \)

\[ \frac{dv}{dt} + \sqrt{\frac{g}{h}} \frac{dh}{dt} = 0 \quad \Rightarrow \quad \frac{dt}{ \frac{dx}{v + \sqrt{gh}}} = \frac{1}{v + \sqrt{gh}} \]

\[ v \div 2 \sqrt{gh} = \text{const} \]
Note: \( C_{-} \Rightarrow v - 2\sqrt{gh} = -2\sqrt{gh_0} \)

So,
\[
v_{1} = V_{b}, \quad \sqrt{gh_{1}} = \sqrt{gh_{0}} + \frac{1}{2} V_{b}
\]

\[\frac{x}{t} = v - \sqrt{gh} = \frac{3}{2} v + \sqrt{gh_{0}}\]

\[v = \frac{2}{3} \left( \frac{x}{t} - \sqrt{gh_{0}} \right) \frac{\sqrt{gh_{0}}}{\frac{x}{t} - \sqrt{gh_{0}} + \frac{3}{2} V_{b}}\]
\[\sqrt{gh} = \sqrt{gh_{0}} + \frac{1}{2} v\]

b) Compressive barrier.

Jump conditions
\[
[h\omega] = 0
\]
\[
[h\omega^{2} + \frac{1}{2} gh^{2}] = 0
\]

\[w < h, \quad r = v - \sqrt{h}\]
Eliminate \( b \), to give

\[
\frac{\omega_1}{\sqrt{g\rho_0}} = \frac{1}{2} \frac{\sqrt{\omega_0}}{\omega_0} \left( 1 - \sqrt{1 + \frac{8\omega_0^2}{g\rho_0}} \right) \tag{*}
\]

Here \( \omega_0 = -V_s \) and \( \omega_1 = V_b - V_s < 0 \)

Solve (\text{\textasteriskcentered}) for \( V_s \) in terms of \( V_b \) and \( g\rho_0 \).

Then

\[
\frac{b_1}{\omega_0} = -1 + \sqrt{1 + \frac{8\omega_0^2}{g\rho_0}}, \quad V_s = V_b
\]

c) The solution of the drop break problem is given by (a) with \( V_b \) so \( b_1 = 0 \), i.e.

\[
V_b = -2\sqrt{g\rho_0}
\]
More or part (b): Jump conditions are

\[ h, \omega, = h, \omega, \]

\[ h, \omega, + \frac{1}{2} g h, ^2 = h, \omega, + \frac{1}{2} g h, ^2 \]

with \( \omega, = V_0 - V_s = - V_s \), \( \omega, = V - V_s \)

Define:

\[ U_s = \frac{V_s}{\sqrt{\omega,}} \]
\[ U_s = \frac{V_s}{\sqrt{\omega,}} \]
\[ z = \frac{h,}{\omega,} \]

\[ \Rightarrow 2, (U_s - U_s) = - U_s \]

\[ 2, (U_s - U_s) ^2 + \frac{1}{2} z, ^2 = U_s, ^2 + \frac{1}{2} \]

\[ (r) \Rightarrow \]

\[ U_s = \frac{2,}{z, - 1} U_i \]

Eliminate \( U_s \) from (2) \Rightarrow

\[ 2, U_i, ^2 = \frac{1}{2} (z, - 1) ^2 (z, + 1) \]

Note that \( U_i \) is known in terms of \( V_i \) and \( g \omega, \) and so (4) is a sign for \( z, \Rightarrow h, \)
There are 2 roots of (***) with \( z, \bar{z} \), one with \( \delta < z, \bar{z} \) and the other \( z, \bar{z} \).

Evidently, the root we seek is the one for which \( \frac{z}{\bar{z}} > 1 \).