Advances and applications of lattice 
supersymmetry

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Several motivations exist for efforts to formulate supersymmetric field theories on a lattice. It is difficult to formulate these theories in a way that avoids fine-tuning of counterterms (besides any bare/physical tuning that already occurs in the continuum). Nevertheless, there have been many promising formulations developed of late. But to what use should/would we put them? Look to motivations.
Say what is meant by

\[ Z[S] = \int \exp(-S). \]

At a formal level we know

\[ Z[S; a] = \int_{D(a)} [d\mu(\phi; a)] \exp(-S[\phi; a]) \]

and \( a \to 0 \) limiting sequence for ingredients.
Definition

- Due to universality, there is no unique definition of $Z[S; a]$.

- But one wants a classification based on the answer to:
  - Perturbative defn.: no CT / local CTs / anomaly
  - Nonperturb. defn.: no CT / local CTs / anomaly

- Favorite answer: no CT.
Q-exact susy-QM:

\[ S = QX, \quad Q^2 = 0. \]

Monte-Carlo evidence (spectrum degeneracy, Ward identities) [Catterall, Gregory 00].

All-order perturbative proof [JG, Poppitz 04].

Nonperturbative proof by transfer matrix calculation [JG, Poppitz 04].

Note: Q-exact → cancellations → no CT.
Spectrum degeneracy in Q-exact SQM. Taken from [Catterall, Gregory 00]. $L = Na = \text{fixed.}$
No CT examples

- **Q-exact 2d Wess-Zumino** (a.k.a. $\mathcal{N} = 2$ Landau-Ginsburg) [Elitzur, Schwimmer 83] [Sakai, Sakamoto 83]:
  - All orders perturbative proof [JG, Poppitz 04].
  - Similar to $Q, Q^\dagger$-preserving spatial [Elitzur, Schwimmer 83].
- Note: Q-exact $\rightarrow$ cancellations $\rightarrow$ no CT.
- Various Monte Carlo evidence: (spectrum degeneracy, Ward identities, R-symmetry) [Beccaria, Curci, D’Ambrosio 98] [Catterall, Karamov 01-02] [JG 05]
$D_{UV} \geq 0$ diagrams
From [Curci, Veneziano 86] we know that \( \mathcal{N} = 1 \) 4d SYM with Ginsparg-Wilson fermions require no CTs.

Overlap-Dirac was proposed [Narayanan, Neuberger 95] and sketched [Maru, Nishimura 97].

But LO simulation studies, such as glueball spectra, have yet to be attempted.
$\mathcal{N} = 1$ 4d SYM w/ chiral fermions

- IR effective theories have been proposed by continuum theorists [Veneziano-Yankielowicz and extensions by Sannino et al., Gabadaze et al., Louis et al.]; it would be nice to test them!

- Domain-wall-Dirac fermions were proposed [Nishimura 97] [Neuberger 98] [Kaplan, Schmaltz 99] and briefly studied [Fleming, Kogut, Vranas 00].

- It deserves another push!
No CT examples

- Deconstruction models (quiver lattice, orbifold matrix model), various extended SYM in 2d, 3d and 4d [Cohen, Kaplan, Katz, Unsal 03].
  - The 2d examples definitely have no fine-tuning in perturbation theory.
  - For the $d > 2$ examples, it is less clear what really happens.
  - If $O(d)$ is recovered, an interesting susy theory is obtained.

- Q-exact compact (2,2) SYM in 2d [Sugino 03-06]. No fine-tuning in perturbation theory.
Extended SYM in 2d & 4d [D’Adda, Kanamori, Kawamoto, Nagata 05].

Modified Leibnitz rule for susy variation:

\[ s_A[\Phi(x_1)P(x_2, \ldots)] = s_A[\Phi(x_1)]P(x_2, \ldots) \]
\[ + (-)^F(\Phi)\Phi(x_1 + a_A)s_A[P_2(x_2, \ldots)]. \]

Action with 4 equivalent forms:

\[ S' = \sum_x \text{Tr} \ s\tilde{s}s_1s_2\Psi_{x,x} = -\sum_x \text{Tr} \ \tilde{s}s_1s_2\Psi_{x,x} \]
\[ = \sum_x \text{Tr} \ s_1s_2\tilde{s}\Psi_{x,x} = -\sum_x \text{Tr} \ s_2s_1\tilde{s}\Psi_{x,x}. \]
Nilpotency:

\[ s^2 = \tilde{s}^2 = s_1^2 = s_2^2 = 0. \]

Thus action invariant under noncommutative (2,2) susy.

Renormalization needs more study. Does noncommutativity matter?
Other no CT examples/claims

- Twisted (Q-exact) geometrical (2,2) SYM in 2d [Catterall 04-06]. MC data seems to indicate no need for CTs.

- 4d WZ with GW fermions [Bonini, Feo 04-05]. Nonlinear, perturbative definition of $Q$.

- Twisted (Q-exact) nonlinear $\sigma$ model, (2,2), 2d [Catterall, Ghadab 03] [JG, Poppitz 04]. MC data seems to indicate no need for CTs [Catterall, Ghadab 06].
Nonholomorphic woes

- Continuum susy tricks usually partly fail to determine IR eff. theory.

- Schematically:

\[
\int d^4 \theta \ K(\Phi, \bar{\Phi}) + \left[ \int d^2 \theta \ W(\Phi) + h.c. \right]
\]

- \( W \) not renormalized in perturbation theory.

- \( W \) sometimes completely determined, once symmetries accounted for.

- Generically none of these nice features hold for \( K \).
Lack of control over nonholomorphic data is distressing.

So-called supersymmetry-breaking soft-terms largely determine superpartner spectra and couplings for the MSSM.

Depend on Kähler potential $K$. 
Lattice Monte Carlo simulations would, as a first step, give us a handle on vevs $\phi_0$ of scalars, and the spectrum of light states.

This constrains:

$$V_\phi, \quad V_{\phi\phi}, \quad V_{\phi\bar{\phi}}$$

evaluated at $\phi_0$. 

Both the Kähler potential $K$ and superpotential $W$ play a role in the scalar potential:

$$V = K^{k\ell} W_k \bar{W}_\ell,$$

where $K^{k\ell}$ is the inverse of the Kähler metric

$$K_{k\ell} = \partial^2 K / \partial \phi^k \partial \bar{\phi}^\ell$$

and

$$W_k = \partial W / \partial \phi_k, \quad \bar{W}_\ell = (W_\ell)^*.$$
Extracting effective $K$

- Hypothesize effective Kähler potential $K$.
- Use known effective superpotential $W$.
- Match microscopic lattice and IR effective lattice data, to fit “phenomenological” constants in $K$. 
These proposals illustrate how lattice simulations have the potential to teach us something about nonperturbative renormalization of nonholomorphic quantities.

Work in progress (w/ Catterall): Determine (2,2) 2d twisted NL$\sigma$M IR effective theory that can reproduce the following (microscopic) constrained effective potentials (of (2,2) 2d SU(2) SYM)...
Constrained effective potential

Potential for $|\phi|$ in $(2,2) \ SU(2)$ SYM, using Catterall’s construction and simulation code.
Constrained effective potential

Potential for $|\text{Tr } \phi^2|$ in $(2,2)$ $SU(2)$ SYM, using Catterall’s construction and simulation code.
Susy breaking

- Third motivation: improve our understanding of **dynamical supersymmetry breaking**.
- Strong susy dynamics often invoked in models of soft susy breaking for **MSSM**.
- Any improvement of our understanding would be helpful.
A simple theory where ground state susy is not yet fully understood.

3d reduction of 4d $\mathcal{N} = 1$ SYM.

Unanswered questions arise in [Affleck, Harvey, Witten 82].

Instanton-generated potential for modulus field $\phi$ (nonpert. renormalization), but uncertainty for small $\phi$.

Only known susy vac. is at $\phi \to \infty$.

Is there a susy vacuum near origin?
Use parity-preserving overlap-Dirac formulation, similar to 3d $\mathcal{N} = 1$ of [Maru, Nishimura 97]: no CT fine-tuning?

Simple content (1 gluon, 1 adj. Maj. fermion, 1 adj. real scalar) $\Rightarrow$ efficient.

Addition of matter leads to very interesting vac. dynamics, even for the $U(1)$ gauge theory [Aharony, Hanany, Intriligator, Seiberg, Strassler 97] [de Boer, Hori, Oz 97]
Quantum gravity

- Evolving understanding of relationship between SYM and string/M-theory.
- Nonperturbative formulation of string/M-theory in general backgrounds is still lacking.
- But there are recent successes for special backgrounds:
Nonperturbative string theory

- D-branes [Polchinski 95].
- M(atrix) theory [Banks, Fischler, Shenker, Susskind 96] [Ishibashi, Kawai, Kitazawa, Tsuchiya 96]
- AdS/CFT correspondence [Maldacena 97] [Gubser, Klebanov, Polyakov 98] [Witten 98]
- PP-wave limit [Metsaev 01] [Metsaev, Tseytlin 02] [Blau, Figueroa-O’Farrill, Hull, Papadopoulos 01] [Berenstein, Maldacena, Nastase 02]
Nonperturbative string theory

- Note that I include nontrivial semiclassical under what I call "nonperturbative."
- It is of considerable interest to study these nonpertubative formulations in relation to SYM on the lattice.
- E.g., Matrix theory and AdS/CFT expressed in terms of dimensionally reduced SYM.
The nontrivial SYM vacuum dynamics takes on a gravitational meaning.

Everybody has heard about $AdS_5 \times S^5$.

A far more interesting example is the Klebanov-Strassler construction [00].

It is based on the Klebanov-Witten construction:

$$AdS_5 \times T^{1,1}.$$ 

KW dual gauge theory: $\mathcal{N} = 1$ SCFT.
Conifold

- Type IIB theory is formulated on:

  4d Minkowski \( \times \) conifold.

- The conifold has a cone-like geometry, with \( T^{1,1} \) base:
$T^{1,1}$ is a quotient manifold:

$$T^{1,1} = [SU(2) \times SU(2)]/U(1)$$

The “1, 1” denotes the $U(1)$ quotient:

$$H = (\sigma_3 \otimes 1) + (1 \otimes \sigma_3).$$
Warping spacetime

- A stack of $N_c$ D3 branes is placed at the tip, where the size of the base shrinks to zero.
- These branes are gravitating. They backreact on the geometry, warping it.
- Not too far from the branes, the geometry is $AdS_5 \times T^{1,1}$. 
One advantage: Type IIB SUGRA on $AdS_5 \times T^{1,1}$ only preserves 8 Killing spinors, whereas

Type IIB on $AdS_5 \times S^5$ preserves 32 Killing spinors.

In the dual gauge theory, we get $\mathcal{N} = 1$ SCFT rather than $\mathcal{N} = 4$ SCFT.
An $\mathcal{N} = 1$ SCFT is a much more promising start, being closer to the real world.

There is a singularity at the tip of the conifold.

In the dual gauge theory, this is reflected by the absence of an IR cutoff.
Klebanov-Strassler resolve the singularity with the deformed conifold.

They show that it is equivalent to confinement in the IR of the dual gauge theory.
Conformal symmetry breaking

- On the gravity side of the duality, this breaks half the Killing spinors.
- On the gauge theory side, this breaks the fermionic conformal charges, reducing to $\mathcal{N} = 1$ SUSY gauge theory.
- Now we are “very close” to the real world.
- I.e., semi-realistic susy $AdS_5$ models.
D7 probes = flavors

- Introduction of $N_f$ D7 probe branes allows for a weakly coupled $U(N_f)$ gauge theory in the dual.

\[ g_f^2 \sim (\text{volume})^{-1}. \]

- Embedding for D7s can generate bare masses for “quarks” of dual gauge theory.

- Low energy partons are really bound states, very much like in technicolor.

- Here, the $U(N_c)$ associated with the D3 branes is the technicolor group.
Recently, studies of the low energy effective 4d and 5d theories have been conducted. [Sakai, Sonnenschein 03] [Kuperstein 04] [Levi, Ouyang 05] [Gherghetta, JG 06].

In the latter work, we also imagined a regulator in the UV, following [Giddings, Kachru, Polchinski 02].

This occurs by capping off the conifold with a compact Calabi-Yau manifold far from the tip.
The CY and the KS tip are, respectively, to be thought of as refinements of the UV and IR branes of, say, Randall-Sundrum 1.

Potential—though challenging—interplay between warped extra dimension models, AdS/CFT, and lattice SYM.
There are many more generalizations of AdS/CFT.

In particular, $AdS_3/CFT_2$ may be accessible through lattice studies.

The duality in the (4,4) 2d SQCD (D1-D5 system) is under study using a deconstruction lattice susy construction with matter [JG 06].
The (4,4) lattice with matter is a generalization of the recent (2,2) SQCD constructions of Kaplan (see talk) and Endres [06].

I now have a construction with 2 exact supercharges and only site/link/diagonal fields.
Conclusions

- Although it is challenging to write down supersymmetric lattice field theories that have a good quantum continuum limit, some examples do exist.

- A wealth of exciting applications await.

- At least for lower dimensional examples, certain nonperturbative features can be studied on the lattice with results that are of broad and of current interest.