MANE 4240 & CIVL 4240
Introduction to Finite Elements

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Numerical Integration in 2D

Reading assignment:
Lecture notes, Logan 10.4

Summary:
• Gauss integration on a 2D square domain
• Integration on a triangular domain
• Recommended order of integration
• "Reduced" vs "Full" integration; concept of "spurious" zero energy modes/ "hour-glass" modes

1D quadrature rule recap

\[ I = \int f(\xi) d\xi = \sum_{i=1}^{M} W_i f(\xi_i) \]

Weight Integration point

Choose the integration points and weights to maximize accuracy

<table>
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<tr>
<th>Newton-Cotes</th>
<th>Gauss quadrature</th>
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<tr>
<td>1. ‘M’ integration points are necessary to exactly integrate a polynomial of degree ‘M-1’</td>
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<td>2. More expensive</td>
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<td>1. ‘M’ integration points are necessary to exactly integrate a polynomial of degree ‘2M-1’</td>
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| 3. Exponential convergence, error proportional to \( \left( \frac{1}{2M} \right)^{2M} \)

Example

A 2-point Gauss quadrature rule

\[ \int f(\xi) d\xi \approx f\left( \frac{1}{\sqrt{3}} \right) + f\left( -\frac{1}{\sqrt{3}} \right) \]

is exact for a polynomial of degree 3 or less.
2D square domain
\[ \left( -\frac{1}{2}, \frac{1}{2} \right) \times \left( -\frac{1}{2}, \frac{1}{2} \right) \]

\[ I = \int \int f(s,t) \, ds \, dt \approx \sum \sum W_{ij} f(s_i, t_j) \]

The rule
\[ I = \int \int f(s,t) \, ds \, dt = \sum \sum W_{ij} f(s_i, t_j) \]

Uses \( M^2 \) integration points on a nonuniform grid inside the parent element and is exact for a polynomial of degree \((2M-1)\) i.e.,
\[ \int_0^1 \int_0^1 s^\alpha t^\beta \, ds \, dt = \sum_{i=1}^M \sum_{j=1}^M W_{ij} s_i^\alpha t_j^\beta \quad \text{for} \quad \alpha + \beta \leq 2M - 1 \]

A \( M^2 \)-point rule is exact for a complete polynomial of degree \((2M-1)\)

For \( M=2 \)
\[ \begin{align*}
I &= \sum_{i=1}^2 \sum_{j=1}^2 W_{ij} f(s_i, t_j) \\
&= W_1 f(0,1) + W_2 f(-1,0) + W_3 f(-1,-1) + W_4 f(0,0) \\
&= f(1,1) + f(-1,1) + f(-1,-1) + f(1,-1)
\end{align*} \]

Number the Gauss points IP=1,2,3,4
\[ I = \int \int f(s,t) \, ds \, dt = \sum_{IP} W_{IP} f_{IP} \]

CASE I: \( M=1 \) (One-point GQ rule)
\[ I = \int \int f(s,t) \, ds \, dt = 4 f(0,0) \]

is exact for a product of two linear polynomials
CASE II: M=2 (2x2 GQ rule)

\[
\begin{align*}
&\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \\
&\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)
\end{align*}
\]

\[
I = \sum_{i=1}^{m} \sum_{j=1}^{n} W_{ij} f(s_{ij}, t_{ij})
\]

is exact for a product of two cubic polynomials.

CASE III: M=3 (3x3 GQ rule)

\[
W_{6} = \frac{64}{81},
\]

\[
W_{5} = W_{4} = W_{3} = \frac{25}{81},
\]

\[
W_{2} = W_{1} = W_{0} = \frac{40}{81}
\]

\[
I = \int_{-1}^{1} \int_{-1}^{1} f(s, t) \, ds \, dt = \sum_{i=1}^{m} \sum_{j=1}^{n} W_{ij} f(s_{ij}, t_{ij})
\]

is exact for a product of two 1D polynomials of degree 5.

Examples

If \(f(s, t) = 1\)

\[
I = \int_{-1}^{1} \int_{-1}^{1} f(s, t) \, ds \, dt = 4
\]

A 1-point GQ scheme is sufficient

If \(f(s, t) = s\)

\[
I = \int_{-1}^{1} \int_{-1}^{1} f(s, t) \, ds \, dt = 0
\]

A 1-point GQ scheme is sufficient

If \(f(s, t) = s^2t^2\)

\[
I = \int_{-1}^{1} \int_{-1}^{1} f(s, t) \, ds \, dt = \frac{4}{9}
\]

A 3x3 GQ scheme is sufficient

2D Gauss quadrature for triangular domains

Remember that the parent element is a right angled triangle with unit sides.

The type of integral encountered

\[
I = \int_{0}^{1} \int_{0}^{1} f(s, t) \, ds \, dt = \sum_{i=1}^{m} \sum_{j=1}^{n} W_{ij} f(s_{ij}, t_{ij})
\]

\[
= \sum_{i=1}^{m} \sum_{j=1}^{n} W_{ij} f(s_{ij}, t_{ij})
\]

\[
= \sum_{i=1}^{m} \sum_{j=1}^{n} W_{ij} f(s_{ij}, t_{ij})
\]
Constraints on the weights
if \( R(s,t) = 1 \)

\[
I = \int_{s=0}^{s=1} f(s,t) \ ds dt = \frac{1}{2}
\]

\[
= \sum_{j=1}^{M} W_{j}^{s} \\
\sum_{j=1}^{M} W_{j}^{s} = \frac{1}{2}
\]

Example 1. A M=1 point rule is exact for a polynomial
\( f(s,t) \sim 1 \)

\[
\int_{s=0}^{s=1} f(s,t) \ ds dt = \frac{1}{2} \int \left( \frac{1}{3} \right)^{2} 
\]

Why?
Assume
\( f(s,t) = \alpha_{1} + \alpha_{2} s + \alpha_{3} t \)

Then
\[
I = \int_{s=0}^{s=1} f(s,t) \ ds dt = \frac{1}{2} \alpha_{1} + \frac{1}{3} \alpha_{2} + \frac{1}{3} \alpha_{3}
\]

But
\[
I = \sum_{j=1}^{M} W_{j}^{s} f(s_{j}, t_{j}) \\
\therefore \frac{1}{2} \alpha_{1} + \frac{1}{3} \alpha_{2} + \frac{1}{3} \alpha_{3} = W_{1}^{s} (\alpha_{1} + \alpha_{2} s_{1} + \alpha_{3} t_{1})
\]

Hence
\[
W_{1}^{s} = \frac{1}{2} ; W_{1}^{s} = \frac{1}{3} ; W_{1}^{s} = \frac{1}{3}
\]

Example 2. A M=3 point rule is exact for a complete polynomial of degree 2
\( f(s,t) \sim 1 \)

\[
\int_{s=0}^{s=1} f(s,t) \ ds dt = \frac{1}{2} \int \left( \frac{1}{3} \right)^{2} + \frac{1}{6} \int \left( \frac{1}{2} \right)^{0} + \frac{1}{6} \int \left( \frac{1}{2} \right)^{0}
\]

\[
\int_{s=0}^{s=1} f(s,t) \ ds dt = \frac{1}{2} \int \left( \frac{1}{3} \right)^{2} + \frac{1}{6} \int \left( \frac{1}{2} \right)^{0} + \frac{1}{6} \int \left( \frac{1}{2} \right)^{0}
\]

\[
\int_{s=0}^{s=1} f(s,t) \ ds dt = \frac{1}{2} \int \left( \frac{1}{3} \right)^{2} + \frac{1}{6} \int \left( \frac{1}{2} \right)^{0} + \frac{1}{6} \int \left( \frac{1}{2} \right)^{0}
\]
Example 4. A M=4 point rule is exact for a complete polynomial of degree 3

\[ f(s,t) \sim \begin{array}{cccc}
1 & s & t & s^2 \cdot st \cdot t^2 \\
& s^3 & s^2 \cdot t & s \cdot t^2 & t^3
\end{array} \]

\[
I = \frac{27}{96} \int \begin{array}{ccc}
1 & 1 & 1 \\
3 & 3 & 3
\end{array} \int \begin{array}{ccc}
1 & 1 & 1 \\
3 & 3 & 3
\end{array}
\]

Recommended order of integration

"Finite Element Procedures" by K. J. Bathe

Which order of GQ to use for full integration?

To compute the stiffness matrix we need to evaluate the following integral

\[
\hat{k} = \int_{-1}^{1} \int_{-1}^{1} B^T D B \det(J) \, ds \, dt
\]

For an "undistorted" element \( \det(J) = \text{constant} \)

Example: 4-noded parallelogram

\[
N_j \sim \begin{array}{ccc}
1 & s & t \\
s & t & s
\end{array}
\]

\[
B = \begin{array}{c}
1 \\
s \\
t
\end{array}
\]

\[
B^T D B = \begin{array}{c}
1 \\
1 & s & t \\
1 & s^2 & st & t^2
\end{array}
\]

"Reduced" vs "Full" integration

**Full integration**: Quadrature scheme sufficient to provide exact integrals of all terms of the stiffness matrix if the element is geometrically undistorted.

**Reduced integration**: An integration scheme of lower order than required by "full" integration.

**Recommendation**: Reduced integration is NOT recommended.
Hence, 2M-1=2
M=3/2
Hence we need at least a 2x2 GQ scheme

Example 2: 8-noded Serendipity element

\[ \begin{pmatrix} N_i \end{pmatrix} = \begin{pmatrix} 1 \\ s \\ s^2 \\ st \\ st^2 \end{pmatrix} \]

\[ \begin{pmatrix} B \end{pmatrix} = \begin{pmatrix} 1 \\ s \\ s^2 \\ st \\ st^2 \end{pmatrix} \]

Hence, 2M-1=4
M=5/2
Hence we need at least a 3x3 GQ scheme

"Spurious" zero energy mode/ "hour-glass" mode

The strain energy of an element

\[ U = \frac{1}{2} \int d^1 \varepsilon dV = \frac{1}{2} \int \varepsilon^T D \varepsilon dV \]

Corresponding to a rigid body mode, \( \varepsilon = 0 \Rightarrow U = 0 \)

If \( U=0 \) for a mode \( \varepsilon \) that is different from a rigid body mode, then \( \varepsilon \) is known as a "spurious" zero energy mode or "hour-glass" mode

Such a mode is undesirable

Reduced integration leads to rank deficiency of the stiffness matrix and "spurious" zero energy modes

Example 1. 4-noded element

\[ U = \frac{1}{2} \int \varepsilon^T D \varepsilon dV \approx \sum_{i} W_i \varepsilon_i^T D \varepsilon_i \]

Full integration: NGAUSS=4
Element has 3 zero energy (rigid body) modes

Reduced integration: e.g., NGAUSS=1

\[ U \approx 4 \left[ \varepsilon^T D \varepsilon \right]_{xy=0} \]
Consider 2 displacement fields
\[
\begin{align*}
    u &= C_{xy} \\
    v &= 0
\end{align*}
\]
\[
\begin{align*}
    u &= 0 \\
    v &= C_{xy}
\end{align*}
\]
At \( x = y = 0 \) \( \varepsilon_x = \varepsilon_y = \gamma_{xy} = 0 \)
\[ \Rightarrow U = 0 \]
We have therefore 2 hour-glass modes.

Example 2. 8-noded serendipity element
\[
U = \frac{1}{2} \int \varepsilon^T D \varepsilon \, dV \approx \sum_{i=1}^{NGAUSS} \int W_i \varepsilon^T D \varepsilon \, dV
\]
Full integration: \( NGAUSS=9 \)
Element has 3 zero energy (rigid body) modes
Reduced integration: e.g., \( NGAUSS=4 \)

Element has one spurious zero energy mode corresponding to the following displacement field
\[
\begin{align*}
    u &= C \ x \ (y^2 - 1/3) \\
    v &= -C \ y \ (x^2 - 1/3)
\end{align*}
\]
Show that the strains corresponding to this displacement field are all zero at the 4 Gauss points
Elements with zero energy modes introduce uncontrolled errors and should NOT be used in engineering practice.