Axially loaded elastic bar

\[ A(x) = \text{cross section at } x \]
\[ b(x) = \text{body force distribution} \]
\[ E(x) = \text{Young's modulus} \]

For each element

Element stiffness matrix

\[ \mathbf{k}_i = \int_{x_i}^{x_{i+1}} \mathbf{B}^T \mathbf{EB} \, dx \]
\[ \mathbf{k}_j = \int_{x_j}^{x_{j+1}} \mathbf{B}^T \mathbf{EB} \, dx \]

where \( \mathbf{B} = \frac{d\mathbf{N}(x)}{dx} \)

Only for a linear finite element

\[ \int_{x_i}^{x_{i+1}} \mathbf{B}^T \mathbf{EB} \, dx = \frac{1}{(x_{i+1} - x_i)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \]

Element nodal load vector

\[ f_i = \int_{x_i}^{x_{i+1}} \mathbf{N}^T \, b \, dx \]

\[ f_{j+1} = \int_{x_j}^{x_{j+1}} \mathbf{N}^T \, b \, dx \]

Question: How do we compute these integrals using a computer?
Any integral from $x_1$ to $x_2$ can be transformed to the following integral on $(-1, 1)$

$$I = \int_{-1}^{1} f(\xi) d\xi$$

Use the following change of variables

$$x = \frac{1-\xi}{2} x_1 + \frac{1+\xi}{2} x_2$$

**Goal:** Obtain a good approximate value of this integral

1. Newton-Cotes Schemes (trapezoidal rule, Simpson’s rule, etc)
2. Gauss Integration Schemes

**NOTE:** Integration schemes in 1D are referred to as “quadrature rules”

**Trapezoidal rule:** Approximate the function $f(\xi)$ by a straight line $g(\xi)$ that passes through the end points and integrate the straight line

$$I = \int_{-1}^{1} f(\xi) d\xi \approx \int_{-1}^{1} g(\xi) d\xi = f(1) + f(-1)$$

- Requires the function $f(x)$ to be evaluated at 2 points $(-1, 1)$
- Constants and linear functions are exactly integrated
- Not good for quadratic and higher order polynomials

How can I make this better?

**Simpson’s rule:** Approximate the function $f(\xi)$ by a parabola $g(\xi)$ that passes through the end points and through $f(0)$ and integrate the parabola

$$I = \int_{-1}^{1} f(\xi) d\xi \approx \int_{-1}^{1} g(\xi) d\xi = f(0) + \frac{2}{3} f(1) + \frac{1}{3} f(-1)$$

$$g(\xi) = \frac{2(\xi-1)}{2} f(-1) + (1 - \xi)(1+\xi)f(0) + \frac{2(1+\xi)}{2} f(1)$$
• Requires the function $f(x)$ to be evaluated at 3 points $(-1, 0, 1)$
• Constants, linear functions and parabolas are exactly integrated
• Not good for cubic and higher order polynomials

How to generalize this formula?

Notice that both the integration formulas had the general form

$$I = \int_{-1}^{1} f(\xi) \, d\xi \approx \sum_{\xi} Wf(\xi)$$

<table>
<thead>
<tr>
<th></th>
<th>Weight</th>
<th>Integration point</th>
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<tbody>
<tr>
<td><strong>Trapezoidal rule:</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$M=2$</td>
<td>$W_1 = 1$</td>
<td>$\xi_1 = -1$</td>
</tr>
<tr>
<td></td>
<td>$W_2 = 1$</td>
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<td><strong>Accurate for polynomial of degree at most 1 (=M-1)</strong></td>
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<td><strong>Simpson’s rule:</strong></td>
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<tr>
<td>$M=3$</td>
<td>$W_1 = 1/3$</td>
<td>$\xi_1 = -1$</td>
</tr>
<tr>
<td></td>
<td>$W_2 = 4/3$</td>
<td>$\xi_2 = 0$</td>
</tr>
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<td>$W_3 = 1/3$</td>
<td>$\xi_3 = 1$</td>
</tr>
<tr>
<td><strong>Accurate for polynomial of degree at most 2 (=M-1)</strong></td>
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Generalization of these two integration rules: **Newton-Cotes**
• Divide the interval $(-1, 1)$ into $M-1$ equal intervals using $M$ points
• Pass a polynomial of degree $M-1$ through these $M$ points (the value of this polynomial will be equal to the value of the function at these $M-1$ points)
• Integrate this polynomial to obtain an approximate value of the integral

With ‘$M$’ points we may integrate a polynomial of degree ‘$M-1$’ exactly.

Is this the best we can do?

With ‘$M$’ integration points and ‘$M$’ weights, I should be able to integrate a polynomial of degree $2M-1$ exactly!!

**Gauss integration rule**

See table 10-1 (p 405) of Logan
Gauss quadrature

\[ \int f(\xi) \, d\xi \approx \sum_{i=1}^{M} W_i f(\xi_i) \]

Weight \hspace{1cm} Integration point

How can we choose the integration points and weights to **exactly** integrate a polynomial of degree \(2M-1\)?

Remember that now we do not know, a priori, the location of the integration points.

### Example: \(M=1\) (Midpoint quadrature)

\[ f = \int f(\xi) \, d\xi = W_i f(\xi_i) \]

How can we choose \(W_i\) and \(\xi_i\) so that we may integrate a \((2M-1=1)\) **linear polynomial** exactly?

\[ f(\xi) = a_0 + a_1 \xi \]

\[ \int f(\xi) \, d\xi = 2a_0 \]

But we want

\[ \int f(\xi) \, d\xi = W_i f(\xi_i) = a_0 W_i + a_1 W_i \xi_i \]

Hence, we obtain the identity

\[ 2a_0 = a_0 W_i + a_1 W_i \xi_i \]

For this to hold for arbitrary \(a_0\) and \(a_1\) we need to satisfy 2 conditions

**Condition 1:** \(W_i = 2\)

**Condition 2:** \(W_i \xi_i = 0\)

i.e., \(W_i = 2; \xi_i = 0\)

For \(M=1\)

\[ I = \int f(\xi) \, d\xi \approx 2f(0) \]

Midpoint quadrature rule:

- Only one evaluation of \(f(\xi)\) is required at the midpoint of the interval.
- Scheme is accurate for constants and linear polynomials (compare with Trapezoidal rule)
Example: M=2

\[ I = \int_{-1}^{1} f(\xi) \, d\xi \approx W_1 f(\xi_1) + W_2 f(\xi_2) \]

How can we choose \( W_1, W_2, \xi_1 \) and \( \xi_2 \) so that we may integrate a polynomial of degree \((2M-1=4-1=3)\) exactly?

\[ f(\xi) = a_0 + a_2 \xi^2 + a_4 \xi^4 \]

\[ \int_{-1}^{1} f(\xi) \, d\xi = 2a_0 + \frac{2}{3}a_2 \]

But we want

\[ \int_{-1}^{1} f(\xi) \, d\xi = W_1 f(\xi_1) + W_2 f(\xi_2) \]

\[ = a_0 (W_1 + W_2) + a_2 [W_1 \xi_1^2 + W_2 \xi_2^2] + a_4 [W_1 \xi_1^4 + W_2 \xi_2^4] \]

Hence, we obtain the 4 conditions to determine the 4 unknowns \((W_1, W_2, \xi_1 \text{ and } \xi_2)\)

**Condition 1:** \( W_1 + W_2 = 2 \)

**Condition 2:** \( W_1 \xi_1 = W_2 \xi_2 = 0 \)

**Condition 3:** \( W_1 \xi_1^3 + W_2 \xi_2^3 = \frac{2}{3} \)

**Condition 4:** \( W_1 \xi_1^5 + W_2 \xi_2^5 = 0 \)

Check that the following is the solution

\[ W_1 = W_2 = 1 \]

\[ \xi_1 = -\frac{1}{\sqrt{3}}; \quad \xi_2 = \frac{1}{\sqrt{3}} \]

Exercise: Derive the 6 conditions required to find the integration points and weights for a 3-point Gauss quadrature rule

For \( M=2 \)

\[ I = \int_{-1}^{1} f(\xi) \, d\xi \approx f(\frac{-1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}}) \]

- Only two evaluations of \( f(\xi) \) is required.
- Scheme is accurate for polynomials of degree at most 3 (compare with Simpson’s rule)

<table>
<thead>
<tr>
<th>Newton-Cotes</th>
<th>Gauss quadrature</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ‘M’ integration points are necessary to exactly integrate a polynomial of degree ‘M-1’</td>
<td>1. ‘M’ integration points are necessary to exactly integrate a polynomial of degree ‘2M-1’</td>
</tr>
<tr>
<td>2. More expensive</td>
<td>2. Less expensive</td>
</tr>
<tr>
<td>3. Exponential convergence, error proportional to ( \frac{1}{M^2} )</td>
<td>3. Exponential convergence, error proportional to ( \frac{1}{5M^2} )</td>
</tr>
</tbody>
</table>
Example

\[ I = \int 2 \xi d\xi \text{ where } f(\xi) = \xi + \frac{1}{\sqrt{\xi}} \]

**Exact integration**

\[ I = \frac{2}{3} \quad \text{Integrate and check!} \]

**Newton-Cotes**

To exactly integrate this I need a 4-point Newton-Cotes formula. Why?

**Gauss**

To exactly integrate this I need a 2-point Gauss formula. Why?

Gauss quadrature:

\[ I = \int 2 \xi d\xi = \left( \int \frac{1}{\sqrt{\xi}} \right) + \left( \int \frac{1}{\sqrt{\xi}} \right) = \frac{2}{3} \quad \text{Exact answer!} \]

Comparison of Gauss quadrature and Newton-Cotes for the integral

\[ I = \int \cos(2\pi x) dx \]

In FEM we ALWAYS use Gauss quadrature

**Linear Element**

\[ \begin{align*}
\xi &= -1 \\
\xi &= 1 
\end{align*} \]

**Stiffness matrix**

\[ k = \int_{-1}^{1} B^T EB \cdot d\xi = \left[ \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right] \left[ \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right] \left[ \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right] \]

**Nodal load vector**

\[ f_i = \int_{-1}^{1} N_i^T b \ d\xi \quad f_s = \int_{-1}^{1} N_s b \ d\xi \]

Usually a 2-point Gauss integration is used. Note that if A, E and b are complex functions of x, they will not be accurately integrated.
Quadratic Element

Nodal shape functions

\[ N_1(\xi) = \frac{\xi}{2} (\xi - 1) \quad \xi = -1 \quad \xi = 0 \quad \xi = 1 \]

\[ N_2(\xi) = (1 - \xi^2) \quad \text{You should be able to derive these!} \]

\[ N_3(\xi) = \frac{\xi}{2} (\xi + 1) \]

Stiffness matrix

\[ \mathbf{k} = \int B^T \mathbf{f} B d\xi = \int B^T \mathbf{f} B d\xi \quad \text{Assuming E and A are constants} \]

\[ \mathbf{B} = \frac{dN}{d\xi} = \begin{bmatrix} \frac{dN_1}{d\xi} & \frac{dN_2}{d\xi} & \frac{dN_3}{d\xi} \end{bmatrix} \begin{bmatrix} \frac{1}{2}(2\xi - 1) & -2\xi & \frac{1}{2}(2\xi + 1) \end{bmatrix} \]

\[ \int B^T \mathbf{f} B d\xi = \begin{bmatrix} (\xi - 1/2)^2 & -2(\xi - 1/2) & (\xi^2 - 1/4) \\
-2\xi(\xi - 1/2) & 4\xi & -2\xi(\xi + 1/2) \\
(\xi^2 - 1/4) & -2\xi(\xi + 1/2) & (\xi^2 + 1/2)^2 \end{bmatrix} \]

Need to exactly integrate quadratic terms.
Hence we need a 2-point Gauss quadrature scheme. Why?

\[ k = \int B^T f B d\xi = \begin{bmatrix} (\xi - 1/2)^2 & -2(\xi - 1/2) & (\xi^2 - 1/4) \\
-2\xi(\xi - 1/2) & 4\xi & -2\xi(\xi + 1/2) \\
(\xi^2 - 1/4) & -2\xi(\xi + 1/2) & (\xi^2 + 1/2)^2 \end{bmatrix} \]