MANE 4240 & CIVL 4240
Introduction to Finite Elements

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Mapped element geometries and shape functions: the isoparametric formulation

Reading assignment:
Chapter 10.1-10.3, 10.6 + Lecture notes

Summary:
• Concept of isoparametric mapping
• 1D isoparametric mapping
• Element matrices and vectors in 1D
• 2D isoparametric mapping : rectangular parent elements
• 2D isoparametric mapping : triangular parent elements
• Element matrices and vectors in 2D

Consider a special 4-noded rectangle in its local coordinate system (s,t)

Displacement interpolation:
\[ u = N_1\mu_1 + N_2\mu_2 + N_3\mu_3 + N_4\mu_4 \]
\[ v = N_1\nu_1 + N_2\nu_2 + N_3\nu_3 + N_4\nu_4 \]

Shape functions in local coord system:
\[ N_1(s,t) = \frac{1}{4}(1+s)(1+t) \]
\[ N_2(s,t) = \frac{1}{4}(1-s)(1+t) \]
\[ N_3(s,t) = \frac{1}{4}(1-s)(1-t) \]
\[ N_4(s,t) = \frac{1}{4}(1+s)(1-t) \]
Recall that

\[ N_1 + N_2 + N_3 + N_4 = 1 \quad \text{Rigid body modes} \]
\[ N_1 s_1 + N_2 s_2 + N_3 s_3 + N_4 s_4 = s \quad \text{Constant strain states} \]
\[ N_1 t_1 + N_2 t_2 + N_3 t_3 + N_4 t_4 = t \]

**Goal** is to map this element from local coords to a general quadrilateral element in global coord system

In the mapped coordinates, the shape functions need to satisfy

1. Kronecker delta property
   Then \[ N_i = \begin{cases} 1 & \text{at node } i \\ 0 & \text{at all other nodes} \end{cases} \]

2. Polynomial completeness
   \[ \sum_i N_i = 1 \]
   \[ \sum_i N_i x_i = x \]
   \[ \sum_i N_i y_i = y \]

The relationship

\[ x = \sum_i N_i(s,t) x_i \]
\[ y = \sum_i N_i(s,t) y_i \]

Provides the required mapping from the local coordinate system to the global coordinate system and is known as **isoparametric mapping**

\((s,t)\): isoparametric coordinates
\((x,y)\): global coordinates
1D isoparametric mapping

3 noded (quadratic) element

Local (isoparametric) coordinates

\[ N_1(s) = \frac{s(1-s)}{2} \]
\[ N_2(s) = \frac{s(1+s)}{2} \]
\[ N_3(s) = 1-s^2 \]

Isoparametric mapping

\[ x = \sum N_i(s)x_i \]
\[ \Rightarrow x = \frac{s(1-s)}{2}x_1 + \frac{s(1+s)}{2}x_2 + (1-s^2)x_3 \]

Notes

1. Given a point in the isoparametric coordinates, I can obtain the corresponding mapped point in the global coordinates using the isoparametric mapping equation

\[ x = \frac{s(1-s)}{2}x_1 + \frac{s(1+s)}{2}x_2 + (1-s^2)x_3 \]

At \( s = -1 \); \( x = x_1 \)
At \( s = 0 \); \( x = x_2 \)
At \( s = 1 \); \( x = x_3 \)

Question

\( x = ? \) at \( s = 0.5 \)?

2. The shape functions themselves get mapped
In the isoparametric coordinates (s) they are polynomials.
In the global coordinates (x) they are in general nonpolynomials
Let’s consider the following numerical example

\[ x_1 = 0; x_2 = 6; x_3 = 4 \]

Isoparametric mapping \( x(s) \)

\[ x = \frac{s(1-s)}{2}v_1 + \frac{s(1+s)}{2}v_2 + (1-s^2)v_3 \]

Inverse mapping \( s(x) \)

\[ s = \frac{3}{2}\sqrt{25-4x} \]

Simple polynomial
Complicated function
Now lets compute the shape functions in the global coordinates

\[ N_1(s) = \frac{s(1 + s)}{2} = \frac{1}{2} \left( \frac{3 - \sqrt{25 - 4s}}{2} \right) \left( \frac{1 + \sqrt{25 - 4s}}{2} \right) \]
\[ = \frac{1}{2} \left( 10s - 2\sqrt{s^2 - 25} \right) = N_2(s) \]

Now lets compute the shape functions in the global coordinates

\[ N_f(x) \]
\[ N_f(x) \]

\[ N_f(s) \text{ is a simple polynomial} \quad N_f(x) = \frac{s(1 + s)}{2} \]
\[ N_f(x) \text{ is a complicated function} \quad N_f(x) = \frac{1}{2} \left( 10 - 2\sqrt{25 - 4s} \right) \]

However, thanks to isoparametric mapping, we always ensure
1. Kronecker delta property
2. Rigid body and constant strain states

Element matrices and vectors for a mapped 1D bar element

Displacement interpolation \( u = N_1 u_1 + N_2 u_2 + N_3 u_3 = N d \)

Strain-displacement relation \( \varepsilon = \frac{du}{dx} + \frac{dN}{dx} u_1 + \frac{dN}{dx} u_2 + \frac{dN}{dx} u_3 = B d \)

Stress \( \sigma = E \varepsilon = E B d \)

The strain-displacement matrix \( B = \left[ \begin{array}{ccc} dN_1 \frac{dN_1}{dx} \frac{dN_1}{dx} \\ dN_2 \frac{dN_2}{dx} \frac{dN_2}{dx} \\ dN_3 \frac{dN_3}{dx} \frac{dN_3}{dx} \end{array} \right] \)

The only difference from before is that the shape functions are in the isoparametric coordinates
\( N_1(s) = \frac{s(1 - s)}{2} \)
\( N_2(s) = \frac{s(1 + s)}{2} \)
\( N_3(s) = 1 - s^2 \)

We know the isoparametric mapping
\[ s = \sum N_i(x) \]

And we will not try to obtain explicitly the inverse map.
How to compute the B matrix?
Using chain rule
\[
\frac{dN_i(s)}{dx} = \frac{dN_i(s)}{ds} \frac{ds}{dx}
\]

(*)

Do I know \(\frac{dN_i(s)}{dx}\)?

Do I know \(\frac{ds}{dx}\)?

I know \(x = \sum_i N_i(s)x_i\)

Hence \(\frac{dx}{ds} = \sum_i \frac{dN_i(s)}{ds} \equiv \frac{1}{J} (\text{Jacobian of mapping})\)

From (*) \(\frac{dN_i(s)}{dx} = \frac{1}{J} \frac{dN_i(s)}{ds}\)

What does the Jacobian do?
\[
\frac{dx}{ds} = J \frac{ds}{dx}
\]

Maps a differential element from the isoparametric coordinates to the global coordinates

The strain-displacement matrix
\[
B = \begin{bmatrix}
\frac{dN_1}{dx} & \frac{dN_2}{dx} & \frac{dN_3}{dx}
\end{bmatrix}
= \frac{1}{J} \begin{bmatrix}
\frac{dN_1}{ds} & \frac{dN_2}{ds} & \frac{dN_3}{ds}
\end{bmatrix}
\]

For the 3-noded element
\[
J = \sum_i \frac{dN_i(s)}{ds} x_i = \frac{2s-1}{2} x_1 + \frac{2s+1}{2} x_2 - 2sx_3
\]

\[
B = \frac{1}{J} \begin{bmatrix}
\frac{2s-1}{2} & \frac{2s+1}{2} & -2s
\end{bmatrix}
\]

The element stiffness matrix
\[
k = \int_0^1 EA B' B \frac{ds}{dx}
= \int_{-1}^1 EA B' B J ds \quad \therefore dx = J ds
\]

NOTES
1. The integral on ANY element in the global coordinates in now an integral from -1 to 1 in the local coordinates
2. The Jacobian is a function of 's' in general and enters the integral. The specific form of 'J' is determined by the values of \(x_1, x_2,\) and \(x_3\). Gaussian quadrature is used to evaluate the stiffness matrix
3. In general \(\hat{B}\) is a vector of rational functions in 's'
Isoparametric mapping in 2D: Rectangular parent elements

Parent element

Mapped element in

Isoparametric mapping

\[ x = \sum_i N_i(s,t)x_i \]
\[ y = \sum_i N_i(s,t)y_i \]

Shape functions of parent element in isoparametric coordinates

\[ N_1(s,t) = \frac{1}{4}(1-s)(1-t) \]
\[ N_2(s,t) = \frac{1}{4}(1-s)(1+t) \]
\[ N_3(s,t) = \frac{1}{4}(1+s)(1-t) \]
\[ N_4(s,t) = \frac{1}{4}(1+s)(1+t) \]

Isoparametric mapping

\[ x = \sum_i N_i(s,t)x_i \]
\[ y = \sum_i N_i(s,t)y_i \]

NOTES:
1. The isoparametric mapping provides the map \((s,t)\) to \((x,y)\), i.e., if you are given a point \((s,t)\) in isoparametric coordinates, then you can compute the coordinates of the point in the \((x,y)\) coordinate system using the equations

\[ x = \sum N_i(s,t)x_i \]
\[ y = \sum N_i(s,t)y_i \]

2. The inverse map will never be explicitly computed.
8-noded Serendipity element: element shape functions in isoparametric coordinates

\[ N_1(s,t) = \frac{1}{4}(1-s)(1-t)(s-t-1) \]

\[ N_2(s,t) = \frac{1}{4}(1+s)(1-t)(s-t-1) \]

\[ N_3(s,t) = \frac{1}{4}(1+s)(1+t)(s+t-1) \]

\[ N_4(s,t) = \frac{1}{4}(1-s)(1+t)(s+t-1) \]

\[ N_5(s,t) = \frac{1}{4}(1-s)(1+s)(1-t) \]

\[ N_6(s,t) = \frac{1}{4}(1+s)(1+t)(1+t) \]

\[ N_7(s,t) = \frac{1}{4}(1-s)(1+t)(1+t) \]

\[ N_8(s,t) = \frac{1}{4}(1-s)(1+t)(1-t) \]

**NOTES**

1. \( N_i(s,t) \) is a simple polynomial in \( s \) and \( t \). But \( N_i(x,y) \) is a complex function of \( x \) and \( y \).
2. The element edges can be curved in the mapped coordinates.
3. A “midside” node in the parent element may not remain as a midside node in the mapped element. An extreme example.

4. Care must be taken to ensure **UNIQUENESS** of mapping.

**Isoparametric mapping in 2D: Triangular parent elements**

**Parent element**: a right angled triangles with arms of unit length

**Key** is to link the isoparametric coordinates with the area coordinates.

\[ \Delta_{123} = \frac{1}{2} \]

\[ \Delta_{s} = \frac{1}{2} \]

\[ \Delta_{t} = \frac{1}{2} \]

\[ \Delta_{s+t} = \frac{1}{2}(1-s-t) \]

\[ \Delta_{s} = \frac{\Delta_{s}}{\Delta_{123}} = s \]

\[ \Delta_{t} = \frac{\Delta_{t}}{\Delta_{123}} = t \]

\[ \Delta_{s+t} = \frac{\Delta_{s+t}}{\Delta_{123}} = 1 - s - t \]
Now replace L₁, L₂, L₃ in the formulas for the shape functions of triangular elements to obtain the shape functions in terms of (s,t)

Example: 3-noded triangle

\[
\begin{align*}
N_1 &= s \\
N_2 &= t \\
N_3 &= 1 - s - t
\end{align*}
\]

Parent shape functions

\[
\begin{align*}
s &= N_i(s,t)x_i + N_j(s,t)x_j + N_k(s,t)x_k \\
y &= N_i(s,t)y_i + N_j(s,t)y_j + N_k(s,t)y_k
\end{align*}
\]

Isoparametric mapping

In isoparametric formulation
1. Shape functions first expressed in (s,t) coordinate system i.e., \(N_i(s,t)\)
2. The isoparametric mapping relates the (s,t) coordinates with the global coordinates (x,y)
3. It is laborious to find the inverse map \(s(x,y)\) and \(t(x,y)\) and we do not do that. Instead we compute the integrals in the domain of the parent element.

Element matrices and vectors for a mapped 2D element

Recall: For each element

\[
\begin{align*}
\mathbf{u} &= \mathbf{N} \mathbf{d} \\
\varepsilon &= \mathbf{B} \mathbf{d} \\
\sigma &= \mathbf{D} \varepsilon
\end{align*}
\]

Element stiffness matrix

\[
\mathbf{k} = \int \mathbf{B}^T \mathbf{D} \mathbf{B} \, dV
\]

Element nodal load vector

\[
\mathbf{f} = \int \mathbf{N}^T dN dV + \int_{S_e} \mathbf{N}^T \mathbf{T} dS
\]

NOTE
1. \(N_i(s,t)\) s are already available as simple polynomial functions
2. The first task is to find \(\frac{\partial N_i}{\partial x}\) and \(\frac{\partial N_i}{\partial y}\)

Use chain rule

\[
\begin{align*}
\frac{\partial N_i}{\partial s} &= \frac{\partial N_i}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial N_i}{\partial y} \frac{\partial y}{\partial s} \\
\frac{\partial N_i}{\partial t} &= \frac{\partial N_i}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial N_i}{\partial y} \frac{\partial y}{\partial t}
\end{align*}
\]
In matrix form
\[
\begin{bmatrix}
\frac{\partial N_i}{\partial x} \\
\frac{\partial N_i}{\partial y} \\
\frac{\partial N_i}{\partial t}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial x}{\partial s} \\
\frac{\partial y}{\partial s} \\
\frac{\partial t}{\partial s}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial N_i}{\partial x} \\
\frac{\partial N_i}{\partial y} \\
\frac{\partial N_i}{\partial t}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial x}{\partial t} \\
\frac{\partial y}{\partial t} \\
\frac{\partial t}{\partial t}
\end{bmatrix}
\]

This is known as the Jacobian matrix \((J)\) for the mapping \((s,t) \rightarrow (x,y)\)

We want to compute these for the \(B\) matrix

\[
\frac{\partial N_i}{\partial x} = \sum \frac{\partial N_i(s,t)}{\partial x} x_i,
\frac{\partial N_i}{\partial y} = \sum \frac{\partial N_i(s,t)}{\partial y} y_i,
\frac{\partial N_i}{\partial t} = \sum \frac{\partial N_i(s,t)}{\partial t} t_i.
\]

How to compute the Jacobian matrix?

Start from
\[
x = \sum N_i(s,t) x_i,
y = \sum N_i(s,t) y_i,
\]

\[
\frac{\partial x}{\partial s} = \sum \frac{\partial N_i(s,t)}{\partial s} x_i ; \frac{\partial x}{\partial t} = \sum \frac{\partial N_i(s,t)}{\partial t} x_i,
\frac{\partial y}{\partial s} = \sum \frac{\partial N_i(s,t)}{\partial s} y_i ; \frac{\partial y}{\partial t} = \sum \frac{\partial N_i(s,t)}{\partial t} y_i.
\]

Need to ensure that \(\text{det}(J) > 0\) for one-to-one mapping

3. Now we need to transform the integrals from \((x,y)\) to \((s,t)\)

**Case 1.** Volume integrals

\[
\int f(x,y) \, dV = \int f(x(s,t), y(s,t)) h \, dA = \int \int f(x(s,t), y(s,t)) h \, \text{det}(J) \, ds dt
\]

\(h=\)thickness of element

This depends on the key result

\[
[dA = dsys = \text{det}(J) \, ds dt]
\]
Proof: $dA = dsdy = \det(J) \, ds \, dt$

\[ dA = \left| \begin{array}{ccc} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{array} \right| \, ds \, dt = \det(J) \, ds \, dt 

\]

Problem: Consider the following isoparametric map

The isoparametric map

\[ x = N_1(s,t)x_1 + N_2(s,t)x_2 + N_3(s,t)x_3 + N_4(s,t)x_4 \]
\[ y = N_1(s,t)y_1 + N_2(s,t)y_2 + N_3(s,t)y_3 + N_4(s,t)y_4 \]

\[
\Rightarrow x = \frac{3t}{4} + \frac{t}{2} \left( 1 + s \right) + \frac{t}{2} \left( 1 - s \right) + \frac{t}{2} \left( 1 + s \right) + \frac{t}{2} \left( 1 - s \right)
\]
\[ y = \frac{3s}{4} + \frac{s}{2} \left( 1 + t \right) + \frac{s}{2} \left( 1 - t \right) + \frac{s}{2} \left( 1 + t \right) + \frac{s}{2} \left( 1 - t \right) \]

In this case, we may compute the inverse map, but we will **NOT** do that!

Displacement interpolation

\[ u = N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4 \]
\[ v = N_1v_1 + N_2v_2 + N_3v_3 + N_4v_4 \]

Shape functions in isoparametric coord system

\[ N_1(s,t) = \frac{1}{4} (1 + s)(1 + t) \]
\[ N_2(s,t) = \frac{1}{4} (1 - s)(1 + t) \]
\[ N_3(s,t) = \frac{1}{4} (1 - s)(1 - t) \]
\[ N_4(s,t) = \frac{1}{4} (1 + s)(1 - t) \]
The **Jacobian matrix**

\[
J = \begin{bmatrix}
\frac{\partial x}{\partial x} & \frac{\partial y}{\partial x} \\
\frac{\partial x}{\partial y} & \frac{\partial y}{\partial y}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\frac{3}{2} & 0 \\
0 & \frac{5}{2}
\end{bmatrix}
\]

since \( x = \frac{3(1 + s)}{2} \) and \( y = \frac{7 + 5t}{2} \)

The diagonal terms are due to stretching of the sides along the x-and y-directions. The off-diagonal terms are zero because the element does not shear.

\[
J^{-1} = \begin{bmatrix}
\frac{2}{3} & 0 \\
0 & \frac{2}{5}
\end{bmatrix}
\]

and \( \text{det}(J) = \frac{15}{4} \)

Hence, if I were to compute the first column of the \( B \) matrix along the positive x-direction

\[
B_1 = \begin{bmatrix}
\frac{\partial N_1}{\partial x} \\
0 \\
\frac{\partial N_1}{\partial y}
\end{bmatrix}
\]

I would use

\[
\begin{bmatrix}
\frac{\partial N_1}{\partial x} \\
\frac{\partial N_1}{\partial y}
\end{bmatrix} = J^{-1} \begin{bmatrix}
\frac{\partial N_1}{\partial x} \\
\frac{\partial N_1}{\partial y}
\end{bmatrix} = \begin{bmatrix}
2/3 & 0 \\
0 & \frac{2}{5}
\end{bmatrix} \begin{bmatrix}
1 + t \\
4 - 10t
\end{bmatrix} = \begin{bmatrix}
1 + t \\
4 - 10t
\end{bmatrix}
\]

Hence

\[
B_1 = \begin{bmatrix}
\frac{1 + t}{6} \\
0 \\
\frac{1 + t}{10}
\end{bmatrix}
\]

The element stiffness matrix

\[
\mathbf{k} = \int B^T \mathbf{D} B \, \text{d}V = \int \mathbf{B}^T \mathbf{D} B \, \text{det}(J) \, \text{d}A
\]

**Case 2. Surface integrals**

For \( \text{d}S_1 \) we consider 2 cases

**Case (a):** \( \mathbf{F} = \pm 1 \)

\[
dS_1 = h \, |dA| = h \left( \left( \frac{\partial y}{\partial s} \right)^2 + \left( \frac{\partial y}{\partial t} \right)^2 \right)^{1/2} \, \text{d}s
\]

**Case (b):** \( \mathbf{F} = \pm 2 \)

\[
dS_1 = 2h \, |dA| = 2h \left( \left( \frac{\partial y}{\partial s} \right)^2 + \left( \frac{\partial y}{\partial t} \right)^2 \right)^{1/2} \, \text{d}s
\]
Case (b): \( t = \pm 1 \)

Summary of element matrices in 2D plane stress/strain

**Quadrilateral element**

\[
k = \int \int B^T D B \ h \det(J) \ ds dt
\]

\[
f_s = \int \int N^T X \ h \det(J) \ ds dt
\]

Suppose \( s = -1 \) gets mapped to \( S_T \)

\[
f_s = \int_{s_1}^{s_2} N^T \left( \left( \frac{\partial x}{\partial s} \right)^2 + \left( \frac{\partial y}{\partial s} \right)^2 \right) \bigg|_{\text{Evaluated on}} \ h ds
\]

**Triangular element**

\[
k = \int_{s_1}^{s_2} B^T D B \ h \det(J) \ ds dt
\]

\[
f_s = \int_{s_1}^{s_2} N^T X \ h \det(J) \ ds dt
\]

Suppose \( t = 0 \) gets mapped to \( S_T \)

\[
f_s = \int_{s_1}^{s_2} N^T \left( \left( \frac{\partial x}{\partial s} \right)^2 + \left( \frac{\partial y}{\partial s} \right)^2 \right) \bigg|_{\text{Evaluated on}} \ h ds
\]