MANE 4240 & CIVL 4240
Introduction to Finite Elements

Prof. Suvranu De

Introduction

Info

Course Instructor:
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Course website:
http://www.rpi.edu/~des/IFEA2017Fall.html

Info

Practicum Instructor:
Professor Jeff Morris
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Info

TA:
Kartik Josyula
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JEC room: JEC 5030
Office hours: M: 4:30-5:30 pm,
R: 3:30-4:30 pm
Course texts and references

Course text (for HW problems and reading assignments):
Title: A First Course in the Finite Element Method
Author: Daryl Logan
Edition: Fifth
Publisher: Cengage Learning
ISBN: 0-534-55298-6

Relevant reference:
Finite Element Procedures, K. J. Bathe, Prentice Hall
A First Course in Finite Elements, J. Fish and T. Belytschko
Lecture notes posted on the course website

Course grades

Grades will be based on:
1. Home works (15 %).
2. Practicum exercises (10 %) to be handed in within a week of assignment.
3. Course project (25 %) to be handed in by December 12th (by noon)
4. Two in-class quizzes (2x25%) on 13th October, 12th December

1) All write ups that you present MUST contain your name and RIN
2) There will be reading quizzes (announced AS WELL AS unannounced) on a regular basis and points from these quizzes will be added on to the homework

Collaboration / academic integrity

1. Students are encouraged to collaborate in the solution of HW problems, but submit independent solutions that are NOT copies of each other.
   Funny solutions (that appear similar/same) will be given zero credit.
   Softwares may be used to verify the HW solutions. But submission of software solution will result in zero credit.
2. Groups of 2 for the projects (no two projects to be the same/similar)
   A single grade will be assigned to the group and not to the individuals.

Homeworks (15%)

1. Be as detailed and explicit as possible. For full credit Do NOT omit steps.
2. Only neatly written homeworks will be graded
3. Late homeworks will NOT be accepted.
4. Two lowest grades will be dropped (except HW #1).
5. Solutions will be posted on the course website
Practicum (10%)

1. Five classes designated as “Practicum”.
2. You will need to download and install NX 10 on your laptops and bring them to class on these days.
3. At the end of each practicum, you will be assigned a single problem (worth 2 points).
4. You will need to hand in the solution to the TA within a week of the assignment.
5. No late submissions will be entertained.

Course Project (25 %)

In this project you will be required to:
• choose an engineering system
• develop a mathematical model for the system
• develop the finite element model
• solve the problem using commercial software
• present a convergence plot and discuss whether the mathematical model you chose gives you physically meaningful results.
• refine the model if necessary.

Course project (25 %).contd.

Logistics:
• Form groups of 2 and email the TA by 22nd September.
• Submit 1-page project proposal latest by 17th October (in class). The earlier the better. Projects will go on a first come first served basis.
• Proceed to work on the project ONLY after it is approved by the course instructor.
• Submit a one-page progress report on November 10th (this will count as 10% of your project grade)
• Submit a project report (hard copy) by noon of 12th December to the instructor.

Major project (25 %).contd.

Project report:
1. Must be professional (Text font Times 11pt with single spacing)
2. Must include the following sections:
   • Introduction
   • Problem statement
   • Analysis
   • Results and Discussions
Project examples:
(two sample project reports from previous year are provided)
1. Analysis of a rocker arm
2. Analysis of a bicycle crank-pedal assembly
3. Design and analysis of a "portable stair climber"
4. Analysis of a gear train
5. Gear tooth stress in a wind-up clock
6. Analysis of a gear box assembly
7. Analysis of an artificial knee
8. Forces acting on the elbow joint
9. Analysis of a soft tissue tumor system
10. Finite element analysis of a skateboard truck

Project grade will depend on
1. Originality of the idea
2. Techniques used
3. Critical discussion
Course content

1. "Direct Stiffness" approach for springs
2. Bar elements and truss analysis
3. Introduction to boundary value problems: strong form, principle of minimum potential energy and principle of virtual work.
6. Discussion on issues in practical FEM modeling
7. Convergence of finite element results
8. Higher order elements
9. Isoparametric formulation
10. Numerical integration in 2D
11. Solution of linear algebraic equations

For next class

Please read Appendix A of Logan for reading quiz next class (10 pts on Hw 1)

Linear Algebra Recap (at the IEA level)

What is a matrix?

A rectangular array of numbers (we will concentrate on real numbers). A nxm matrix has ‘n’ rows and ‘m’ columns

$$
M_{3x4} = \begin{bmatrix}
M_{11} & M_{12} & M_{13} & M_{14} \\
M_{21} & M_{22} & M_{23} & M_{24} \\
M_{31} & M_{32} & M_{33} & M_{34}
\end{bmatrix}
$$

First row
Second row
Third row
Fourth row
First column
Second column
Third column
Fourth column
Row number
Column number
What is a vector?

A vector is an array of ‘n’ numbers

A row vector of length ‘n’ is a 1xn matrix

\[
\begin{bmatrix}
 a_1 \\
 a_2 \\
 a_3 \\
 a_4
\end{bmatrix}
\]

A column vector of length ‘m’ is a mx1 matrix

\[
\begin{bmatrix}
 a_1 \\
 a_2 \\
 a_3
\end{bmatrix}
\]

Special matrices

Zero matrix: A matrix all of whose entries are zero

\[
0_{3\times4} = 
\begin{bmatrix}
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0
\end{bmatrix}
\]

Identity matrix: A square matrix which has ‘1’ s on the diagonal and zeros everywhere else.

\[
I_{3\times3} = 
\begin{bmatrix}
 1 & 0 & 0 \\
 0 & 1 & 0 \\
 0 & 0 & 1
\end{bmatrix}
\]

Matrix operations

Equality of matrices

If \( A \) and \( B \) are two matrices of the same size, then they are “equal” if each and every entry of one matrix equals the corresponding entry of the other.

\[
A = 
\begin{bmatrix}
 1 & 2 & 4 \\
 -3 & 0 & 7 \\
 9 & 1 & 5
\end{bmatrix}
\]

\[
B = 
\begin{bmatrix}
 a & b & c \\
 d & e & f \\
 g & h & i
\end{bmatrix}
\]

\( a = 1, \quad b = 2, \quad c = 4, \)

\( A = B \iff d = -3, \quad e = 0, \quad f = 7, \)

\( g = 9, \quad h = 1, \quad i = 5. \)

Addition of two matrices

If \( A \) and \( B \) are two matrices of the same size, then the sum of the matrices is a matrix \( C = A + B \) whose entries are the sums of the corresponding entries of \( A \) and \( B \).

\[
A = 
\begin{bmatrix}
 1 & 2 & 4 \\
 -3 & 0 & 7 \\
 9 & 1 & 5
\end{bmatrix}
\]

\[
B = 
\begin{bmatrix}
 -1 & 3 & 10 \\
 -3 & 1 & 0 \\
 1 & 0 & 6
\end{bmatrix}
\]

\[
C = A + B = 
\begin{bmatrix}
 0 & 5 & 14 \\
 -6 & 1 & 7 \\
 10 & 1 & 11
\end{bmatrix}
\]
Matrix operations

Properties of matrix addition:
1. Matrix addition is **commutative** (order of addition does not matter)
   \[ \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \]
2. Matrix addition is **associative**
   \[ \mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C} \]
3. Addition of the zero matrix
   \[ \mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A} \]

Matrix operations

### Multiplication by a scalar

If \( \mathbf{A} \) is a matrix and \( c \) is a scalar, then the product \( c\mathbf{A} \) is a matrix whose entries are obtained by multiplying each of the entries of \( \mathbf{A} \) by \( c \)

\[
\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & 7 \\ 9 & 1 & 5 \end{bmatrix}, \quad c = 3
\]

\[
c\mathbf{A} = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & 21 \\ 27 & 3 & 15 \end{bmatrix}
\]

Matrix operations

### Special case

If \( \mathbf{A} \) is a matrix and \( c = -1 \) is a scalar, then the product \(-\mathbf{A} \) is a matrix whose entries are obtained by multiplying each of the entries of \( \mathbf{A} \) by \(-1\)

\[
\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & 7 \\ 9 & 1 & 5 \end{bmatrix}, \quad c = -1
\]

\[
c\mathbf{A} = -\mathbf{A} = \begin{bmatrix} -1 & -2 & -4 \\ 3 & 0 & -7 \\ -9 & -1 & -5 \end{bmatrix}
\]

Matrix operations

### Subtraction

If \( \mathbf{A} \) and \( \mathbf{B} \) are two square matrices of the same size, then \( \mathbf{A} - \mathbf{B} \) is defined as the sum \( \mathbf{A} + (\mathbf{-B}) \)

\[
\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & 7 \\ 9 & 1 & 5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -1 & 3 & 10 \\ -3 & 1 & 0 \\ 9 & 1 & 5 \end{bmatrix}
\]

\[
\mathbf{C} = \mathbf{A} - \mathbf{B} = \begin{bmatrix} 2 & -1 & -6 \\ 0 & -1 & 7 \\ 8 & 1 & -1 \end{bmatrix}
\]

Note that \( \mathbf{A} - \mathbf{A} = \mathbf{0} \) and \( \mathbf{0} - \mathbf{A} = -\mathbf{A} \)
If $A$ is a $m \times n$ matrix, then the transpose of $A$ is the $n \times m$ matrix whose first column is the first row of $A$, whose second column is the second column of $A$ and so on.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & 7 \\ 9 & 1 & 5 \end{bmatrix} \quad \Leftrightarrow \quad A^T = \begin{bmatrix} 1 & -3 & 9 \\ 2 & 0 & 1 \\ 4 & 7 & 5 \end{bmatrix}$$

If $A$ is a square matrix ($m \times m$), it is called symmetric if $A = A^T$.

If $a$ and $b$ are two vectors of the same size

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The scalar (dot) product of $a$ and $b$ is a scalar obtained by adding the products of corresponding entries of the two vectors

$$a^Tb = (a_1b_1 + a_2b_2 + a_3b_3)$$

For a product to be defined, the number of columns of $A$ must be equal to the number of rows of $B$.

$$A_{m \times r} \quad B_{r \times n} \quad AB_{m \times n}$$
If $A$ is an $m \times r$ matrix and $B$ is an $r \times n$ matrix, then the product $C = AB$ is an $m \times n$ matrix whose entries are obtained as follows. The entry corresponding to row $'i'$ and column $'j'$ of $C$ is the dot product of the vectors formed by the row $'i'$ of $A$ and column $'j'$ of $B$.

\[
A_{3 \times 3} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & 7 \\ 9 & 1 & 5 \end{bmatrix}, \quad B_{2 \times 2} = \begin{bmatrix} 1 & 3 \\ 5 & 1 \end{bmatrix}
\]

\[
C_{3 \times 2} = AB = \begin{bmatrix} 10 & -9 \\ -7 & 28 \end{bmatrix}
\]

Notice, $C = AB$ is a $3 \times 2$ matrix, but $BA$ does not exist (e.g., in the previous example $C = AB$ is a $3 \times 2$ matrix, but $BA$ does not exist).

Even if the product exists, the products $AB$ and $BA$ are not generally the same.

### Properties of Matrix Multiplication:

1. **Noncommutative** (order of addition does matter)
   
   $AB \neq BA$ in general

   - It may be that the product $AB$ exists but $BA$ does not (e.g., in the previous example $C = AB$ is a $3 \times 2$ matrix, but $BA$ does not exist).
   - Even if the product exists, the products $AB$ and $BA$ are not generally the same.

2. **Associative**
   
   $A(BC) = (AB)C$

3. **Distributive Law**
   
   $A(B + C) = AB + AC$
   
   $(B + C)A = BA + CA$

4. **Multiplication by Identity Matrix**
   
   $AI = A; \quad IA = A$

5. **Multiplication by Zero Matrix**
   
   $A0 = 0; \quad 0A = 0$

6. **Transpose**
   
   $(AB)^T = B^T A^T$

### Miscellaneous Properties

1. If $A$, $B$, and $C$ are square matrices of the same size, and $A \neq 0$ then $AB = AC$ does not necessarily mean that $B = C$.

2. $AB = 0$ does not necessarily imply that either $A$ or $B$ is zero.
### Inverse of a Matrix

**Definition**

If \( A \) is any *square matrix* and \( B \) is another square matrix satisfying the conditions

\[
AB = BA = I
\]

Then

(a) The matrix \( A \) is called *invertible*, and

(b) the matrix \( B \) is the inverse of \( A \) and is denoted as \( A^{-1} \).

The inverse of a matrix is *unique*.

### Uniqueness

The inverse of a matrix is unique

Assume that \( B \) and \( C \) are both inverses of \( A \)

\[
AB = BA = I, \\
AC = CA = I, \\
(BC)A = AC = C, \\
B(AC) = BI = B \\
\therefore B = C
\]

Hence a matrix cannot have two or more inverses.

### Some Properties

**Property 1:** If \( A \) is any invertible square matrix the inverse of its inverse is the matrix \( A \) itself

\[
(A^{-1})^{-1} = A
\]

**Property 2:** If \( A \) is any invertible square matrix and \( k \) is any scalar then

\[
(kA)^{-1} = \frac{1}{k} A^{-1}
\]

### Properties

**Property 3:** If \( A \) and \( B \) are invertible square matrices then

\[
(AB)^{-1} = B^{-1} A^{-1}
\]

Premultiplying both sides by \( A^{-1} \)

\[
A^{-1}(AB) = A^{-1} A^{-1}
\]

Premultiplying both sides by \( B^{-1} \)

\[
B^{-1} A^{-1} A^{-1} = B^{-1} A^{-1}
\]
The determinant of a square matrix is a number obtained in a specific manner from the matrix.

For a 1x1 matrix:
\[ A = [a_{11}] ; \quad \det(A) = a_{11} \]

For a 2x2 matrix:
\[ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} ; \quad \det(A) = a_{11}a_{22} - a_{12}a_{21} \]

Product along red arrow minus product along blue arrow.

**Example 1**
Consider the matrix
\[ A = \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} \]

Notice (1) A matrix is an array of numbers
(2) A matrix is enclosed by square brackets

\[ \det(A) = \begin{vmatrix} 1 & 3 \\ 5 & 7 \end{vmatrix} = 1 \times 7 - 3 \times 5 = -8 \]

Notice (1) The determinant of a matrix is a number
(2) The symbol for the determinant of a matrix is a pair of parallel lines.

**Duplicate column method for 3x3 matrix**
For **ONLY** a 3x3 matrix write down the first two columns after the third column

\[ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \]

Sum of products along red arrow minus sum of products along blue arrow

\[ \det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \]

This technique works only for 3x3 matrices.

**Example**
\[ A = \begin{bmatrix} 2 & 4 & -3 \\ 1 & 0 & 4 \\ 2 & -1 & 2 \end{bmatrix} \]

Sum of red terms = 0 + 32 + 3 = 35
Sum of blue terms = 0 – 8 + 8 = 0
Determinant of matrix \( A \) = \( \det(A) = 35 - 0 = 35 \)
Finding determinant using inspection

Special case. If two rows or two columns are proportional (i.e. multiples of each other), then the determinant of the matrix is zero

\[
\begin{vmatrix}
2 & 7 & 8 \\
3 & 2 & 4 \\
-2 & -7 & -8
\end{vmatrix} = 0
\]

because rows 1 and 3 are proportional to each other

If the determinant of a matrix is zero, it is called a singular matrix

Cofactor method

If A is a square matrix

\[
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

The minor, \(M_{ij}\), of entry \(a_{ij}\) is the determinant of the submatrix that remains after the \(i^{th}\) row and \(j^{th}\) column are deleted from \(A\).

The cofactor of entry \(a_{ij}\) is \(C_{ij} = (-1)^{(i+j)} M_{ij}\)

\[
M_{12} = \begin{vmatrix}
a_{21} & a_{23} \\
a_{31} & a_{33}
\end{vmatrix} = a_{21}a_{33} - a_{23}a_{31} = -M_{12} = -\begin{vmatrix}
a_{11} & a_{13} \\
a_{11} & a_{13}
\end{vmatrix}
\]

What is a cofactor?

Sign of cofactor

\[
\begin{bmatrix}
+ & - & + \\
- & + & - \\
+ & - & +
\end{bmatrix}
\]

Find the minor and cofactor of \(a_{33}\)

\[
A = \begin{bmatrix}
2 & 4 & -3 \\
1 & 0 & 4 \\
2 & -1 & 2
\end{bmatrix}
\]

Minors

\[
M_{33} = \begin{vmatrix}
2 & 4 \\
1 & 0
\end{vmatrix} = 2 \times 0 - 4 \times 1 = -4
\]

Cofactor

\[
C_{33} = (-1)^{(3+3)} M_{33} = M_{33} = -4
\]

Cofactor method of obtaining the determinant of a matrix

The determinant of a \(n \times n\) matrix \(A\) can be computed by multiplying ALL the entries in ANY row (or column) by their cofactors and adding the resulting products. That is, for each \(1 \leq i \leq n\) and \(1 \leq j \leq n\)

Cofactor expansion along the \(j^{th}\) column

\[
\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}
\]

Cofactor expansion along the \(i^{th}\) row

\[
\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}
\]
Example: evaluate det(A) for:

\[
A = \begin{pmatrix}
1 & 0 & 2 & -3 \\
3 & 4 & 0 & 1 \\
-1 & 5 & 2 & -2 \\
0 & 1 & 1 & 3
\end{pmatrix}
\]

\[
det(A) = det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14}
\]

\[
det(A) = (1)
\]

\[
4 & 0 & 1 \\
5 & 2 & -2 \\
1 & 1 & 3 \\
3 & 4 & 0
\]

\[
0 & 1 & 3 \\
-1 & 1 & 1 \\
0 & 1 & 1
\]

\[
det(A) = (1)(35) - 0 + 2(62) - (-3)(13) = 198
\]

Example: evaluate

\[
det(A) = \begin{pmatrix}
1 & 5 & -3 \\
1 & 0 & 2 \\
3 & -1 & 2
\end{pmatrix}
\]

By a cofactor along the third column

\[
det(A) = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}
\]

\[
det(A) = -3((-1)^3 + 2((-1)^3 + 1)) = 198
\]

Quadratic form

The scalar

\[
U = d^T k d
\]

\[
d = \text{vector}
\]

\[
k = \text{square matrix}
\]

Is known as a **quadratic form**

If U>0: Matrix k is known as **positive definite**

If U≥0: Matrix k is known as **positive semidefinite**

Quadratic form

Let

\[
d = \begin{pmatrix}
d_1 \\
d_2
\end{pmatrix}
\]

\[
k = \begin{pmatrix}
k_{11} & k_{12} \\
k_{21} & k_{22}
\end{pmatrix}
\]

Then

\[
U = d^T k d = d_1^2 k_{11} + k_{12} d_1 d_2 + d_2^2 k_{22}
\]

\[
= d_1^2 (k_{11}d_1^2 + k_{12}d_2) + d_2^2 (k_{11}d_1 + k_{22}d_2)
\]

\[
= k_{11}d_1^2 + 2k_{12}d_1 d_2 + k_{22}d_2^2
\]
Differentiation of quadratic form

Differentiate \( U \) wrt \( d_1 \)
\[
\frac{\partial U}{\partial d_1} = 2k_{11}d_1 + 2k_{12}d_2
\]

Differentiate \( U \) wrt \( d_2 \)
\[
\frac{\partial U}{\partial d_2} = 2k_{12}d_1 + 2k_{22}d_2
\]

Hence
\[
\frac{\partial U}{\partial d} = \begin{bmatrix}
\frac{\partial U}{\partial d_1} \\
\frac{\partial U}{\partial d_2}
\end{bmatrix} = 2\begin{bmatrix}
k_{11} & k_{12} \\
k_{12} & k_{22}
\end{bmatrix}\begin{bmatrix}
d_1 \\
d_2
\end{bmatrix} = 2k_{dd}
\]

Outline

- Role of FEM simulation in Engineering Design
- Course Philosophy

Role of simulation in design:
Boeing 777

Another success ..in failure:

Airbus A380


Drag Force Analysis of Aircraft

- Question
  What is the drag force distribution on the aircraft?
- Solve
  - Navier-Stokes Partial Differential Equations.
- Recent Developments
  - Multigrid Methods for Unstructured Grids

San Francisco Oakland Bay Bridge

Before the 1989 Loma Prieta earthquake

San Francisco Oakland Bay Bridge

After the earthquake
San Francisco Oakland Bay Bridge

A finite element model to analyze the bridge under seismic loads
Courtesy: ADINA R&D

Crush Analysis of Ford Windstar

- Question
  - What is the load-deformation relation?
- Solve
  - Partial Differential Equations of Continuum Mechanics
- Recent Developments
  - Meshless Methods, Iterative methods, Automatic Error Control

Engine Thermal Analysis

Picture from http://www.adina.com

- Question
  - What is the temperature distribution in the engine block?
- Solve
  - Poisson Partial Differential Equation.
- Recent Developments
  - Fast Integral Equation Solvers, Monte-Carlo Methods

Electromagnetic Analysis of Packages

Thanks to Coventor http://www.coventer.com

- Solve
  - Maxwell's Partial Differential Equations
- Recent Developments
  - Fast Solvers for Integral Formulations
Micromachine Device Performance Analysis

From www.memscap.com

- Equations
  - Elastomechanics, Electrostatics, Stokes Flow
- Recent Developments

Radiation Therapy of Lung Cancer

http://www.simulia.com/academics/research_lung.html

Virtual Surgery

Engineering design

Physical Problem

Question regarding the problem
...how large are the deformations?
...how much is the heat transfer?

Mathematical model

Governed by differential equations

Assumptions regarding
Geometry
Kinematics
Material law
Loading
Boundary conditions
Etc.
Example: A bracket

(Physical problem)

Questions:
1. What is the bending moment at section AA?
2. What is the deflection at the pin?

(Mathematical model 1: beam)

Moment at section AA
\[ M = WL \]
\[ = 27,500 \text{ N cm} \]

Deflection at load
\[ \delta_{\text{load}} = \frac{1}{3} \frac{W(L + f_0)^2}{EI} + \frac{W(L + f_0)}{6AG} \]
\[ = 0.053 \text{ cm} \]

How reliable is this model?
How effective is this model?

(Mathematical model 2: plane stress)

Difficult to solve by hand!

..General scenario..

Physical Problem

Mathematical model

Governed by differential equations

Numerical model

e.g., finite element model
PREPROCESSING
1. Create a geometric model
2. Develop the finite element model

FINITE ELEMENT MODEL (FEM)

FEM analysis scheme

Step 1: Divide the problem domain into non-overlapping regions ("elements") connected to each other through special points ("nodes")

Step 2: Describe the behavior of each element

Step 3: Describe the behavior of the entire body by putting together the behavior of each of the elements (this is a process known as "assembly")

POSTPROCESSING
Compute moment at section AA
FEM solution to mathematical model 2 (plane stress)
Moment at section AA \( M = 27,500 \text{ N cm} \)
Deflection at load \( \delta_{\text{at load}} = 0.064 \text{ cm} \)

**Conclusion:** With respect to the questions we posed, the beam model is **reliable** if the required bending moment is to be predicted within 1% and the deflection is to be predicted within 20%. The beam model is also highly **effective** since it can be solved easily (by hand).

What if we asked: what is the maximum stress in the bracket? would the beam model be of any use?

---

1. The **selection** of the mathematical model depends on the response to be predicted.
2. The **most effective** mathematical model is the one that delivers the answers to the questions in reliable manner with least effort.
3. **The numerical solution is only as accurate as the mathematical model.**
Critical assessment of the FEM

For a well-posed mathematical problem the numerical technique should always, for a reasonable discretization, give a reasonable solution which must converge to the accurate solution as the discretization is refined.

e.g., use of reduced integration in FEM results in an unreliable analysis procedure.

The performance of the numerical method should not be unduly sensitive to the material data, the boundary conditions, and the loading conditions used.

e.g., displacement based formulation for incompressible problems in elasticity.