Recall that the finite element shape functions need to satisfy the following properties

1. Kronecker delta property

\[ N_i = \begin{cases} 1 & \text{at node } i \\ 0 & \text{at all other nodes} \end{cases} \]

Inside an element

\[ u = N_1 \mu_1 + N_2 \mu_2 + \ldots \]

At node 1, \( N_1 = 1, N_2 = N_3 = \ldots = 0 \), hence

\[ u_{\text{node 1}} = u_i \]

Facilitates the imposition of boundary conditions

2. Polynomial completeness

If \( u = \alpha_1 + \alpha_2 x + \alpha_3 y \)

Then

\[ \sum N_i = 1 \]
\[ \sum N_i \alpha_i = x \]
\[ \sum N_i \beta_i = y \]
Higher order elements in 1D

2-noded (linear) element:

\[ x_1 \quad x_2 \quad x \]
\[ 1 \quad 2 \]

\[ N_i = \frac{x-x_1}{x_2-x_1} \]
\[ N_j = \frac{x-x_2}{x_1-x_2} \]

In "local" coordinate system (shifted to center of element)

\[ x \]
\[ 1 \quad a \quad a \quad 2 \]

\[ N_i = \frac{a-x}{2a} \]
\[ N_j = \frac{a+x}{2a} \]

3-noded (quadratic) element:

\[ x_1 \quad x_3 \quad x_2 \quad x \]
\[ 1 \quad 3 \quad 2 \]

\[ N_i = \frac{(x-x_1)(x-x_3)(x-x_2)}{(x_3-x_1)(x_2-x_1)(x_1-x_2)} \]
\[ N_j = \frac{(x-x_3)(x-x_1)(x-x_2)}{(x_2-x_3)(x_1-x_2)(x_3-x_1)} \]
\[ N_k = \frac{(x-x_2)(x-x_3)(x-x_1)}{(x_1-x_2)(x_3-x_1)(x_2-x_3)} \]

In "local" coordinate system (shifted to center of element)

\[ x \]
\[ 1 \quad a \quad 3 \quad 2 \]

\[ N_i = -\frac{(a-x)}{2a^2} \]
\[ N_j = \frac{x(a+x)}{2a^2} \]
\[ N_k = \frac{a^3-x^3}{a} \]

4-noded (cubic) element:

\[ x_1 \quad x_3 \quad x_4 \quad x_2 \quad x \]
\[ 1 \quad 3 \quad 4 \quad 2 \]

\[ N_i = \frac{(x-x_1)(x-x_3)(x-x_4)(x-x_2)}{(x_3-x_1)(x_2-x_1)(x_1-x_2)(x_2-x_4)} \]
\[ N_j = \frac{(x-x_3)(x-x_1)(x-x_4)(x-x_2)}{(x_2-x_3)(x_1-x_2)(x_3-x_1)(x_1-x_4)} \]
\[ N_k = \frac{(x-x_4)(x-x_1)(x-x_3)(x-x_2)}{(x_1-x_4)(x_3-x_1)(x_2-x_3)(x_3-x_2)} \]
\[ N_l = \frac{(x-x_2)(x-x_1)(x-x_3)(x-x_4)}{(x_4-x_2)(x_1-x_4)(x_3-x_2)(x_2-x_3)} \]

In "local" coordinate system (shifted to center of element)

\[ x \]
\[ 1 \quad 3 \quad 4 \quad a \quad 2 \]

\[ N_i = \frac{-9}{16a}(a-x)(x-a/3)(x+a/3) \]
\[ N_j = \frac{9}{16a}(x+a)(x-a/3)(x+a/3) \]
\[ N_k = \frac{-27}{16a}(a-x)(a/3-x)(a+x) \]
\[ N_l = \frac{27}{16a}(x-a)(x-a/3)(x+a/3) \]

Polynomial completeness

1 node; k=1; p=2

2 node; 3 node; 4 node; k=2; p=3 k=3; p=4

Recall that the convergence in displacements

\[ |u-u_h| \leq C h^k \]
\[ k=\text{order of complete polynomial} \]
**Triangular elements**

Area coordinates \((L_1, L_2, L_3)\)

Total area of the triangle \(A = A_1 + A_2 + A_3\)

At any point \(P(x,y)\) inside the triangle, we define

\[
L_i = \frac{A_i}{A}
\]

Note: Only 2 of the three area coordinates are independent, since \(L_1 + L_2 + L_3 = 1\)

\[
A = \frac{1}{2} \det \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix}
\]

\[
a_1 = x_2 y_3 - x_3 y_2 \\
b_1 = y_3 - y_2 \\
c_1 = x_3 - x_2 \\
\ldots\]

Check that

\[
L_1 + L_2 + L_3 = 1 \\
L_1 x_1 + L_2 x_2 + L_3 x_3 = x \\
L_1 y_1 + L_2 y_2 + L_3 y_3 = y
\]

Lines parallel to the base of the triangle are lines of constant ‘L’
We will develop the shape functions of triangular elements in terms of the area coordinates.

For a 3-noded triangle:

\[ N_1 = L_1 \]
\[ N_2 = L_2 \]
\[ N_3 = L_3 \]

How to write down the expression for \( N_1 \)?

Realize the \( N_1 \) must be zero along edge 2-3 (i.e., \( L_1=0 \)) and at nodes 4&6 (which lie on \( L_1=1/2 \))

Determine the constant ‘c’ from the condition that \( N_1=1 \) at node 1 (i.e., \( L_1=1 \))

\[ N_1 = c \left( L_1 - 0 \right) \left( L_1 - 1/2 \right) \]

\[ N_1 \text{ (at } L_1 = 1) = c \left( 1 - 1/2 \right) = 1 \]

\[ \Rightarrow c = 2 \]

\[ \therefore N_1 = 2 L_1 \left( L_1 - 1/2 \right) \]
\[ N_1 = 2L_1(L_1 - 1/2) \]
\[ N_2 = 2L_2(L_2 - 1/2) \]
\[ N_3 = 2L_3(L_3 - 1/2) \]
\[ N_4 = 4L_1L_2 \]
\[ N_5 = 4L_3L_2 \]
\[ N_6 = 4L_3L_1 \]

For a 10-noded triangle

\[ L_1 = \frac{2}{3} \]
\[ L_2 = \frac{2}{3} \]
\[ L_3 = \frac{1}{3} \]
\[ L_4 = \frac{1}{3} \]
\[ L_5 = 0 \]
\[ L_6 = \frac{1}{3} \]
\[ L_7 = \frac{2}{3} \]
\[ L_8 = \frac{2}{3} \]
\[ L_9 = 0 \]
\[ L_{10} = 1 \]

NOTES:
1. Polynomial completeness

\[ \begin{array}{ccc}
   x & y & x^2 \\
   xy & y^2 & x^2y \\
x^3 & y^3 & x^3y^3
\end{array} \]

Convergence rate (displacement)

3 node; \( k=1; p=2 \)
6 node; \( k=2; p=3 \)
10 node; \( k=3; p=4 \)
2. Integration on triangular domain

1. \( \int L_1^m L_2^m L_3^m \, dA = 2A \cdot \frac{k!m!n!}{(2 + k + m + n)!} \)

2. \( \int_{L_{1;2}} L_1^m \, dS = l_{1;2} \cdot \frac{k!m!}{(1 + k + m)!} \)

3. Computation of derivatives of shape functions: use chain rule
e.g.,
\[ \frac{\partial N_i}{\partial x} \]
But
\[ \frac{\partial L_1}{\partial x} = \frac{b_1}{2A}, \quad \frac{\partial L_2}{\partial x} = \frac{b_2}{2A} \]
e.g., for the 6-noded triangle
\[ N_i = 4L_1L_2 \]
\[ \frac{\partial N_i}{\partial x} = 4L_1 \frac{b_1}{2A} + 4L_2 \frac{b_2}{2A} \]

Rectangular elements

Lagrange family

Serendipity family

Lagrange family

4-noded rectangle

In local coordinate system
\[ N_1 = \frac{(a + x)(b + y)}{4ab}, \quad N_2 = \frac{(a - x)(b + y)}{4ab}, \quad N_3 = \frac{(a - x)(b - y)}{4ab}, \quad N_4 = \frac{(a + x)(b - y)}{4ab} \]

9-noded quadratic

Corner nodes
\[ N_1 = \frac{(ab + x)(ab + y)}{2a}, \quad N_2 = \frac{(a-x)(y+y)}{2b}, \quad N_3 = \frac{(a-x)(y-y)}{2b}, \quad N_4 = \frac{(a-x)(y+y)}{2b} \]
Midside nodes
\[ N_1 = \frac{(a-x)(y+y)}{2b}, \quad N_2 = \frac{(a-x)(y-y)}{2b}, \quad N_3 = \frac{(a-x)(y+y)}{2b}, \quad N_4 = \frac{(a-x)(y-y)}{2b} \]
Center node
\[ N_1 = \frac{(a-x)(y-y)}{2b}, \quad N_2 = \frac{(a-x)(y+y)}{2b} \]
NOTES:
1. Polynomial completeness

Convergence rate (displacement)

4 node; p=2
9 node; p=3

Lagrange shape functions contain higher order terms but miss out lower order terms

Serendipity family

4-noded same as Lagrange

8-noded rectangle: how to generate the shape functions?

First generate the shape functions of the midside nodes as appropriate products of 1D shape functions, e.g.,

8-noded rectangle:

Then go to the corner nodes. At each corner node, first assume a bilinear shape function as in a 4-noded element and then modify:

“bilinear” shape fn at node 1: \( \hat{N}_i = \frac{(a+x)(b+y)}{4ab} \)

actual shape fn at node 1: \( N_i = \hat{N}_i - \frac{N_2}{2} - \frac{N_4}{2} \)

8-noded rectangle:

Midside nodes

Corner nodes

NOTES:
1. Polynomial completeness

Convergence rate (displacement)

4 node; p=2
8 node; p=3
12 node; p=4
16 node; p=4

More even distribution of polynomial terms than Lagrange shape functions but ‘p’ cannot exceed 4!