MANE 4240 & CIVL 4240
Introduction to Finite Elements

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Finite element formulation for
1D elasticity using the
Rayleigh-Ritz Principle

Reading assignment:
Lecture notes, Logan 3.10

Summary:
• Stiffness matrix and nodal load vectors for 1D elasticity problem

Axially loaded elastic bar

\[ A(x) = \text{cross section at } x \]
\[ b(x) = \text{body force distribution (force per unit length)} \]
\[ E(x) = \text{Young’s modulus} \]

Potential energy of the axially loaded bar corresponding to the
exact solution \( u(x) \)

\[ \Pi(u) = \frac{1}{2} \int_{0}^{L} E(x) \left( \frac{du}{dx} \right)^2 \, dx - \int_{0}^{L} b(x) \, dx - F(x = L) \]

Potential energy of the bar corresponding to an admissible

\[ \Pi(w) = \frac{1}{2} \int_{0}^{L} E(x) \left( \frac{dw}{dx} \right)^2 \, dx - \int_{0}^{L} b(x) \, dx - F(x = L) \]

Finite element idea:

**Step 1:** Divide the truss into finite elements connected to each other through special points ("nodes")

Total potential energy= sum of potential energies of the elements

\[ \Pi(w) = \frac{1}{2} \int_{0}^{L} E(x) \left( \frac{dw}{dx} \right)^2 \, dx - \int_{0}^{L} b(x) \, dx - F(x = L) \]
**Total potential energy**

\[ \Pi(w) = \frac{1}{2} \int_x^y EA \left( \frac{dw}{dx} \right)^2 dx - \int_x^y bw \, dx - Fw(x = L) \]

**Potential energy of element 1:**

\[ \Pi_1(w) = \frac{1}{2} \int_{x_1}^{x_2} EA \left( \frac{dw}{dx} \right)^2 dx - \int_{x_1}^{x_2} bw \, dx \]

**Potential energy of element 2:**

\[ \Pi_2(w) = \frac{1}{2} \int_{x_3}^{x_4} EA \left( \frac{dw}{dx} \right)^2 dx - \int_{x_3}^{x_4} bw \, dx \]

**Potential energy of element 3:**

\[ \Pi_3(w) = \frac{1}{2} \int_{x_5}^{x_6} EA \left( \frac{dw}{dx} \right)^2 dx - \int_{x_5}^{x_6} bw \, dx - Fw(x = L) \]

Total potential energy = sum of potential energies of the elements

\[ \Pi(w) = \Pi_1(w) + \Pi_2(w) + \Pi_3(w) \]

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**Step 2:** Describe the behavior of each element

In the “direct stiffness” approach, we derived the stiffness matrix of each element directly (See lecture on Springs/Trusses).

Now, we will first approximate the displacement inside each element and then show you a systematic way of deriving the stiffness matrix (sections 2.2 and 3.1 of Logan).

**Task 1:** Approximate the displacement within each element

**Task 2:** Approximate the strain and stress within each element

**Task 3:** Derive the stiffness matrix of each element (this class) using the Rayleigh-Ritz principle

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**Summary**

Inside an element, the three most important approximations in terms of the nodal displacements \( d \) are:

**Displacement approximation** in terms of shape functions

\[ w(x) = N \, d \]  \hspace{1cm} (1)

**Strain approximation** in terms of strain-displacement matrix

\[ \varepsilon(x) = B \, d \]  \hspace{1cm} (2)

**Stress approximation** in terms of strain-displacement matrix and Young’s modulus

\[ \sigma = EB \, d \]  \hspace{1cm} (3)
The shape functions for a 1D linear element

\[ N_1(x) = \frac{x_2 - x}{x_2 - x_1} \]

\[ N_1(x) = \frac{x - x_1}{x_2 - x_1} \]

Within the element, the displacement approximation is

\[ w(x) = \frac{x_2 - x}{x_2 - x_1} d_{1x} + \frac{x - x_1}{x_2 - x_1} d_{2x} \]

For a linear element

Displacement approximation in terms of shape functions

\[ w(x) = \begin{bmatrix} \frac{x_2 - x}{x_2 - x_1} & \frac{x - x_1}{x_2 - x_1} \end{bmatrix} \begin{bmatrix} d_{1x} \\ d_{2x} \end{bmatrix} \]

Strain approximation

\[ \varepsilon = \frac{1}{E} \frac{d w}{d x} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} d_{1x} \\ d_{2x} \end{bmatrix} \]

Stress approximation

\[ \sigma = E \varepsilon = E \begin{bmatrix} \frac{d w}{d x} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} d_{1x} \\ d_{2x} \end{bmatrix} \]

Why is the approximation “admissible”? For the entire bar, the displacement approximation is

\[ w(x) = w^{(1)}(x) + w^{(2)}(x) + w^{(3)}(x) \]

Where \( w^{(i)}(x) \) is the displacement approximation within element \( i \). Let use set \( d_{i=0} = 0 \). Then, can you see that the above approximation does satisfy the two conditions of being an admissible function on the entire bar, i.e.,

1. \( w(x = 0) = 0 \)
2. \( \frac{d w}{d x} \) exists

TASK 3: DERIVE THE STIFFNESS MATRIX OF EACH ELEMENT USING THE RAYLEIGH-RITZ PRINCIPLE

Potential energy of element 1:

\[ \Pi_i(w) = 0 \int_{x_i}^{x_{i+1}} \sigma A dx - \int_{x_i}^{x_{i+1}} b w dx \]

Let's plug in the approximation

\[ w(x) = N d \]

\[ \varepsilon(x) = B d \]

\[ \sigma = E B d \]

\[ \Pi_i(d) = \frac{1}{2} d^T \left( \int_{x_i}^{x_{i+1}} B^T E B \right) d - d^T \left( \int_{x_i}^{x_{i+1}} N^T b \right) dx \]
Let's see what the matrix
\[ \int B^T E B \, A \, dx \]
is for a 1D linear element
Recall that
\[ B = \frac{1}{x_2 - x_1} \begin{bmatrix} -1 & 1 \end{bmatrix} \]
Hence
\[ B^T E B = \frac{1}{x_2 - x_1} \begin{bmatrix} -1 & 1 \end{bmatrix} E \begin{bmatrix} -1 & 1 \end{bmatrix} \]
\[ = \frac{E}{(x_2 - x_1)^2} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} \]
\[ = \frac{E}{(x_2 - x_1)^2} \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \]

Then why is it necessary to go through this complicated procedure??
1. Easy to handle nonuniform E and A
2. Easy to handle distributed loads
For nonuniform E and A, i.e. E(x) and A(x), the **stiffness matrix** of the linear element will NOT be
\[ \frac{E A}{(x_2 - x_1)} \begin{bmatrix} 1 & -1 \end{bmatrix} \]
But it will **ALWAYS be**
\[ \int [B^T E B] \, A \, dx \]

Now, if we assume E and A are constant
\[ \int [B^T E B] \, A \, dx = \int \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} \]
\[ = \frac{AE}{(x_2 - x_1)^2} \begin{bmatrix} 1 & -1 \end{bmatrix} \]
Remembering that \((x_2 - x_1)\) is the length of the element, this is the **stiffness matrix** we had derived directly before using the **direct stiffness** approach!!

Now let's go back to
\[ \Pi([d]) = \int \frac{1}{2} \int [B^T E B] \, A \, dx \, d - d' \int [N^T] b \, dx \]
Element stiffness matrix
\[ \begin{bmatrix} [B^T E B] \, A \, dx \end{bmatrix} \]
Element nodal load vector due to distributed body force
\[ f_b = \int [N^T] b \, dx \]
Apply Rayleigh-Ritz principle for the 1D linear element
\[
\frac{\partial \Pi_1(d)}{\partial d_1} = 0 \quad \frac{\partial \Pi_1(d)}{\partial d_2} = 0
\]
\[\Rightarrow \frac{\partial \Pi_1(d)}{\partial d} = 0\]

Recall from linear algebra (Lecture notes on Linear Algebra)

\[\Pi_1(d) = \frac{1}{2} d^T k d - d^T f_1\]
\[\Rightarrow \frac{\partial \Pi_1(d)}{\partial d} = k d - f_1\]

Hence
\[\frac{\partial \Pi_1(d)}{\partial d} = 0\]
\[\Rightarrow k d = f_1\]

Exactly the same equation that we had before, except that the stiffness matrix and nodal force vectors are more general.

Recap of the properties of the element stiffness matrix

\[k = \int_0^1 B^T E B \, \text{d}x\]

1. The stiffness matrix is singular and is therefore non-invertible
2. The stiffness matrix is symmetric
3. Sum of any row (or column) of the stiffness matrix is zero!

Why?

Sum of any row (or column) of the stiffness matrix is zero

Consider a rigid body motion of the element

\[d_1 = 1 \quad d_2 = 1\]

Element strain \(\varepsilon = 0 = B \, d\)

\[\Rightarrow k \, d = \int_0^1 B^T E B \, \text{d}x \quad \text{d}x = \int_0^1 B^T E (\text{d}x) \, \text{d}x\]

\[\Rightarrow k_1 d_1 + k_2 d_2 = \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}\]

\[\Rightarrow k_1 + k_2 = 0\] and \(k_2 + k_2 = 0\)
**The nodal load vector**

\[ f_b = \int_{x_1}^{x_2} N^T b \, dx \]

**A distributed load is represented by two nodal loads in a consistent manner**

**Summary:** For each element

Displacement approximation in terms of shape functions

\[ w(x) = N \, d \]

Strain approximation in terms of strain-displacement matrix

\[ \varepsilon = B \, d \]

Stress approximation

\[ \sigma = E \, B \, d \]

Element stiffness matrix

\[ k = \int_{x_1}^{x_2} B^T E B \, dx \]

Element nodal load vector

\[ f_b = \int_{x_1}^{x_2} N^T b \, dx \]

**What happens for element #3?**

\[ \Pi_e(w) = \frac{1}{2} \int_{x_1}^{x_2} E A \left( \frac{dw}{dx} \right)^2 \, dx - \int_{x_1}^{x_2} b w \, dx - F w(x = L) \]

For element 3

\[ w(x) = \begin{bmatrix} x_4 - x \\ x_4 - x_3 \\ x_4 - x_3 \end{bmatrix} \begin{bmatrix} d_{1x} \\ d_{2x} \end{bmatrix} \]

\[ \Rightarrow w(x = L) = d_{ax} \]

The discretized form of the potential energy

\[ \Pi_e(d) = \frac{1}{2} \left( \int_{x_1}^{x_2} B^T E B \, dx \right) d - \int_{x_1}^{x_2} N^T b \, dx \cdot F d_{ax} \]
What happens for element #3?

Now apply Rayleigh-Ritz principle
\[ \frac{\partial H_{1}(d)}{\partial d} = 0 \]
\[ \Rightarrow k d = f + \begin{bmatrix} 0 \\ F \end{bmatrix} \]

Hence there is an extra load term on the right hand side due to the concentrated force \( F \) applied to the right end of the bar.

**NOTE** that whenever you have a concentrated load at ANY node, that load should be applied as an extra right hand side term.

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**Step3: Assembly** exactly as you had done before, assemble the global stiffness matrix and global load vector and solve the resulting set of equations by properly taking into account the displacement boundary conditions.

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**Problem:**

- **E=30\times10^6 \text{ psi}**
- \( \rho=0.2836 \text{ lb/in}^3 \)
- **Thickness of plate, \( t=1'' \)**

Model the plate as 2 finite elements and

1. Write the expression for element stiffness matrix and body force vectors
2. Assemble the global stiffness matrix and load vector
3. Solve for the unknown displacements
4. Evaluate the stress in each element
5. Evaluate the reaction in each support

**Solution (1)**

- **Finite element model**
- **Stiffness matrix of El #1**

\[
\begin{bmatrix}
1 & -1 & 1 \\
-1 & 1 & -1 \\
1 & -1 & 1
\end{bmatrix}
\]

- **Node-element connectivity chart**
Stiffness matrix of El #2

\[
k^{(2)} = \int_0^{24} \frac{E}{(12)^2} \frac{B^2}{A} dx = \frac{E}{(12)^2} \left[ \int_0^{24} \frac{B^2}{A} dx \right]
\]

\[
\int_0^{24} \frac{B^2}{A} dx = \int_0^{24} \frac{B^2}{A} (6-0.125x) dx = \int_0^{24} \frac{B^2}{A} (6-0.125x) dx = 45 A
\]

\[
k^{(2)} = \frac{E}{(12)^2} \left[ \begin{array}{cc}
1 & -1 \\
-1 & 1
\end{array} \right]
\]

Now compute the element load vector due to distributed body force (weight)

\[
f_1 = \int_0^{24} N'^T b \ dx
\]
**Solution (3)**

Hence we need to solve

\[
\begin{bmatrix}
13.125 & -13.125 & 0 \\
10^6 & -13.125 & 22.5 & -9.375 \\
0 & -9.375 & 9.375 & 13.125
\end{bmatrix}
\begin{bmatrix}
d_{x}\n \\
d_{x}\n \\
d_{x}\n \\
d_{x}\n \\
\end{bmatrix}
\begin{bmatrix}
9.3588 + R \\
115.3144 \\
5.9556
\end{bmatrix}
\]

\[R_1\] is the reaction at node 1.

Notice that since the boundary condition at x=0 (d1x=0) has not been taken into account, the system matrix is not invertible.

Incorporating the boundary condition d1x=0 we need to solve the following set of equations

\[
\begin{bmatrix}
22.5 & -9.375 & d_{x}\n
-9.375 & 9.375 & d_{x}\n\end{bmatrix}
\begin{bmatrix}
d_{x}\n
\end{bmatrix}
= \begin{bmatrix}
115.3144 \\
5.9556
\end{bmatrix}
\]

Solve to obtain

\[
\begin{bmatrix}
d_{x}\n
\end{bmatrix} = \begin{bmatrix}
0.92396 \times 10^5 \\
0.98749 \times 10^5
\end{bmatrix}
\]

**Solution (4) Stress in elements**

Notice that since we are using linear elements, the stress within each element is constant.

**In element #1**

\[
\sigma^{(1)} = \frac{E}{x_2 - x_1} \begin{bmatrix}
10^6 \\
-13.125 \\
22.5 \\
-9.375
\end{bmatrix}
\begin{bmatrix}
d_{x}\n
\end{bmatrix}
\begin{bmatrix}
115.3144 \\
5.9556
\end{bmatrix}
\]

\[
\sigma^{(1)} = \frac{E}{x_2 - x_1} \begin{bmatrix}
30 \times 10^6 \\
12
\end{bmatrix}
\begin{bmatrix}
d_{x}\n
\end{bmatrix}
\begin{bmatrix}
0
\end{bmatrix} = 23.099 \text{ psi}
\]

**In element #2**

\[
\sigma^{(2)} = \frac{E}{x_2 - x_1} \begin{bmatrix}
10^6 & -13.125 & 0 \\
-13.125 & 22.5 & -9.375 \\
0 & -9.375 & 9.375
\end{bmatrix}
\begin{bmatrix}
d_{x}\n
\end{bmatrix}
\begin{bmatrix}
115.3144 \\
5.9556
\end{bmatrix}
\]

\[
\sigma^{(2)} = \frac{E}{x_2 - x_1} \begin{bmatrix}
30 \times 10^6 \\
12
\end{bmatrix}
\begin{bmatrix}
d_{x}\n
\end{bmatrix}
\begin{bmatrix}
0
\end{bmatrix}
= 1.5882 \text{ psi}
\]

**Solution (5) Reaction at support**

Go back to the first line of the global equilibrium equations...

\[
\begin{bmatrix}
13.125 & -13.125 & 0 \\
10^6 & -13.125 & 22.5 & -9.375 \\
0 & -9.375 & 9.375 & 13.125
\end{bmatrix}
\begin{bmatrix}
d_{x}\n
\end{bmatrix}
\begin{bmatrix}
9.3588 + R \\
115.3144 \\
5.9556
\end{bmatrix}
\]

\[\Rightarrow R = -130.6288 \text{ lb} \text{ (The –ve sign indicates that the force is in the –ve x-direction)}
\]

**Check**

The reaction at the wall from force equilibrium in the x-direction

\[
R = P + \int_{0}^{x} (6000A) \, dx
\]

\[
= 100 \times \int_{0}^{6} (6-0.125x) \, dx
\]

\[
= 130.6288 \text{ lb}
\]
Problem: Can you solve for the displacement and stresses analytically?

Check out

\[ u_{\text{anal}} = \begin{cases} -4.727 \times 10^3 x^2 + 9.487 \times 10^5 x & \text{for } 0 \leq x < 12 \\ -4.727 \times 10^3 x^2 + 2.0797 \times 10^5 x + 8.89 \times 10^6 & \text{for } 12 \leq x \leq 24 \end{cases} \]

Stress

\[ \sigma(x)_{\text{anal}} = E \frac{du}{dx} = 30 \times 10^3 \frac{du}{dx} \]

Notice:
1. Slope discontinuity at \( x = 12 \) (why?)
2. The finite element solution does not produce the exact solution even at the nodes
3. We may improve the solution by
   (1) Increasing the number of elements
   (2) Using higher order elements (e.g., quadratic instead of linear)

The analytical as well as the finite element stresses are discontinuous across the elements