So far, structural mechanics using Direct Stiffness approach.

Finite element method is used to solve physical problems:
- Solid Mechanics
- Fluid Mechanics
- Heat Transfer
- Electrostatics
- Electromagnetism

Physical problems are governed by differential equations which satisfy
- Boundary conditions
- Initial conditions

One variable: Ordinary differential equation (ODE)
Multiple independent variables: Partial differential equation (PDE)
Axially loaded elastic bar

\[ A(x) = \text{cross section at } x \]
\[ b(x) = \text{body force distribution (force per unit length)} \]
\[ E(x) = \text{Young's modulus} \]
\[ u(x) = \text{displacement of the bar at } x \]

Differential equation governing the response of the bar

\[ \frac{d}{dx}\left( E(x) \frac{du}{dx} \right) + b(x) = 0; \quad 0 < x < L \]

Second order differential equations
Requires 2 boundary conditions for solution

Flexible string

\[ S = \text{tensile force in string} \]
\[ p(x) = \text{lateral force distribution (force per unit length)} \]
\[ w(x) = \text{lateral deflection of the string in the y-direction} \]

Differential equation governing the response of the bar

\[ E(x) \frac{d^2 w}{dx^2} + p(x) = 0; \quad 0 < x < L \]

Second order differential equations
Requires 2 boundary conditions for solution

Heat conduction in a fin

\[ A(x) = \text{cross section at } x \]
\[ Q(x) = \text{heat input per unit length per unit time [J/sm]} \]
\[ k(x) = \text{thermal conductivity [J/C ms]} \]
\[ T(x) = \text{temperature of the fin at } x \]

Differential equation governing the response of the fin

\[ \frac{d}{dx}\left( k(x) \frac{dT}{dx} \right) + Q(x) = 0; \quad 0 < x < L \]

Second order differential equations
Requires 2 boundary conditions for solution
Boundary conditions (examples)

**Dirichlet/ displacement bc**

\[ T = 0 \quad \text{at} \quad x = 0 \]  
\[ -k \frac{dT}{dx} = h \quad \text{at} \quad x = L. \]  

**Neumann/ force bc**

\[ \frac{d}{dx}(k \frac{d\phi}{dx}) + Q = 0; \quad 0 < x < L. \]

Fluid flow through a porous medium (e.g., flow of water through a dam)

- **A(x) = cross section at x**
- **Q(x) = fluid input per unit volume per unit time**
- **k(x) = permeability constant**
- **\( \phi(x) = \text{fluid head} \)**

Differential equation

Second order differential equations  
Requires 2 boundary conditions for solution

Boundary conditions (examples)

\[ \phi = 0 \quad \text{at} \quad x = 0 \]  
**Known head**

\[ -k \frac{d\phi}{dx} = h \quad \text{at} \quad x = L \]  
**Known velocity**
Observe:
1. All the cases we considered lead to very similar differential equations and boundary conditions.
2. In 1D it is easy to analytically solve these equations.
3. Not so in 2 and 3D especially when the geometry of the domain is complex: need to solve **approximately**.
4. We’ll learn how to solve these equations in 1D. The approximation techniques easily translate to 2 and 3D, no matter how complex the geometry.

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**A generic problem in 1D**

A general algorithm for approximate solution:

Guess

\[ u(x) = a_0 \phi_0(x) + a_1 \phi_1(x) + \ldots + \text{...} \]

where \( \phi_0(x), \phi_1(x), \ldots \) are “known” functions and \( a_0, a_1, \ldots \) are constants chosen such that the approximate solution

Satisfies the differential equation

Satisfies the boundary conditions

i.e.,

\[
\begin{align*}
ad_0 \frac{d^2 \phi_0(x)}{dx^2} + a_1 \frac{d^2 \phi_1(x)}{dx^2} + \ldots + a_n \frac{d^2 \phi_n(x)}{dx^2} + \ldots + x = 0; & \quad 0 < x < 1 \\
a_0 \phi_0(0) + a_1 \phi_1(0) + \ldots & = 0 \\
a_0 \phi_0(1) + a_1 \phi_1(1) + \ldots & = 1
\end{align*}
\]

Solve for unknowns \( a_0, a_1, \ldots \) and plug them back into

\[ u(x) = a_0 \phi_0(x) + a_1 \phi_1(x) + \ldots \]

This is your **approximate solution** to the strong form.

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**Analytical solution**

\[ u(x) = -\frac{1}{6} x^3 + \frac{7}{6} x \]

Assume that we **do not know** this solution.