Introduction to 3D Elasticity

Summary:
- 3D elasticity problem
  - Governing differential equation + boundary conditions
  - Strain-displacement relationship
  - Stress-strain relationship
- Special cases
  - 2D (plane stress, plane strain)
  - Axisymmetric body with axisymmetric loading
- Principle of minimum potential energy

1. Strong formulation: Equilibrium equation + boundary conditions

<table>
<thead>
<tr>
<th>Equilibrium equation</th>
<th>Boundary conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{d\sigma}{dx} + b = 0$; $0 &lt; x &lt; L$</td>
<td>$u = 0$ at $x = 0$</td>
</tr>
<tr>
<td>$E \frac{du}{dx} = F$ at $x = L$</td>
<td></td>
</tr>
</tbody>
</table>

2. Strain-displacement relationship: $\varepsilon(x) = \frac{du}{dx}$

3. Stress-strain (constitutive) relation: $\sigma(x) = E \varepsilon(x)$
   - $E$: Elastic (Young’s) modulus of bar
Problem definition

3D Elasticity

V: Volume of body
S: Total surface of the body

The deformation at point $x = [x, y, z]^T$ is given by the 3 components of its displacement $\mathbf{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$.

NOTE: $y = u(x, y, z)$, i.e., each displacement component is a function of position.

3D Elasticity: EXTERNAL FORCES ACTING ON THE BODY

Two basic types of external forces act on a body:
1. **Body force** (force per unit volume) e.g., weight, inertia, etc.
2. **Surface traction** (force per unit surface area) e.g., friction.

BODY FORCE

**Body force**: distributed force per unit volume (e.g., weight, inertia, etc)

$$\mathbf{X} = \begin{pmatrix} X_x \\ X_y \\ X_z \end{pmatrix}$$

NOTE: If the body is accelerating, then the **inertia force** $\rho \mathbf{u}$ may be considered as part of $\mathbf{X}$.

$$\mathbf{X} = \mathbf{X} - \rho \mathbf{u}$$

SURFACE TRACTION

**Traction**: Distributed force per unit surface area

$$\mathbf{T}_S = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$
3D Elasticity:
INTERNAL FORCES

If I take out a chunk of material from the body, I will see that, due to the external forces applied to it, there are reaction forces (e.g., due to the loads applied to a truss structure, internal forces develop in each truss member). For the cube in the figure, the internal reaction forces per unit area (red arrows), on each surface, may be decomposed into three orthogonal components.

Strains: 6 independent strain components

Consider the equilibrium of a differential volume element to obtain the 3 equilibrium equations of elasticity

\[
\begin{align*}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X_x &= 0 \\
\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + X_y &= 0 \\
\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + X_z &= 0
\end{align*}
\]

Compactly;

EQUILIBRIUM EQUATIONS

\[
\begin{bmatrix}
\frac{\partial}{\partial x} & 0 & 0 \\
0 & \frac{\partial}{\partial y} & 0 \\
0 & 0 & \frac{\partial}{\partial z}
\end{bmatrix}
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{yx} \\
\tau_{zy} \\
\tau_{zx}
\end{bmatrix} + \begin{bmatrix}
X_x \\
X_y \\
X_z
\end{bmatrix} = 0
\]

σ_x, σ_y, and σ_z are normal stresses. The rest 6 are the shear stresses. Convention

\[
\tau_{yx} \text{ is the stress on the face perpendicular to the x-axis and points in the +ve y direction}
\]

Total of 9 stress components of which only 6 are independent since \(\tau_{xy} = \tau_{yx}\), \(\tau_{xz} = \tau_{zx}\), \(\tau_{zy} = \tau_{yz}\).

The stress vector is therefore

\[
\sigma = \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{yx} \\
\tau_{zy} \\
\tau_{zx}
\end{bmatrix}
\]
3D elasticity problem is completely defined once we understand the following three concepts:

- **Strong formulation (governing differential equation + boundary conditions)**
- **Strain-displacement relationship**
- **Stress-strain relationship**

1. **Strong formulation of the 3D elasticity problem:** "Given the externally applied loads (on ST and in V) and the specified displacements (on Su) we want to solve for the resultant displacements, strains and stresses required to maintain equilibrium of the body."

**Equilibrium equations**

\[
\nabla \cdot \sigma + \chi = 0 \text{ in } V \quad (1)
\]

**Boundary conditions**

1. **Displacement boundary conditions:** Displacements are specified on portion Su of the boundary

\[
u = u \text{ specified on } S_u
\]

2. **Traction (force) boundary conditions:** **Traction** are specified on portion St of the boundary

Now, how do I express this mathematically?
3D elasticity problem is completely defined once we understand the following three concepts:
- Strong formulation (governing differential equation + boundary conditions)
- Strain-displacement relationship
- Stress-strain relationship

2. Strain-displacement relationships:
\[
\begin{align*}
\varepsilon_x &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \\
\varepsilon_y &= \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \\
\varepsilon_z &= \frac{\partial w}{\partial z} \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\
\gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial x} \\
\gamma_{xz} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}
\end{align*}
\]
3D elasticity problem is completely defined once we understand the following three concepts:

- **Strong formulation** (governing differential equation + boundary conditions)
- **Strain-displacement relationship**
- **Stress-strain relationship**

### 3. Stress-Strain relationship:

Linear elastic material (Hooke’s Law)

\[ \sigma = D \varepsilon \]  

(3)

Linear elastic isotropic material:

\[ D = \frac{E}{(1+v)(1-2v)} \]

\[
\begin{bmatrix}
1-v & v & 0 & 0 & 0 \\
-v & 1-v & 0 & 0 & 0 \\
v & 0 & 1-2v & 0 & 0 \\
0 & 0 & 0 & \frac{1-2v}{2} & 0 \\
0 & 0 & 0 & 0 & \frac{1-2v}{2}
\end{bmatrix}
\]
Special cases:
1. **1D elastic bar** (only 1 component of the stress (stress) is nonzero. All other stress (strain) components are zero)
   Recall the (1) equilibrium, (2) strain-displacement and (3) stress-strain laws
2. **2D elastic problems**: 2 situations
   - **PLANE STRESS**
   - **PLANE STRAIN**
3. **3D elastic problem**: special case - axisymmetric body with axisymmetric loading (we will skip this)

**PLANE STRESS** Examples:
1. Thin plate with a hole
2. Thin cantilever plate

**MONOSTRESS**

Nonzero stresses: \( \sigma_x, \sigma_y, \tau_{xy} \)
Nonzero strains: \( \varepsilon_x, \varepsilon_y, \gamma_{xy} \)

Isotropic linear elastic stress-strain law
\[
\sigma = D \varepsilon
\]

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} =
\begin{bmatrix}
E & 0 & 0 \\
0 & E & 0 \\
0 & 0 & \frac{E}{1-\nu}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}

\]

\( \varepsilon_z = -\frac{\nu}{1-\nu} (\varepsilon_x + \varepsilon_y) \)

Hence, the \( D \) matrix for the **plane stress case** is
\[
D = \frac{E}{1-\nu^2}
\begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & \frac{1-\nu}{2}
\end{bmatrix}
\]
**PLANE STRAIN**

Nonzero strain components: $\varepsilon_x, \varepsilon_y, \gamma_{xy}$

Area element $dA$

Nonzero strain components $\varepsilon_x, \varepsilon_y, \gamma_{xy}$

Assumptions:
1. Displacement components $u, v$ functions of $(x, y)$ only and $w=0$
2. Top and bottom surfaces are fixed
3. $X_c=0$
4. $p_x$ and $p_y$ do not vary with $z$

**PLANE STRAIN**

Examples:

1. Dam

Slice of unit thickness

2. Long cylindrical pressure vessel subjected to internal/external pressure and constrained at the ends

**Nonzero stress:**

$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} E \\ E \\ E \end{bmatrix} \begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & 1-2v \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$

$\sigma_y = v\sigma_x + \sigma_y$

Hence, the $D$ matrix for the plane strain case is

$D = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & 1-2v \end{bmatrix}$

**Example problem**

The square block is in plane strain and is subjected to the following strains

$\varepsilon_x = 2\gamma y$

$\varepsilon_y = 3\gamma y^2$

$\gamma_{xy} = x^2 + y^2$

Compute the displacement field (i.e., displacement components $u(x,y)$ and $v(x,y)$) within the block
Solution
Recall from definition
\[ \varepsilon_y = \frac{\partial u}{\partial x} = 2xy \]  
(1)
\[ \varepsilon_x = \frac{\partial v}{\partial y} = 3xy^2 \]  
(2)
\[ \gamma_0 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = x^2 + y^3 \]  
(3)

Integrating (1) and (2)
\[ u(x, y) = x^2y + C_1(y) \]  
(4)
\[ v(x, y) = xy^3 + C_2(x) \]  
(5)
Arbitrary function of 'x'
Arbitrary function of 'y'

Hence
\[ \frac{\partial C_1(y)}{\partial y} = \frac{\partial C_2(x)}{\partial x} = C \] (a constant)
Integrate to obtain
\[ C_1(y) = Cy + D_1 \] \( D_1 \) and \( D_2 \) are two constants of integration
\[ C_2(x) = Cx + D_2 \]
Plug these back into equations (4) and (5)
\[ u(x, y) = x^2y + Cy + D_1 \]
\[ v(x, y) = xy^3 - Cx + D_2 \]

How to find \( C, D_1 \) and \( D_2 \)?

Plug expressions in (4) and (5) into equation (3)
\[ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = x^2 + y^3 \]  
(3)
\[ \Rightarrow \frac{\partial [x^2y + C_1(y)]}{\partial y} + \frac{\partial [xy^3 + C_2(x)]}{\partial x} = x^2 + y^3 \]
\[ \Rightarrow x^2 + \frac{\partial C_1(y)}{\partial y} + y^3 + \frac{\partial C_2(x)}{\partial x} = x^2 + y^3 \]
\[ \Rightarrow \frac{\partial C_1(y)}{\partial y} + \frac{\partial C_2(x)}{\partial x} = 0 \]
Function of 'y'
Function of 'x'

Use the 3 boundary conditions
\[ u(0,0) = 0 \]
\[ v(0,0) = 0 \]
\[ v(2,0) = 0 \]
To obtain
\[ C = 0 \]
\[ D_1 = 0 \]
\[ D_2 = 0 \]
Hence the solution is
\[ u(x, y) = x^2y \]
\[ v(x, y) = xy^3 \]
**Principle of Minimum Potential Energy**

**Definition:** For a linear elastic body subjected to body forces \( \mathbf{X} = [X_a, X_b, X_c]^T \) and surface tractions \( \mathbf{T}_S = [p_x, p_y, p_z]^T \), causing displacements \( \mathbf{u} = [u, v, w]^T \) and strains \( \mathbf{\varepsilon} \) and stresses \( \mathbf{\sigma} \), the potential energy \( \Pi \) is defined as the strain energy minus the potential energy of the loads involving \( \mathbf{X} \) and \( \mathbf{T}_S \)

\[
\Pi = U - W
\]

- **Strain energy of the elastic body**
  - Using the stress-strain law \( \mathbf{\sigma} = \mathbf{D} \mathbf{\varepsilon} \)
  - \( U = \frac{1}{2} \int_{V} \mathbf{\varepsilon}^T \mathbf{D} \mathbf{\varepsilon} \, dV \)
  - In 1D
    \( U = \frac{1}{2} \int_{L} \mathbf{\varepsilon} \, dx \)
  - In 2D plane stress and plane strain
    \( U = \frac{1}{2} \int_{V} \left( \sigma_{xx}, \varepsilon_{x} + \sigma_{yy}, \varepsilon_{y} + \tau_{xy}, \gamma_{xy} \right) \, dV \)

  *Why?*

- **Principle of minimum potential energy:** Among all admissible displacement fields the one that satisfies the equilibrium equations also render the potential energy \( \Pi \) a minimum.

  *“admissible displacement field”:
  1. first derivative of the displacement components exist
  2. satisfies the boundary conditions on \( S_u \)