A REVIEW OF ERROR ANALYSIS

NEEP Laboratory

ENVE-4860 / MANE-4370

Updated 2006
Error Analysis

• In the laboratory we measure physical quantities.
• All measurements are subject to some uncertainties.
• Error analysis is the study and evaluation of these uncertainties.
• When mathematically manipulating measured quantities, a proper manipulation is required for the uncertainties.
Errors in Measurements

• Measure length with ruler.
• Measure voltage with digital multimeter.
• Measure time.
• Measure radiation decay (counting statistics).
• Stated instrumentation accuracy.
• Counting ??
Types of Uncertainties

- **RANDOM** – arising from a random effect.
  - Example: radioactive nuclear decay.

- **SYSTEMATIC** – arising from a systematic effect.
  - Example: instrument calibration error.

Examples

- Random: Small
  - Systematic: Small
- Random: Small
  - Systematic: Large
- Random: Large
  - Systematic: Small
- Random: Large
  - Systematic: Large
Reporting Uncertainties

- (Measured value of $x$) = $x \pm \delta x$
  - Example $V=2.5\pm0.2$

- Round error to one significant digit.
  - $g=9.82\pm0.02385 \, \text{m/s}^2 \rightarrow g=9.82\pm0.02 \, \text{m/s}^2$

- The last significant digit of the quantity should be of the same order of the uncertainty.
  - $I=4.35\pm0.2 \, \text{A} \rightarrow I=4.4\pm0.2 \, \text{A}$
Average and Variance

- Given $N$ measurements of a quantity $x_i$, we can estimate the mean of the distribution of $x$ by calculating the average:

$$\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i$$

- The estimate for the variance of the distribution is:

$$\sigma^2 = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})^2$$

- The estimated standard deviation is $\sigma$.

- The more samples we have (larger $N$), the average $\bar{x}$ will get closer to the real average $\bar{x}$ of the distribution.
Different measurements

- 4 laboratories measured the absorption cross section of the same isotope.
- Which value should I use?
Weighted Average

• When we measure the quantity $x_i$ with an associated error $\sigma_i$, then the best estimate of the of that quantity is calculated by the weighted average:

$$\bar{x}_w = \frac{\sum_{i=1}^{N} w_i x_i}{\sum_{i=1}^{N} w_i}$$

• Where the weight is taken as $w_i = \frac{1}{\sigma_i^2}$

• The error in that estimated average is be given by:

$$\sigma_w = \frac{1}{\sqrt{\sum_{i=1}^{N} w_i}}$$

• This calculation gives more weight to measurements with small errors.
Average and Variance - Example

Following are results of the same measurement from several students

\[
\begin{pmatrix}
13 \\
12 \\
16 \\
11.5
\end{pmatrix}
\]
\[
\begin{pmatrix}
0.5 \\
0.4 \\
2 \\
0.3
\end{pmatrix}
\]
\[N:= \text{length}(x)\]

**Non Weighted**

\[
x_{av} := \frac{1}{N} \sum_{i=1}^{N} x_i
\]
\[x_{av} = 13.125\]
\[\text{std} := \sqrt{\frac{1}{N} \sum_{i=1}^{4} (x_i - x_{av})^2}
\]
\[\text{std} = 1.746\]

**Weighted**

\[
i := 1..N
\]
\[w_i := \frac{1}{(\sigma_i)^2}\]
\[\sum_{i=1}^{N} \left( w_i \cdot x_i \right)\]
\[x_{wav} := \frac{1}{\sum_{i=1}^{N} w_i} \sum_{i=1}^{N} w_i \cdot x_i
\]
\[x_{wav} = 11.974\]
\[\sigma_{wav} := \frac{1}{\sum_{i=1}^{N} w_i} \sqrt{\sum_{i=1}^{N} w_i \cdot (x_{i} - x_{wav})^2}
\]
\[\sigma_{wav} = 0.215\]
Measurement Distribution I

- In most cases the distribution of a measured quantity is Gaussian (or Normal when $\sigma = \sqrt{x}$).

$$G(x, \mu, \sigma) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$

- Where $\mu$ is the average and $\sigma$ is the standard deviation.

$$\mu = \int_{-\infty}^{\infty} G(x, \mu, \sigma) dx$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 G(x, \mu, \sigma) dx$$

- If $x$ is sampled from a Normal distribution then
  - 68% of the samples will be between $\mu - \sigma$ to $\mu + \sigma$.
  - 95.5% between $\mu - 2\sigma$ to $\mu + 2\sigma$.
  - 99.7% between $\mu - 3\sigma$ to $\mu + 3\sigma$. 
Error Propagation I

• Consider a function \( q(x_i, y_i) \) \( I=1, \ldots N \).
• The first order Taylor series expansion of \( q(x_i, y_i) \) at the point \((\bar{x}, \bar{y})\):

\[
q_i = q(x_i, y_i) \approx q(\bar{x}, \bar{y}) + \frac{\partial q}{\partial x}_{|\bar{x}, \bar{y}} (x_i - \bar{x}) + \frac{\partial q}{\partial y}_{|\bar{x}, \bar{y}} (y_i - \bar{y})
\]

• We can calculate the mean of \( q(x_i, y_i) \):

\[
\bar{q} = \frac{1}{N} \sum_{i=1}^{N} q_i \approx \frac{1}{N} \sum_{i=1}^{N} \left[ q(\bar{x}, \bar{y}) + \frac{\partial q}{\partial x}_{|\bar{x}, \bar{y}} (x_i - \bar{x}) + \frac{\partial q}{\partial y}_{|\bar{x}, \bar{y}} (y_i - \bar{y}) \right]
\]

• Which can be written as:

\[
\bar{q} = \frac{1}{N} \sum_{i=1}^{N} q_i \approx \frac{1}{N} \sum_{i=1}^{N} q(\bar{x}, \bar{y}) + \frac{\partial q}{\partial x}_{|\bar{x}, \bar{y}} \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x}) + \frac{\partial q}{\partial y}_{|\bar{x}, \bar{y}} \frac{1}{N} \sum_{i=1}^{N} (y_i - \bar{y})
\]

• From the definition of the average \( \frac{\sum_{i=1}^{N} (x_i - \bar{x})}{N} = 0, \frac{\sum_{i=1}^{N} (y_i - \bar{y})}{N} = 0 \) and thus

\[
\bar{q} = \frac{1}{N} \sum_{i=1}^{N} q(\bar{x}, \bar{y}) = \frac{q(\bar{x}, \bar{y})}{N} \sum_{i=1}^{N} 1 \Rightarrow \bar{q} = q(\bar{x}, \bar{y})
\]
The variance of $q$ is defined as: 
\[ \sigma_q^2 = \frac{1}{N} \sum_{i=1}^{N} (q_i - \bar{q})^2 \]

Evaluating the variance we get:

\[
\sigma_q^2 = \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{\partial q}{\partial x} (x_i - \bar{x}) + \frac{\partial q}{\partial y} (y_i - \bar{y}) \right]^2 = \left( \frac{\partial q}{\partial x} \right)^2 \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2 + \left( \frac{\partial q}{\partial y} \right)^2 \frac{1}{N} \sum_{i=1}^{N} (y_i - \bar{y})^2
\]

\[
+ 2 \frac{\partial q}{\partial x} \frac{\partial q}{\partial y} \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y})
\]

We define the covariance:

\[
\sigma_{xy} = \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y})
\]

And finally the standard deviation $\sigma_q$ is given by:

\[
\sigma_q^2 = \left( \frac{\partial q}{\partial x} \right)^2 \sigma_x^2 + \left( \frac{\partial q}{\partial y} \right)^2 \sigma_y^2 + 2 \frac{\partial q}{\partial x} \frac{\partial q}{\partial y} \sigma_{xy}
\]
Error Propagation – Examples I

• In many cases we can assume that the variables are independent. For a function with \( n \) variables \( q(x_1, x_2, \ldots, x_n) \) the variance is given by:

\[
\sigma_q^2 = \sum_{i=1}^{n} \left( \frac{\partial q}{\partial x_i} \right)^2 \sigma_i^2
\]

• Using this equation, we can derive simple helpful relations for the propagation of errors:

• Addition and subtraction: \( u = x \pm y \) \( \frac{\partial u}{\partial x} = 1 \) \( \frac{\partial u}{\partial y} = 1 \) \( \Rightarrow \Delta u = \sqrt{\Delta x^2 + \Delta y^2} \)

• Multiplication by a constant: \( u = Ax \) \( \frac{\partial u}{\partial x} = A \) \( \Rightarrow \Delta u = A \Delta x \)

• Multiplication or division: \( u = xy \) or \( u = \frac{x}{y} \) \( \left( \frac{\Delta u}{u} \right)^2 = \left( \frac{\Delta x}{x} \right)^2 + \left( \frac{\Delta y}{y} \right)^2 \)
Error Propagation – Example II

• we measured $V=1.51\pm0.02$ V across a resistor with $R=900\pm5$ Ω What is the current $I$.

$$I = \frac{V}{R} = \frac{1.51}{900} = 1.67778 \times 10^{-3} \text{ A}$$

• Use the error propagation for division, the fractional error of $I$ is:

$$\frac{\Delta I}{I} = \sqrt{\left(\frac{\Delta R}{R}\right)^2 + \left(\frac{\Delta V}{V}\right)^2} = \sqrt{\left(\frac{5}{900}\right)^2 + \left(\frac{0.02}{1.51}\right)^2} = 0.01436$$

• The error in $I$ is then $\Delta I = I \frac{\Delta I}{I} = 0.01436 \times 1.67778 \times 10^{-3} = 0.024 \times 10^{-3} \text{ A}$

• We report: $I = 1.68\pm0.02$ mA.
Covariance

- A covariance value that is different from zero indicates that data are correlated. Here is an example.

\[
\sigma_{xy} = \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y})
\]

Assume we want to find the average sum \( \langle z \rangle = \langle x \rangle + \langle y \rangle \)

\[
x_{av} := \text{mean}(x) \quad y_{av} := \text{mean}(y)
\]

\[
x_{av} = 29 \quad y_{av} = 95.8
\]

\[
\sigma_{xy} := \frac{1}{N} \sum_{i=1}^{N} (x_i - x_{av})(y_i - y_{av})
\]

\[
\sigma_{xy} = 14
\]

Assume we want to find the average sum \( \langle z \rangle = \langle x \rangle + \langle y \rangle \)

\[
z_{av} := x_{av} + y_{av} \quad z_{av} = 124.8
\]

\[
\sigma z1 := \sqrt{\sigma_x^2 + \sigma_y^2} \quad \sigma z2 := \sqrt{\sigma_x^2 + \sigma_y^2 + 2\sigma_{xy}}
\]

\[
\sigma z1 = 6.112 \quad \sigma z2 = 8.085
\]
Counting Statistics I

- For a process with very small success probability \( p \ll 1 \), if we carry \( n \) experiments, the distribution having \( x \) successes is Binomial. It can be approximated by the Poisson distribution.

\[
p(x) = \frac{(pn)^x e^{-pn}}{x!}
\]

- The average and variance of this distribution is \( pn \).
- In nuclear decay, large number of nuclei make up a sample or numbers of tries (\( n \)) but only a relatively small fraction of them give rise to a success event (small \( p \)).
- In a counting experiment we record the number of counts, \( n \) in a given counting time \( t \). The distribution of \( n \) is Poisson:

\[
p(n) = \frac{N^n e^{-N}}{n!}
\]

- Where \( N \) is the expected counts (the mean) and the error in \( N \) is \( \sigma = \sqrt{N} \)
For large $N$ the Poisson distribution can be approximated by the Gaussian distribution.

**Example**: we make $N$ experiment and record $x_i$ counts in each. What is the average number of counts and expected error of the average?

$$\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i$$

$$\sigma = \frac{1}{N} \sqrt{\sum_{i=1}^{N} \left(\sqrt{x_i}\right)^2} = \frac{1}{N} \sqrt{\sum_{i=1}^{N} x_i} = \frac{1}{N} \sqrt{N\bar{X}} = \sqrt{\frac{\bar{x}}{N}}$$
Modeling of Data-I

- In many cases we have some theoretical background on the physical behavior of the phenomena we are measuring.
- In these cases we can try to check if the experiment agrees with the theory and also extract parameters from it.
- For example we would like to measure the attenuation of gamma rays through a slab of Al. We setup the following experiment:

\[
\begin{array}{c|c|c}
\text{Source} & \text{Pb} & \text{Pb} \\
\text{Collimator} & \text{Al} & \text{Collimator} \\
\text{Thickness } x & & \\
\text{Detector} & & \\
\end{array}
\]

- We repeat the experiment \(N\) times \((N>3)\) for several thickness \(x_i\) of Aluminum
- We have no background (or corrected for it).
Modeling of Data-II

• If we count for sufficient time we except:
  \[ C_i = I_0 e^{-\mu x_i} \]

• Where \( C_i \) are the counts for sample \( i \), \( \mu \) is the attenuation coefficient in units of \( \text{cm}^{-1} \) and \( x_i \) is the sample thickness.

• We take the log of both sides of the equation we get:
  \[ \log(C_i) = \log(I_0) - \mu x_i \]

• Define \( y_i = \log(C_i) \quad b = -\mu \quad a = \log(I_0) \)

• The equation can be rewritten as:
  \[ y_i = bx_i + a \]

• Find a procedure that can use all our measurements of \( y_i \) to find \( a \) and \( b \) that best fit the data.

• Knowing the counting error in \( C_i \), we will try to estimate the error in \( a \) and \( b \).
Least-Squares Fitting I

• Given a series of \( N \) measurements of \( x_i \) and \( y_i \pm \sigma_i \), fit the model \( y_i = a + bx_i \) to the data.

• We would like find \( a \) and \( b \) that will minimize the expression.

\[
\chi^2 = \sum_{i=1}^{N} \left( \frac{y_i - a - bx_i}{\sigma_i} \right)^2
\]

• To do that we take the first derivative with respect to \( a \) and \( b \) and set the derivatives equal to zero:

\[
\frac{\partial \chi^2}{\partial a} = -2 \sum_{i=1}^{N} \frac{y_i - a - bx_i}{\sigma_i} = 0
\]

\[
\frac{\partial \chi^2}{\partial b} = -2 \sum_{i=1}^{N} \frac{y_i - a - bx_i}{\sigma_i} x_i = 0
\]

• We can define:

\[
S \equiv \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \quad S_x \equiv \sum_{i=1}^{N} \frac{x_i}{\sigma_i^2} \quad S_y \equiv \sum_{i=1}^{N} \frac{y_i}{\sigma_i^2}
\]

\[
S_{xx} \equiv \sum_{i=1}^{N} \frac{x_i^2}{\sigma_i^2} \quad S_{xy} \equiv \sum_{i=1}^{N} \frac{x_i y_i}{\sigma_i^2}
\]
Least-Squares Fitting II

- We can then rewrite the equations as:

\[ aS + bS_x = S_y \]
\[ aS_x + bS_{xx} = S_{xy} \]

- We solve the equations for \( a \) and \( b \):

\[
a = \frac{S_{xx}S_y - S_xS_{xy}}{\Delta} \quad b = \frac{SS_{xy} - S_xS_y}{\Delta}
\]
\[ \Delta = SS_{xx} - (S_x)^2 \]

- The error in \( a \) and \( b \) can be estimated by propagating the errors in the above equations (independent case), the result is:

\[
\sigma_a = \sqrt{\frac{S_{xx}}{\Delta}} \quad \sigma_b = \sqrt{\frac{S}{\Delta}}
\]
Least-Squares Fitting - Example

\[
\begin{align*}
\vec{x} &= \begin{pmatrix} 2 \\ 4 \\ 6 \\ 8 \\ 10 \end{pmatrix}, \\
y_1 &= \begin{pmatrix} 431.219 \\ 184.518 \\ 81.06 \\ 31.792 \\ 9.71 \end{pmatrix}, \\
\sigma_1 &= \sqrt{y_1}, \\
N &= \text{length}(x).
\end{align*}
\]

**Transform the data**

\[
y := \ln(y_1) \\
\sigma := \sigma_1 / y_1\]

**Remember:** \( y = I_0 \exp(-\mu x) \Rightarrow \ln(y) = \ln(y_0) - \mu x \)

**Fit the data**

\[
\begin{align*}
S &:= \sum_{i=1}^{N} \frac{1}{(\sigma_i)^2}, \\
S_x &:= \sum_{i=1}^{N} \frac{X_i}{(\sigma_i)^2}, \\
S_y &:= \sum_{i=1}^{N} \frac{Y_i}{(\sigma_i)^2}, \\
S_{xx} &:= \sum_{i=1}^{N} \frac{(X_i)^2}{(\sigma_i)^2}, \\
S_{xy} &:= \sum_{i=1}^{N} \frac{X_i Y_i}{(\sigma_i)^2}.
\end{align*}
\]

\[
\Delta := S S_{xx} - S_x^2, \quad a := \frac{S S_{xy} - S_x S_y}{\Delta}, \quad b := \frac{S S_{xy} - S_y S_x}{\Delta}, \quad \sigma_a := \sqrt{\frac{S_x^2}{\Delta}}, \quad \sigma_b := \sqrt{\frac{S_y}{\Delta}}.
\]

**Transforming the results of the fit**

\[
\begin{align*}
\mu &= -b, \\
\sigma_\mu &= \sigma_b, \\
I_0 &= \exp(a), \\
I_0 &= \exp(a) \cdot \sigma_a.
\end{align*}
\]

\[
\begin{align*}
\mu &= 0.44, \\
\sigma_\mu &= 0.02, \\
I_0 &= 1041, \\
\sigma_I &= 78.
\end{align*}
\]

\[
\begin{align*}
j &:= 1, \ldots, 100, \\
x_j &:= 0.1 j, \\
J_j &:= I_0 \exp(-\mu x_j).
\end{align*}
\]